\[
\Lambda = \begin{bmatrix}
-\sin \phi & -\cos \phi \\
\cos \phi & -\sin \phi
\end{bmatrix},
\]  
(A5)

and

\[
\dot{\mathbf{T}} = -\omega^2 \mathbf{T}.
\]  
(A6)

Hence, substitution of equation (10) into equation (14) gives:

\[
\mathbf{T}^{-1} \mathbf{M} \dot{\mathbf{x}} + (2\omega \mathbf{T}^{-1} \mathbf{M}^* + \mathbf{T}^{-1} \mathbf{C}) \dot{\mathbf{x}} + (\omega \mathbf{T}^{-1} \mathbf{C}^* + \mathbf{T}^{-1} \mathbf{K} - \omega^2 \mathbf{T}^{-1} \mathbf{M}) \mathbf{x} = \mathbf{F}_d.
\]  
(A7)

To evaluate \( \mathbf{T}^{-1} \mathbf{M} \), it is convenient to partition it into \( 2 \times 2 \) submatrices. Any one of these submatrices will have the form \( \Lambda^{-1} \mathbf{mA} \), where \( \mathbf{m} \) is the corresponding submatrix of \( \mathbf{M} \). Then it is easy to show that \( \Lambda^{-1} \mathbf{mA} \) equals \( \mathbf{m} \) if, and only if, the elements of \( \mathbf{m} \) are of the form:

\[
\mathbf{m} = \begin{bmatrix}
m_1 - m_2 \\
m_2 - m_1
\end{bmatrix}.
\]  
(A8)

Hence, if equation (A8) is satisfied for all submatrices of \( \mathbf{M} \), then \( \mathbf{T}^{-1} \mathbf{M} \) equals \( \mathbf{M} \). Conditions similar to equation (A8) are required for \( \mathbf{T}^{-1} \mathbf{C} \) and \( \mathbf{T}^{-1} \mathbf{K} \) to equal \( \mathbf{C} \) and \( \mathbf{K} \), respectively. As may be seen from reference [10], such conditions are satisfied for \( \mathbf{M} \), \( \mathbf{C} \), and \( \mathbf{K} \) matrices in general, even when gyroscopic effects are present.

To evaluate \( \mathbf{T}^{-1} \mathbf{M}^* \), it is again convenient to partition it into \( 2 \times 2 \) submatrices. Any one of these submatrices will have the form \( \Lambda^{-1} \mathbf{mA}^* \), where \( \mathbf{m} \) is the corresponding submatrix of \( \mathbf{M} \). Hence, noting that \( \mathbf{m} \) will be of the form given by equation (A8), \( \Lambda^{-1} \mathbf{mA}^* \) will equal \( \mathbf{m} \) where \( \mathbf{m} \) is given by:

\[
\mathbf{m} = \begin{bmatrix}
-m_2 - m_1 \\
m_1 - m_2
\end{bmatrix}.
\]  
(A9)

Thus \( \mathbf{T}^{-1} \mathbf{M}^* \) equals \( \mathbf{M} \), where the \( 2 \times 2 \) submatrices \( \mathbf{m} \) of \( \mathbf{M} \) are formed from the corresponding submatrices \( \mathbf{m} \) of \( \mathbf{M} \) according to equations (A8) and (A9). Similarly \( \mathbf{T}^{-1} \mathbf{C}^* \) equals \( \mathbf{C} \). Hence, equation (A7) simplifies to equation (15).

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**DISCUSSION**

E. J. Gunter\(^1\)

The topic of dynamic characteristics of squeeze film dampers in multi-mass rotors is one of considerable importance in the field of turbomachinery. The squeeze film damper is currently being extensively employed in a number of aircraft engines with rolling element bearings. Without the squeeze film damper, many of the modern aircraft engines would suffer from serious vibrations.

In addition to the application of squeeze film dampers for attenuation of critical speed and balancing, the squeeze film damper has also been employed in centrifugal compressors for the control of self-excited instability. The journal bearing is mounted in a damper bearing. The damper bearing may be supported by either "O" rings or centering springs. For the case of horizontal rotors of moderate to high weight, centering springs are normally employed. In certain applications of aircraft gas turbines, centering springs may not necessarily be employed. Under these circumstances, the lift-off characteristics depend upon the actual rotor unbalance and the supply pressure or cavitation conditions in the damper.

