Proof of the Factorizability Theorem
Conjectured by Sciarrino and Toller

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A general proof of a symmetry property of the residues of the transformation function between irreducible representations of $SU(2)$ and $SU(1,1)$ which is encountered in the decomposition of the principal series of the $SL(2,C)$ functions is presented.

§ 1. Statement of the problem

Recently, Sciarrino and Toller\(^1\) investigated the decomposition of unitary irreducible representations of $SL(2,C)$ into those of $SU(1,1)$ in order to expand a Lorentz-pole contribution into a family of Regge poles. Under the assumption that the Lorentz-pole residue is factorizable, the Regge-pole residues of the corresponding family are shown to be also factorizable if

$$W_{l,n}^{M,\lambda, \mu} = (-1)^{n+f-m} W_{l,m}^{M,\lambda, \mu},$$

(1.1)

where $W_{l,m}^{M,\lambda, \mu}$ denotes the residue at $l=\lambda-n-1$, ($n=0,1,2,\cdots$), of the continued transformation function,\(^2\) $K_{J,l}^{M,\lambda}(l)$, between unitary irreducible representations of $SU(2)$ and $SU(1,1)$ in a principal-series representation $J^{M,\lambda}$ of $SL(2,C)$. The function $K_{J,l}^{M,\lambda}(l)$ for $\text{Re} \ l = -\frac{1}{2}$ and $\text{Re} \ \lambda = 0$ is defined by

$$K_{J,l}^{M,\lambda}(l) = \sqrt{2j+1} (-1)^{n-M} \sum \int_{0}^{\pi} (\cosh \zeta)^{l-m} r_{M,m}(\theta)$$

$$\times d_{m,M}(\zeta)^{\frac{1}{2}} \sinh \zeta d\zeta$$

(1.2)

with

$$\tan \frac{\theta}{2} = \tanh \frac{\zeta}{2}, \quad (0 \leq \theta < \frac{\pi}{2}),$$

(1.3)

where $r_{M,m}(\theta)$ is the well-known function appearing in the generalized spherical harmonics,\(^3\) and $d_{m,M}(\zeta)$ is a representation function\(^4\) of the continuous classes of $SU(1,1)$. Since (1.2) has a symmetry property

$$K_{J,-l}^{M,\lambda}(l) = (-1)^{M-m} K_{J,l}^{M,\lambda}(l),$$

(1.4)

\(^*) K_{J,l}^{M,\lambda}(l)$, identical with $K_{M}^{\lambda}(e, J, \pm j)$ of reference 1, is an analytic function of the complex variables $l$ and $\lambda$; $2j$, $f=M$ and $f=m$ are non-negative integers.
we may confine ourselves to the case $m \geq M$ without loss of generality. Then
the explicit expressions for $r^j_{M,m}(\theta)$ and $d^j_{m,M}(\zeta)$ read
\[ r^j_{M,m}(\theta) = \sum_{m \geq M} (\cos \frac{1}{2} \theta)^{M+m+1} (\sin \frac{1}{2} \theta)^{m-M} \frac{(j-m)!(j+M)!}{(j-M)!} \times F(-j+m, -j-M; m-M+1; -\tan^2 \frac{1}{2} \theta) 
\] (1.5)
with
\[ (c^j_{M,m})^\gamma = (j+M)!(j-M)!(j+m)!(j-m)! 
\] (1.6)
and
\[ d^j_{m,M}(\zeta) = \frac{\Gamma(l+m+1)}{(m-M)! \Gamma(l+M+1)} \left( (\cosh \frac{1}{2} \zeta)^{m+M} (\sinh \frac{1}{2} \zeta)^{m-M} \right) \times F(-l+m, l+m+1; m-M+1; -\sinh^2 \frac{1}{2} \zeta), 
\] (1.7)
respectively.

