On the Solutions of the Bethe-Salpeter Equation in the Unequal-Mass Wick-Cutkosky Model in the Moving System

Kenji SETO

Department of Physics, Faculty of Science
Hokkaido University, Sapporo

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The stereographic projection method is applied to obtain the bound state amplitude in the moving system of the Bethe-Salpeter equation in the unequal-mass Wick-Cutkosky model. The timelike, spacelike, lightlike and spurionlike solutions are derived from the one in the moving system. The normalization properties of these solutions are also discussed.

§ 1. Introduction

It has been found in an early paper that the stereographic projection method, which was first introduced by Cutkosky, is extremely useful to solve the Bethe-Salpeter (B-S) equation in the unequal-mass Wick-Cutkosky model for the bound state amplitude in the rest system. The method leads us to the $O(4)$ symmetry in the converted momentum space, and hence it becomes possible to solve the equation by use of the four-dimensional solid harmonic.

In this paper, the same method is applied to solve the bound state equation in the moving system by the extended use of the prescriptions given in the early paper. It will be shown that the method is also useful for this case and the solution in the moving system obtained in this way contains the timelike, spacelike, lightlike and spurionlike solutions which are the analytic continuities of one another with respect to the total momentum of the bound state. The solutions have also been presented by Nakanishi without using the stereographic projection method; however, the method adopted in this paper to derive these solutions is much simpler than that of Nakanishi's and the connection between these solutions can easily be seen.

In the next section we apply the stereographic projection method to the B-S equation in the moving system and its solutions are explicitly given for the timelike, spacelike, lightlike and spurionlike bound states in § 3. The B-S equation in the unequal-mass Wick-Cutkosky model presents a useful mathematical tool to reveal the analytical behaviors of the bound state such as the negative norm and the multiple poles of the corresponding Green's function. These analytical behaviors are discussed in the final section.
§ 2. B-S equation in the moving system and the stereographic projection method

The total momentum \((2k)\) of the bound state has at least two nonvanishing components in the moving system. We take the frame where its zero-th and third components are nonvanishing while other ones are vanishing. The B-S equation for the bound state amplitude \(\phi(p, p_0)\) in the unequal-mass Wick-Cutkosky model reads

\[
[(1 + \Delta)\sin \theta + (p - k)] [(1 - \Delta)\sin \theta + (p + k)] \phi(p, p_0) = \frac{\lambda}{\pi^2 i} \int d^4p' \phi(p', p_0') \frac{\delta(p' - p)^3 - i\epsilon}{(p - p')^3 - i\epsilon},
\]

where \(\Delta, 1 - \Delta\) are the masses of the constituent particles, \(\lambda\) is the coupling constant squared and \((p, p_0)\) is the relative momentum. The Wick-rotated form of (2·1) is given by

\[
[(1 + \Delta)\sin \theta + (p - i\eta)] [(1 - \Delta)\sin \theta + (p + i\eta)] \phi(p, p_0) = \frac{\lambda}{\pi^2} \int d^4p' \phi(p', p_0') \frac{\delta(p' - p)^3 - i\epsilon}{(p - p')^3 - i\epsilon}.
\]

Here the Euclidean \((\eta, \eta_0)\) is defined by

\[
(\eta, \eta_0) = (0, 0, ik_3, k_0)
\]

and is related to the total energy squared \(s\) by

\[
\eta^2 = k_3^2 - k_0^2 = -k^2 = s/4.
\]

In order to manipulate the B-S equation so that it exhibits the complete rotational invariance, we can adopt the stereographic projection method in much the same manner as in reference 1). First the four-momentum \((p, p_0)\) can be mapped upon the point \((\xi)\) on the surface of a five-dimensional sphere set up by introducing the fifth axis in the Wick-rotated space. If the five-vector \((\xi)\) is described in the polar coordinate, the polar angle \(\zeta\) between \((\xi)\) and the fifth axis is related to \(\pi (\frac{\rho}{\rho + \rho_0})\) and the projection radius \(|\xi|\) by \(\tan \frac{\pi}{2\rho} = \frac{|\xi|}{|\rho|}\) and the other three polar angles are the same as those of the four-dimensional momentum \((p, p_0)\). If we fix the projection radius \(|\xi|\) by

\[
|\xi| = (1 + \Delta^2 - \eta^2)^{1/2} = (1 + \Delta^2 - \frac{1}{2}s)^{1/2},
\]

