Field Current Identity
and
the Anomalous Magnetic Moment of the Muon

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We estimate the contribution of the hadron current to the anomalous magnetic moment of the muon by the hypothesis “Field Current Identity.” The result is too small to explain the discrepancy between experimental and the theoretical values.

§ 1. Introduction

The new experimental result on the g-factor of the muon has been reported to be $\kappa_{\text{exp.}} = (g - 2)/2 = (11666 \pm 5) \times 10^{-7}$, which is slightly above the theoretical value of the pure electrodynamics. The theoretical Q.E.D. value up to fourth order in $e$ leads to $(\kappa_{\text{exp.}} - \kappa_{\text{Q.E.D.}}) = (11 \pm 5) \times 10^{-7}$. To explain this difference we estimate the effect of the hadron current on the anomalous magnetic moment of the muon by the hypothesis “Field Current Identity,” which is essentially equivalent to the vector dominance model in the electromagnetic interaction. The electric current of the hadron is expressed by the linear combination of the three neutral vector fields $\rho$, $\phi$, and $\omega$.

In this calculation the $\phi$-$\omega$ mixing is also considered explicitly.

§ 2. Calculation

The electric current of the hadron $j_\mu^{(h)}$ is expressed, using the “Field Current Identity”, as follows,

$$j_\mu^{(h)} = e \left[ \frac{m_\rho^2}{g_\rho} \rho_\mu(x) + \frac{1}{2g_\gamma} \left( \cos \theta_F m_\phi \phi_\mu(x) - \sin \theta_F m_\omega \omega_\mu(x) \right) \right],$$

(2.1)

where $g_\rho$ and $g_\gamma$ are the coupling constants of vector mesons to hadron current, and $\theta_F$ is the $\phi$-$\omega$ mixing angle. The photon propagator has to be modified by the coupling of the electromagnetic field to the current $j_\mu^{(h)}$ of (2.1). After renormalization of the photon mass and charge we have the modified photon propagator.
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propagator to second order in $\alpha$

$$D_{\mu\nu} = D_{\mu\nu}^{\text{free}} + D_{\mu\nu}^{(h)}$$  \hspace{1cm} (2.2)

where $D_{\mu\nu}^{\text{free}}$ is the free photon propagator and $D_{\mu\nu}^{(h)}$ is the hadronic contribution to second order in $\alpha$ which is expressed in the (vector meson) mass spectral representation, as follows,

$$D_{\mu\nu}^{(h)}(x) = -\frac{i e^2}{(2\pi)^4} \int_0^\infty \frac{da}{a^2} \left\{ \frac{m_F^4}{g_{\rho a}} \sigma^a(a) + \frac{1}{4} \left( g^{-1} M^2 \sigma^a(a) M^2 g^{-1\tau} \right) \right\}$$

$$\times \int \frac{d^4 k}{k^2 - a + i\varepsilon} e^{-i k x}. \hspace{1cm} (2.3)$$

The spectral function $\sigma^a$ of (2.3) is related to the matrix element of the spatial components of $p_{\rho a}^\dagger$ by

$$\sigma^a(a) = \frac{(2\pi)^3}{3} \sum_a \delta(p^a - a) \langle 0 | p(0) | P = 0, P_{\rho a}^\dagger = a, s = 1, \alpha \rangle^2, \hspace{1cm} (2.4)$$

where $s$ and $\alpha$ represent spin and other quantum numbers of vector mesons, $g$ and $M^2$ are the $(2 \times 2)$ matrices given by

$$M^2 = \begin{pmatrix} m^2_{\phi} & 0 \\ 0 & m^2_{\omega} \end{pmatrix}, \quad g = T^{-1} P^\dagger, \quad T = \begin{pmatrix} \cos \theta_T & -\sin \theta_T \\ \sin \theta_T & \cos \theta_T \end{pmatrix}, \quad P_{\rho a}^\dagger = \begin{pmatrix} g_\rho & 0 \\ 0 & g_a \end{pmatrix}, \hspace{1cm} (2.5)$$

and the $(2 \times 2)$ matrix $\sigma_{\rho a}(a)$ is the mixed spectral function. The subscript 11 denotes the matrix element of the first row and first column.