If we put
\[ z = \sinh^2 \frac{1}{2} \zeta, 
\] (1.8)
then (1.2) together with (1.5) and (1.7) becomes
\[ K_{j,m}^{\lambda}(l) = A^j_{M,m}(l) \int_0^\infty dx (1+2x)^{\lambda-j-1} (1+z)^{j+M} \times F(-j+m, -j-M; m-M+1; -z) \] (1.9)
with
\[ A^j_{M,m}(l) = \frac{(-1)^{M-l} \Gamma(l+m+1)}{(j-m)! (j+M)! [m-M)! \Gamma(l+M+1)}, 
\] (1.10)
\[ A^j_{M,m} = \sqrt{2j+1} c^j_{M,m}. 
\] (1.11)

Sciarrino and Toller\(^1\) explicitly carried out the integration in (1.9) by expanding the hypergeometric function of the integral parameters into a finite series. Their explicit formula shows that $K_{j,m}^{\lambda}(l)$ has simple poles at
\[ l = \lambda - n - 1, \]
\[ l = -\lambda + n', \]
where $n$ and $n'$ are non-negative integers. We may confine ourselves to the former alone because the residue of the latter is simply related to that of the former. Sciarrino and Toller explicitly calculated the residue
\[ W^M_{j,m} = \lim_{l \to \lambda - n - 1} (l - \lambda + n + 1) K_{j,m}^{\lambda}(l), 
\] (1.13)
but their result contains a triple summation. They proposed the identity (1·1) by verifying it for \( n=0, 1 \) algebraically and for many choices of parameters by means of a computer. Very recently, Delbourgo, Koller and Mahanta\(^6\) and Akyeampong, Boyce and Rashid\(^9\) have found simpler expressions for \( K_{f,m}^{K,\alpha,n}(l) \), but they are still not suitable for proving (1·1).

In the present paper, we present a general proof of (1·1). For this purpose, it is convenient to calculate the residue (1·13) before carrying out the integration in (1·9). In this way we directly find an expression for \( W_{f',\alpha,n}^{K,\alpha,n} \) in closed form. On the the basis of it, we prove (1·1) by only using some linear transformation properties of the hypergeometric function.

\section{Proof of (1·1)}

By putting \( x=1/z \), (1·9) becomes

\[
K_{f,m}^{K,\alpha}(l) = \Lambda^{K}_{f,m}(l) \int_{0}^{\infty} dx \ x^{-l-m-1}(2+x)^{\lambda-j-1}(1+x)^{j+m} \\
\times F(-j+m, -j-M; m-M+1; -1/(1+x)) \\
\times F(-l+m, l+m+1; m-M+1; -1/x). \quad (2·1)
\]

This integral can be divergent only in the neighbourhood of \( x=0 \). Because of (A·1), the last factor of (2·1) can be rewritten as

\[
F(-l+m, l+m+1; m-M+1; -1/x) \\
= \frac{(m-M)! \Gamma(2l+1)}{\Gamma(l+m+1) \Gamma(-l-M+1)} x^{-l+m} F(-l+m, -l+M; -2l; -x) \\
+ \frac{(m-M)! \Gamma(-2l-1)}{\Gamma(-l+m+1) \Gamma(-l-M)} x^{l+m+1} F(l+m+1, l+M+1; 2l+2; -x). \quad (2·2)
\]

In order to continue \( K_{f,m}^{K,\alpha}(l) \) analytically beyond the domain of the convergence of (2·1), we employ a pseudofunction\(^6\)

\[
Y_{\nu}(x) = [1/\Gamma(\nu)] Pf. x^{-\nu} \theta(x) \quad \text{for} \quad \nu \neq 0, -1, -2, \cdots, \\
= \delta^{(-\nu)}(x) \quad \text{for} \quad \nu = 0, -1, -2, \cdots, \quad (2·3)
\]

where Pf. denotes Hadamard's finite part. Since our integral (2·1) has the form

\[
K_{f,m}^{K,\alpha}(l) = \int_{0}^{a} dx \ x^{j-m} \varphi(x, l) + \int_{0}^{a} dx \ x^{-l-m-1} \psi(x, l) \\
+ \int_{a}^{\infty} dx \chi(x, l), \quad (a>0), \quad (2·4)
\]