Eq. (2·2) can be written in terms of the new variable \((\xi)\):

\[
\left[|\xi|^2 - |\xi| - \frac{i}{2}(|\xi| \xi_\eta + \xi_\eta \xi_\eta + d \xi_\xi - d|\xi|^2)\right] H(\xi) = \frac{\lambda}{8\pi^2} \int dQ_{\xi'} \frac{H(\xi')}{1 - \cos \gamma}
\]

with

\[
H(\xi) = \phi(p, p_0)/\sin^2(\frac{\pi}{2} \xi_\eta),
\]

where \(dQ_{\xi'}\) is an element of the solid angle in the five-dimensional space and \(\gamma\) is the angle between \((\xi)\) and \((\xi')\). In order to simplify the equation further, we introduce a new variable \((\tilde{\xi})\) by rotating the coordinate in the \(\xi_3 \xi_4 \xi_5\) space. The transformation is explicitly given by
\[ \hat{\xi}_i = \xi_i \text{ for } i = 1, 2 \]

and

\[
\begin{pmatrix}
\hat{\xi}_3 \\
\hat{\xi}_4 \\
\hat{\xi}_5
\end{pmatrix} =
\begin{pmatrix}
\cos \delta & \sin \delta & 0 \\
- \sin \delta & \cos \delta & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos \hat{\delta}_1 & 0 & - \sin \hat{\delta}_1 \\
0 & 1 & 0 \\
- \sin \hat{\delta}_1 & \cos \hat{\delta}_1 & 0
\end{pmatrix}
\begin{pmatrix}
\xi_3 \\
\xi_4 \\
\xi_5
\end{pmatrix}
\]

(2.8)

with

\[
\tan \hat{\delta}_1 = \gamma_i / \eta_i ,
\]

(2.9)

and an arbitrary complex angle \( \delta \) which will be fixed later. Now Eq. (2.6) becomes

\[
[|\hat{\xi}|^2 - |\hat{\xi}|^{-4}\{\sqrt{(1 - \eta^2)}(\hat{d}^2 - \eta^2)\hat{\xi}_5 - \hat{A}|\hat{\xi}|^2\}]\hat{H}(\hat{\xi}) = \frac{\lambda}{8\pi^2} \int d\hat{\Omega}_a \frac{\hat{H}(\hat{\xi}')}{1 - \cos \hat{\gamma}} ,
\]

(2.10)

where \( \hat{H}(\hat{\xi}) = H(\xi(\hat{\xi})) \), and \( d\hat{\Omega}_a \) and \( \hat{\gamma} \) correspond to \( d\Omega_a \) and \( \gamma \) in the \( (\hat{\xi}) \) system, respectively. Equation (2.10) is just the same as Eq. (3) of reference 1), where the bound state was treated in the rest system. Thus we see that the same prescriptions as in reference 1) can be applied to exhibit the \( O(4) \) symmetry of the equation. Our manipulation is to come back to a flat four-dimensional space \( (q, q_4) \) from \( (\hat{\xi}) \) by the inverse stereographic projection with the unchanged projection radius where \( \hat{\xi}_5 \) is the new polar axis instead of \( \xi_5 \). Normalizing the variable \( (q, q_4) \) by

\[
(\hat{\rho}', \hat{\rho}_4) = (q, q_4) / M
\]

(2.11)

with

\[
M = |\hat{\xi}|^2 / (1 + \hat{d})
\]

(2.12)

and

\[
\hat{d} = [(1 - \eta^2) / (1 - \eta^2)]^{1/2},
\]

(2.13)

we have an equation very similar to Eq. (2.2):

\[
[ (1 + \hat{d}^2 + \hat{\rho}^2) [ (1 - \hat{d}^2 + \hat{\rho}^2) ] \hat{\phi}(\hat{\rho}', \hat{\rho}_4) = (1 - \hat{d}) \hat{\lambda} \int d^4\hat{\rho}' \hat{\phi}(\hat{\rho}', \hat{\rho}_4) ,
\]

(2.14)

where

\[
\hat{\phi}(\hat{\rho}', \hat{\rho}_4) = \sin^\delta(\frac{1}{2}\hat{\Theta}) \hat{H}(\hat{\xi})
\]

(2.15)

and

\[
\hat{\lambda} = \lambda / (1 - \hat{d}) .
\]

(2.16)*

*) The definition (2.16) of \( \hat{\lambda} \) is different from that of the early paper\(^1\) and is coincident with that of Nakanishi's paper\(^4\).
Equation (2·14) has the complete rotational invariance in the Euclidean \((\hat{p}, \hat{p}_4)\) space, the solutions of which will be discussed in the next section.