Using the photon propagator of (2.2) we can calculate the anomalous magnetic moment of the muon including the effect of the hadron current and find the contribution of the hadron current of second order in $\alpha$ as follows,

$$\kappa^{(h)} = 4 m^2_{\rho a} \alpha^2 \int_0^\infty \frac{da}{a^2} \left\{ \frac{m_F^4}{g_{\rho a}} \sigma^a(a) + \frac{1}{4} \left( g^{-1} M^2 \sigma^a(a) M^2 g^{-1\tau} \right) \right\}$$

$$\times \int \frac{1}{m^2_{\rho a} x + a(1-x)} dx. \hspace{1cm} (2.6)$$

This is the most general form for the contribution of hadron current to the anomalous magnetic moment of the muon.

The $(2 \times 2)$ matrix spectral function $\sigma_{\rho a}(a)$ can be diagonalized by the same method given in reference 2 in the following form,

$$\sigma_{\rho a}(a) = \begin{pmatrix} \sigma_\phi(a) & 0 \\ 0 & \sigma_\omega(a) \end{pmatrix}, \hspace{1cm} (2.7)$$
where \( \sigma_\phi(a) \) and \( \sigma_\omega(a) \) are the spectral functions of the \( \phi \) and \( \omega \) mesons respectively. We express the \( \kappa^{(b)} \) in terms of this \( \sigma_\phi(a) \),

\[
\kappa^{(b)} = -\frac{2\pi^2 a^2}{\alpha^2} \frac{da}{\alpha^2} \left\{ \frac{m_\phi^4}{g_\phi^2} \sigma_\phi(a) + \frac{m_\phi^4}{4g_{\phi'}^2} \cos^2 \theta_{\phi} \sigma_\phi(a) + \frac{m_\phi^4}{4g_{\phi'}^2} \sin^2 \theta_{\phi} \sigma_\phi(a) \right\}
\]

\[
\times \left[ \frac{x^2(1-x)dx}{m_\phi^2 x^2 + a(1-x)} \right].
\]

If the spectral functions \( \sigma_\phi(a), \sigma_\omega(a) \) and \( \sigma_{\omega}(a) \) are known, we can calculate \( \kappa^{(b)} \) from (2.8).

First we assume the spectral functions to be \( \delta \)-function;

\[
\sigma_\phi(a) = \delta(a - m_\phi^2), \quad \sigma_\omega(a) = \delta(a - m_\omega^2), \quad \sigma_{\omega}(a) = \delta(a - m_\omega^2).
\]

Then we can obtain from (2.8) and (2.9),

\[
\kappa^{(b)} = \frac{2\pi}{3} \left[ \frac{4\pi}{g_\phi^2} K(m_\mu/m_\phi) + \frac{1}{4} \frac{4\pi}{g_{\phi'}^2} \{ \cos^2 \theta_{\phi} K(m_\mu/m_\phi) + \sin^2 \theta_{\phi} K(m_\mu/m_\omega) \} \right],
\]

where the function \( K(z) \) is defined as follows,

\[
K(z) = z^3 \left[ 1 + z^2 \left( \frac{25}{4} + 6 \ln z \right) + O(z^3) \right].
\]

In deriving (2.10) we used the approximation

\[
-\int_0^1 \frac{x^3(1-x)dx}{b x^3 - x + 1} = \frac{1}{3} \left( 1 + \frac{25}{4} b + 3 b \ln b \right) \quad \text{for } b \ll 1.
\]

The form of the \( \kappa^{(b)} \) of (2.10) is the same as that which was derived in the dispersion theory.\(^3\)

Next we consider that these neutral vector mesons are unstable particles. In this case the assumption (2.9) of zero width is not valid. We derive the spectral functions for the unstable vector mesons, in order to estimate the effect of finite width of unstable particle. As the first step we write the propagator of an unstable vector meson \( \nu_\mu \) in the two forms,

\[
\langle 0|T(\nu_\mu(x)\nu_\mu(0))|0\rangle = -\frac{i}{(2\pi)^4} \int_0^\infty \frac{da}{a^2} \sigma_{\nu}(a) \int d^4k \left( g_{\mu\nu} - \frac{2 k_\mu k_\nu}{k^2} \right) \frac{e^{-ikx}}{k^2 - a + i\varepsilon}.\]