where \( \varphi(x, l) \) and \( \psi(x, l) \) are analytic and non-vanishing at \( x=0 \), we obtain its
analytic continuation in \( l \) and \( \lambda \) by replacing \( x^{-1} \) by \( \text{Pf.}x^{-1} \), namely,

\[
K_{l,\lambda}^{M,n}(l) = \Gamma(l-\lambda+1) \int_{-a}^{a} dx \, Y_{l-\lambda+1}(x) \varphi(x, l) + \Gamma(-l-\lambda) \int_{-a}^{a} dx \, Y_{l+\lambda}(x) \psi(x, l) + \int_{a}^{\infty} dx \, \chi(x, l).
\]

(2.5)

All integrals in (2.5) are now finite for any values of \( l \) and \( \lambda \), and hence it is clear that \( K_{l,\lambda}^{M,n}(l) \) has simple poles at the locations given by (1.12). We consider a pole at \( l^2 = \lambda - n - 1 \). Since

\[
\lim_{z \to n} (z+n) \Gamma(z) = (-1)^{n}/n!,
\]

(2.6), (1.13), (2.5) and (2.3) imply

\[
W_{l,\lambda}^{M,n} = (-1)^{n} \frac{1}{n!} \int_{0}^{a} dx \, \delta^{(n)}(x) \varphi(x, \lambda - n - 1) = (1/n!) (d/dx)^{n} \varphi(x, \lambda - n - 1) \Big|_{x=0}.
\]

(2.7)

On substituting the concrete expression for \( \varphi(x, l) \), we obtain an expression for \( W_{l,\lambda}^{M,n} \) in closed form:

\[
W_{l,\lambda}^{M,n} = \left(-\frac{1}{n!}\right)^{n} \int_{0}^{a} dx \, \delta^{(n)}(x) \varphi(x, \lambda - n - 1)
\]

(2.8)

with

\[
\mu = \lambda - n,
\]

(2.9)

\[
\alpha_{M,n}^{l,\lambda} - \alpha_{M,n}^{l,\lambda} = \frac{\Delta_{M,n}^{l,\lambda} \Gamma(-2\mu + 1)}{(j-m)! \Gamma(m+1) \Gamma(-\mu - m + 1)}.
\]

(2.10)

From (A.1) again we have\(^*\)

\(^*\) If the differentiations are explicitly carried out, (2.8) reproduces (5.22) of reference 1). Furthermore, from (2.8) we can easily find the generating function \( \sum_{n} W_{l,\lambda}^{M,n} y^n \) in closed form by means of Cauchy's theorem and a transformation \( x = 2y/(1-y) \).

\(^*\) Here we consider the case \( m + M > 0 \). For \( m + M < 0 \), the right-hand side of (2.11) should be replaced by

\[
(-1)^{j+M} (j-m)! (j+M)! (-m-M)! \Gamma(-j-m, -j-M; -m-M+1; 1/(1+x)).
\]

We change the sign of \( m \) instead of \( M \), taking account of (2.10) and an identity

\[
F(\mu + m, \mu + M; 2\mu; -x) = (1+x)^{\mu-M} \mu F(\mu + m, \mu - M; 2\mu; -x),
\]

which follows from (A.2). Then we obtain (2.12) on account of \( W_{l,\lambda}^{M,n} = (-1)^{m-n} W_{l,\lambda}^{M,n} \), which follows from (1.4). Thus (2.12) is always valid. The author thanks Dr. N. Seto for pointing out the inadequacy of (2.11) for \( m + M < 0 \).
Proof of the Factorizability Theorem

\[ \frac{(1+x)^{j+M}}{(j+M)!} F(-j+m, -j-M; m-M+1; -1/(1+x)) \]

\[ = \frac{(-1)^{j-m}(1+x)^{m+M}}{(j-M)! (m+M)!} F(-j+m, -j-M; m+M+1; -1/(1+x)). \] (2·11)