In order to clarify the reality of the variables in (2·14), we introduce the pseudo-Euclidean momentum \((\hat{p}, \hat{p}_0)\) from \((\hat{p}, \hat{p}_4)\) by \(\hat{p}_0 = i\hat{p}_1\). Then the connection between \((\hat{p}, \hat{p}_0)\) and \((\hat{p}, \hat{p}_0)\) is given explicitly by

\[
(\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_0) = \frac{1}{A(p, p_0)} (p_1, p_2, x_1, y_1) \tag{2·17}
\]

with

\[
\begin{pmatrix}
x_1 \\
y_1
\end{pmatrix} = \begin{pmatrix}
\cos \delta & -i \sin \delta \\
-i \sin \delta & \cos \delta
\end{pmatrix} \begin{pmatrix}
x_0 \\
y_0
\end{pmatrix}, \tag{2·18}
\]

where \(\delta\) is the arbitrary complex angle appearing in the transformation equation (2·8) and \((x_0, y_0)\) is defined by

\[
x_0 = \frac{8A(k_2 p_3 - k_0 p_3) - s (p^2 - \xi^2)}{[s (s - 4 \xi^2) (s - 4)]^{1/2}} \tag{2·19}
\]

and

\[
y_0 = \frac{2(k_3 p_0 - k_0 p_0)}{s^{1/2}}. \tag{2·20}
\]

\(A(p, p_0)\) in Eq. (2·17) is given by

\[
A(p, p_0) = \left[\frac{s (s - 4 \xi^2) (s - 4)}{[s (s - 4 \xi^2) (s - 4)]^{1/2}}\right]^{1/2} (p^2 + \xi^2) - 4A(p^2 - \xi^2) + 8s^2 (k_3 p_3 - k_0 p_3) M \tag{2·21}
\]

with \(|\xi|\) and \(M\) given in (2·5) and (2·12), respectively. \(\hat{\phi}(\hat{p}, \hat{p}_0)\) is related to the original B-S amplitude \(\phi(p, p_0)\) by

\[
\hat{\phi}(\hat{p}, \hat{p}_0) = [A(p, p_0)/M]^{1/2} \phi(p, p_0). \tag{2·22}
\]

It is clear from Eqs. (2·17) \sim (2·21) that the condition necessary for \((\hat{p}, \hat{p}_0)\) to be real is

**Condition (A)** \(\cos \delta\) and \(\sin \delta\) are pure imaginary and real, respectively, for \(0 < s < 4 \xi^2\)

and \(\cos \delta\) and \(\sin \delta\) are pure real and imaginary, respectively, for \(s < 0\).

If the condition (A) is satisfied, the variable \((\hat{p}, \hat{p}_0)\) in Eq. (2·14) should be real along the Wick-rotated contour.

§ 3. **Solutions to the B-S equation**

In this section, we start to solve Eq. (2·14) under the condition (A) discussed in the last of the previous section.
The solutions of Eq. (2·14) with quantum number \( \kappa, n, l, m \) can be represented in the spectral representation by

\[
\mathcal{H}_{\kappa, n, l, m} (\hat{\mathbf{p}}, \hat{\mathbf{p}}) = -iB_{\kappa, n, l, m} (\hat{\mathbf{p}}, \hat{\mathbf{p}}) \mathcal{H}_{\kappa-1, n, l, m} (\hat{\mathbf{p}}, \hat{\mathbf{p}})
\times \int_{-1}^{1} dz \frac{g_{\kappa, n} (z, \hat{\mathbf{s}})}{\frac{1}{2} (1 + z) (1 - \hat{\mathbf{A}}) \left\{ (1 + \hat{\mathbf{A}})^2 + \hat{\mathbf{p}}^2 \right\} + \frac{1}{2} (1 - z) (1 - \hat{\mathbf{A}}) \left\{ (1 - \hat{\mathbf{A}})^2 + \hat{\mathbf{p}}^2 \right\} }^{(n+1/2)},
\]