The simplest model of a flexible rotor in a damper supports is given by the author in equation (31). These equations have been extensively examined and presented in the report "Dynamic Stability of Rotor-Bearing Systems," NASA SP113 1965. For the case of the linear system, equation (31) represents a stable system.

With a centering spring, the squeeze film damper for small perturbations is always stable. However, with the application of large rotating unbalance loads, nonlinear jumps may occur in the response. The response, however, is still synchronous motion, whereas, self-excited motion represents an excitation of the lowest system eigenvalue. Self-excited motion is normally caused by fluid film bearings, aerodynamic cross-coupling or rotor internal friction. I have never encountered a case in which the squeeze film dampers caused a self-excited instability. One normally considers instability in the linearized case as the condition in which an eigenvalue has a real positive root. For self-excited motion in the nonlinear case, a limit cycle motion is achieved. Analysis of this limit cycle motion shows a component of synchronous and non-synchronous motion.

The characteristics of a squeeze film damper with a centering spring have proven to be superior to that of the damper with no centering spring. If the unbalance causes the eccentricity to exceed approximately 40 percent of the damper clearance, then the nonlinear forces in the damper build up rapidly. In the case of a fully cavitated damper, a radial stiffness effect is generated. In the case of a damper with no centering spring, cavitation is required in order to achieve a lift-off effect. Computer simulations of an uncentered supported damper with no cavitation and gravitational loading produce poor results since lift-off cannot be achieved. If a pressurized squeeze film damper is employed such that cavitation does not occur, then a centering spring must be employed to achieve satisfactory dynamical characteristics.

To fully understand the characteristics of the nonlinear damper, it is desirable to perform time transient studies in which the pressure profile is integrated at each time step. One can then see the change in orbit as affected by unbalance. However, under no circumstances, has self-excited instability been observed.

In general, I feel that the word "stability" in the title to describe the characteristics of the squeeze film damper may not be appropriate as this may lead to confusion with the concepts of self-excited instability of linear systems. I feel that the term "nonlinear response" may be more appropriate. I would also recommend that the authors extend their investigation to cover the nonlinear response of squeeze film dampers by the direct numerical integration of the Reynolds equation. This approach would yield additional insight than that which can be obtained by using the perturbed solution about a circular orbit. For example, with gravitational loading, the orbit is no longer circular and secondary harmonics can be developed.

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The authors thank Professor Gunter for his interesting discussion. The utility of squeeze film dampers in attenuating unbalance vibrations as well as for promoting stability in otherwise unstable rotor bearing systems, both in the presence and absence of central preload, has been widely recognized for some time now. Though the relative merits of centrally preloaded dampers vis a vis dampers without centering springs are arguable, there is no doubt that parametric design analysis of unbalanced axially symmetric rotor bearing systems utilizing the former is significantly simpler and indeed as shown in Ref. [2] and in this paper, quite feasible for the case of a single damper, though certainly not trivial.

The claim that "with a centering spring, the squeeze film damper for small perturbations is always stable," cannot, unfortunately, be substantiated. Fig. 6 of the paper shows several instances of unstable operation as did Ref. [3]. Whether the instability can be termed "self-excited" is a question of definition, and as far as the authors can see, of no immediate relevance to this paper which is concerned with the stability of the non-linear equilibrium solutions, with instability, in agreement with the discussor, being present whenever an eigenvalue of the linearized perturbed system has a positive real part. In view of this, the authors are at a loss to understand why the use of the term "stability" in the title is anything but appropriate. After all, the non-linear response has already been dealt with Ref. [2], and it is the stability of this response which is here being investigated.

The authors do not doubt the value of transient solution techniques for analyzing squeeze film damped rotor bearing systems. They are also conscious of the limitations of such an approach, as well as the potentially huge computational effort involved in arriving at a reasonably accurate steady state solution, particularly in situations of low damping, thereby often rendering the generation of parametric design data prohibitively time consuming. It is precisely to overcome the problems associated with transient solutions that circular orbit solution techniques have been developed. Insofar as such solutions actually correspond to many practical physical systems, such circular orbit solutions have proved to be an extremely useful design tool and indeed, can even occasionally be gainfully used to predict potentially unsuitable damper designs for practical systems with resultant unidirectional loading.