Comparing the imaginary parts of two forms of (2.12), we can express the spectral function in terms of the width \( \Gamma \) of the unstable vector meson,
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\[
\sigma_V(a) = \frac{1}{\pi} \frac{\Gamma m_V}{(a-m_V)^2 + \Gamma^2 m_V^2} \theta(a-a_o), \quad (2.13)
\]

where \(a_o\) is a lower limit of integration. We substitute these spectral functions of (2.13) into (2.8) and obtain

\[
\kappa^{(b)} = \frac{e^2}{3\pi} \left[ \frac{4\pi}{g_\rho^2} I(\xi_\rho, b_\rho) + \frac{1}{4} \frac{4\pi}{g_\phi^2} \left( \cos^2 \theta \Gamma I(\tilde{\xi}_\phi, b_\phi) + \sin^2 \theta \Gamma I(\tilde{\xi}_\omega, b_\omega) \right) \right],
\]

where the function \(I(\xi, b)\) is defined as follows,

\[
I(\xi, b) = \frac{m^2}{m_V^2} b \int \frac{dx}{x^3 (x-1)^2 + b^2},
\]

where \(b\) is the ratio of width of the vector meson to the mass \(\Gamma/m_V\), and the lower limit of integration is given by

\[
\xi_\rho = \frac{4m_\rho^2}{m_V^2}, \quad \tilde{\xi}_\phi = \frac{9m_\phi^2}{m_V^2}, \quad \tilde{\xi}_\omega = \frac{9m_\omega^2}{m_V^2}.
\]

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\(\tilde{\xi}_\omega = \frac{9m_\omega^2}{m_V^2}.
\)

\section{Results and discussion}

In the previous section we obtain the expression (2.8) for the contribution of hadrons to the anomalous magnetic moment of the muon. Here we estimate the value \(\kappa^{(b)}\), using the experimental values of the coupling constants \(g_\rho, g_\phi\), the mixing angle \(\theta\) and the widths \(\Gamma_\rho, \Gamma_\phi, \Gamma_\omega\). Though these experimental data have not yet been determined conclusively, we adopt the values for the \(\rho^0\) meson,

\[
m_\rho = 770\text{ MeV}, \quad \Gamma_\rho = 140\text{ MeV}, \quad \frac{g_\rho^2}{4\pi} = 2.3.
\]

But, for the \(\phi\) and \(\omega\) mesons, the zero width approximation (2.9) is certainly valid. Then the final results are

\[
\kappa^{(\rho)} = 1.6 \times 10^{-7},
\]

\[
\kappa^{(\phi)} + \kappa^{(\omega)} \sim 8 \times 10^{-9},
\]

thus \(\kappa^{(b)} = 1.6 \times 10^{-7}\).

These values are not definite because of the uncertainty of the above experimental data for vector mesons, but seem to be too small to explain the discrepancy \((\kappa_{\exp} - \kappa_{\text{Q.E.D.}}) = (11 \pm 5) \times 10^{-7}\).

If we use the sum rules for the spectral functions, we can estimate the upper limit of the contribution of the hadron (2.8) approximately. The estimated upper limit is found to be not so different from (3.1).

From our analysis we conclude, though not definitely, that there is a significant difference between theoretical and experimental values. If this discrepancy
were definitely established, we would like to consider the effect of bosons other than vector mesons.4)

We would like to thank Professor Y. Miyamoto for helpful advice.

References

   This result was also derived by the same methods as this paper. H. Terazawa, Prog. Theor. Phys. 39 (1968), 1326.
4) Y. Miyamoto, Soryushiron Kenkyu (mimeographed circular in Japanese) 37 (1968), 227.
   Y. Miyamoto anticipates the existence of massive axial photon from the chiral invariance of Q.E.D.