Substitution of (2·11) in (2·8) yields another expression for \( W_{j,m}^{k,n} \). By changing the sign of \( M \) in it, we find

\[ W_{j,m}^{k,n} = \frac{(-1)^{j-m} \alpha_{j,m,k}^{n}}{(j+M)! (m-M)!} \left[ (2+x)^{n+j-1}(1+x)^{m-M} \right. \]
\[ \times F(-j+m, -j-M; m-M+1; -1/(1+x)) \]
\[ \left. \times F(\mu+m, \mu-M; 2\mu; -x) \right|_{x=0}. \] (2·12)

Hence, in order to prove (1·1), we have only to show

\[ \left. \frac{1}{n!} \left( \frac{d}{dx} \right)^n [(2+x)^nf(x)] \right|_{x=0} = \frac{(-1)^n}{n!} \left( \frac{d}{dx} \right)^n \left[ (2+x)^ng(x) \right] \right|_{x=0}, \] (2·13)

where

\[ f(x) = (2+x)^{n+j-1}(1+x)^{j+M}F(-j+m, -j-M; m-M+1; -1/(1+x)) \]
\[ \times F(\mu+m, \mu-M; 2\mu; -x), \] (2·14)
\[ g(x) = (2+x)^{n+j-1}(1+x)^{m-M}F(-j+m, -j-M; m-M+1; -1/(1+x)) \]
\[ \times F(\mu+m, \mu-M; 2\mu; -x). \] (2·15)

Carrying out the differentiations in (2·13), we have

\[ \sum_{k=0}^{n} C_{k}(2^{k}/k!)f^{(k)}(0) = (-1)^{n} \sum_{k=0}^{n} C_{k}(2^{k}/k!)g^{(k)}(0). \] (2·16)

Of course (2·16) is uniquely solved with respect to \( g^{(n)}(0) \), which is given by

\[ g^{(n)}(0) = (-1)^{n} \sum_{k=0}^{n} C_{k}(n!/k!)f^{(k)}(0), \] (2·17)

as is verified by inserting it into (2·16). We rewrite (2·17) in terms of the generating function \( g(x) \):

\[ g(x) = \sum_{n=0}^{\infty} (x^{n}/n!) g^{(n)}(0) \]
\[ = \sum_{k=0}^{n} \frac{(-x)^{k}}{k!} f^{(k)}(0) (1+x)^{-k-1} \]
\[ = \frac{1}{1+x} f\left( \frac{-x}{1+x} \right). \] (2·18)

Therefore, to prove (1·1) is equivalent to showing

\[ f\left( -x/(1+x) \right) = (1+x)g(x). \] (2·19)
Since

\[ F(\mu + m, \mu + M; 2\mu; x/(1 + x)) = (1 + x)^{\mu+m} F(\mu + m, -\mu - M; 2\mu; -x) \]  \hspace{1cm} (2·20)

because of (A·2), the identity (2·19) is immediately verified by substituting (2·14) and (2·15) in it. Thus (1·1) has been established for all \( n \).

**Appendix**

The following formulas for the hypergeometric function are well known:

\[
F(a, b; c; z) = \frac{\Gamma(c) \Gamma(b-a)}{\Gamma(b) \Gamma(c-a)} (-z)^{-a} F(a, 1-c+a; 1-b+a; 1/z) \\
+ \frac{\Gamma(c) \Gamma(a-b)}{\Gamma(a) \Gamma(c-b)} (-z)^{-b} F(b, 1-c+b; 1-a+b; 1/z),
\]  \hspace{1cm} (A·1)

\[
F(a, b; c; z) = (1-z)^{-a} F(a, c-b; c; z/(z-1)).
\]  \hspace{1cm} (A·2)

**References**


**Note added in proof:** W. Rühl appears to have very recently given another proof of the factorizability theorem, which will be published in his forthcoming book entitled “Lectures on the Lorentz Group and Harmonic Analysis.” The author is very grateful to Professor M. Toller for informing him of this matter.