(3·1)

where \( \mathcal{H}_{\kappa-1, n, l, m} (\hat{\mathbf{p}}, \hat{\mathbf{p}}) \) is the \((n-1)\)-th order solid harmonic of the \( \text{O}(4) \) group (see the Appendix). Substituting (3·1) in (2·14), we see that the weight function \( g_{\kappa, n} (z, \hat{\mathbf{s}}) \), the so-called Cutkosky function, satisfies the following integral equation:

\[
g_{\kappa, n} (z, \hat{\mathbf{s}}) = \frac{\mathcal{H}_{\kappa, n} (\hat{\mathbf{s}})}{2n} \int_{-1}^{1} dz' \left\{ R(z, z') \right\}^{n} g_{\kappa, n} (z', \hat{\mathbf{s}}) \frac{1}{1 - \frac{1}{4} (1 - z'^2) \hat{\mathbf{s}}},
\]

(3·2)

where

\[
\hat{\mathbf{s}} = -4 \hat{\mathbf{A}}^2 / (1 - \hat{\mathbf{A}}^2) = (s - 4 \hat{\mathbf{A}}^2) / (1 - \hat{\mathbf{A}}^2),
\]

(3·3)

\[
R(z, z') = (1 + z) / (1 + z') \quad \text{for} \quad z \leq z',
\]

(3·4)

and \( \mathcal{H}_{\kappa, n} (\hat{\mathbf{s}}) \) is the eigenvalue of \( \hat{\lambda} \) with respect to fixed \( \hat{\mathbf{s}} \). Equation (3·2) can be transformed into the differential equation of the form

\[
\left[ (1 - z^2) \left( \frac{d}{dz} \right)^2 + 2 (n - 1) z \left( \frac{d}{dz} \right) - n (n - 1) \right] g_{\kappa, n} (z, \hat{\mathbf{s}}) + \mathcal{H}_{\kappa, n} (\hat{\mathbf{s}}) \left\{ 1 - \frac{1}{4} (1 - z^2) \hat{\mathbf{s}}^{-1} \right\} g_{\kappa, n} (z, \hat{\mathbf{s}}) = 0
\]

(3·5)

with \( g_{\kappa, n} (\pm 1, \hat{\mathbf{s}}) = 0 \). Equation (3·5) for the zero value of \( \hat{\mathbf{s}} \) has already been presented by Cutkosky:\tnumber{3·6}

\[
g_{\kappa, n} (z, 0) = (1 - z^2)^n C_n^{\kappa + 1/2} (z)
\]

(3·6)

with

\[
\mathcal{H}_{\kappa, n} (0) = (\kappa + n) (\kappa + n + 1),
\]

(3·7)

where \( C_n^{\kappa} (z) \) denotes a Gegenbauer polynomial. The eigenvalue equation (3·5) has been solved only for the infinitesimal \( \hat{\mathbf{s}} \) for the nonzero value of \( \hat{\mathbf{s}} \) and the eigenvalue thus calculated is given by\tnumber{3·8}

\[
\mathcal{H}_{\kappa, n} (\hat{\mathbf{s}}) = \mathcal{H}_{\kappa, n} (0) \left[ 1 - \frac{1}{2} (\kappa + n) (\kappa + n + 1) + n^2 - 1 \right] \hat{\mathbf{s}} + O (\hat{\mathbf{s}}^3).
\]

(3·8)

By tracing back the manipulation of the previous section, we can get a spectral solution for the original B-S amplitude \( \phi (\mathbf{p}, \mathbf{p}_0) \) satisfying (2·1) through \( \mathcal{H}_{\kappa, n, l, m} (\hat{\mathbf{p}}, \hat{\mathbf{p}}) \) given by (3·1). The amplitude \( \phi (\mathbf{p}, \mathbf{p}_0) \) thus obtained is, by explicitly showing the variable \((k_5, k_6)\) and the subsidiary parameter \( \hat{\theta} \), written by

\[
\phi_{\kappa, n, l, m} (\mathbf{p}, \mathbf{p}_0; k_5, k_6) = -iB_{\kappa, n, l, m} (\hat{\mathbf{p}}, \hat{\mathbf{p}}) \left[ \rho (\mathbf{z}; \mathbf{p}, \mathbf{p}_0; k_5, k_6) \right]^{n+1/2}
\]

(3·9)
On the Solutions of the Bethe-Salpeter Equation

\[ \rho(x; p, p_0; k_3, k_0) = \frac{1 + z}{2} (1 - \delta) \left\{ (1 + \delta)^2 + (p + k)^2 \right\} \]

\[ + \frac{1 - z}{2} (1 + \delta) \left\{ (1 - \delta)^2 + (p - k)^2 \right\} \]  

(3.10)

and \((x_3, y_3), g_{3n}(z, s)\) given in (2.17) and (3.1), respectively. \(Z_{n-1, l, m}(p_1, p_2, x_3, y_3)\) is the solid harmonic of the \(O(3, 1)\) group which is related to that of the \(O(4)\) group by

\[ \mathcal{Z}_{n-1, l, m}(p_1, p_2, x_3, y_3) = \mathcal{H}_{n-1, l, m}(p_1, p_2, x_3, -iy_3). \]  

(3.11)

It is to be noticed that the analyticity of the expression (3.9) with respect to \(s\) and \(\delta\) assures that this expression will be valid even if the condition (A) on \(\delta\) is removed out. It is also noticed that the expression (3.9) with finite \(\delta\) becomes linearly dependent on the ones with vanishing \(\delta\) belonging to the same eigenvalues, so that the appearance of \(\delta\) in (3.9) has no essential meaning. In fact, with the help of boost matrix (3) see also the Appendix), we have

\[ \phi^{(p)}_{\text{relm}}(p, p_0; k_3, k_0) = \sum_{\nu=m}^{n-1} d^{(n-1, 0)}_{\nu m} (\delta) \phi^{(0)}_{\text{relm}}(p, p_0; k_3, k_0) \]  

(3.12)

which comes from the fact that the normalization constant \(B_{3n}(\delta)\) is independent of \(l\) and \(\delta\), as will be discussed in the next section, and from the formula on solid harmonic:

\[ \mathcal{Z}_{n-1, l, m}(p_1, p_2, x_3, y_3) = \sum_{\nu=m}^{n-1} d^{(n-1, 0)}_{\nu m} (\delta) \mathcal{Z}_{n-1, \nu, m}(p_1, p_2, x_3, y_3), \]  

(3.13)

where \((x_3, y_3)\) and \((x_0, y_0)\) are related by (2.18).

From now on, we make a special choice \(\delta'\) for \(\delta\) satisfying the condition (A), which is given by

\[ \sin \delta' = \frac{-2Jk_0}{s(d^2 + k_3^2)} \quad \text{and} \quad \cos \delta' = \frac{ik_3[(s - 4d^2)(s - 4)]^{1/2}}{2[s(d^2 + k_3^2)]^{1/2}}. \]  

(3.14)

For the choice of \(\delta\) given above, \((x_3, y_3)\) defined by (2.18) becomes

\[ x_3 = \frac{d \rho_3 + \frac{1}{2} (p^2 - \delta^2) k_3}{(d^2 + k_3^2)^{1/2}} \]  

(3.15)

and

\[ y_3 = \frac{4d \rho_3 + 2J (p^2 - \delta^2) k_3 + 4\delta^2 (k_3 p_0 - k_0 p_3) k_3}{[(d^2 + k_3^2)(s - 4d^2)(s - 4)]^{1/2}}. \]  

(3.16)

with \(|\delta|\) defined by (2.5). It should be noted that \((x_3, y_3)\) is free from the singularity at \(s = 0\) which appears in \((x_0, y_0)\). In the following, we write down explicitly the solutions of the B-S amplitude in the following typical four cases, by substituting \((x_3, y_3)\) specified by (3.15) and (3.16) to Eq. (3.9).
The case $k_s = 0, k_o = 0$ ($s = 4k_o^2$)

The bound state becomes timelike and its amplitude is given by

$$\phi_{\text{en}}^{(0)}(p, p_o; 0, k_o) = -iB_en(s) Z_{n-1,1,m}(p_1, p_2, p_o, y_{ys}^s)$$

\[ \times \int dz \frac{g_{en}(z, s)}{[\rho(z; p, p_o; 0, k_o)]^{n+2}}, \tag{3.17} \]

where

$$y_{ys}^s = \frac{4A p_o + 2(\beta^2 - 2) k_o}{[s - 4A^2]^{1/2}}. \tag{3.18}$$

The case $k_s = 0, k_o = 0$ ($s = -4k_o^2$)

The bound state becomes spacelike and its amplitude is given by

$$\phi_{\text{en}}^{(0)}(p, p_o; k_s, 0) = -iB_en(s) Z_{n-1,1,m}(p_1, p_2, x_{ys}^s, p_o)$$

\[ \times \int dz \frac{g_{en}(z, s)}{[\rho(z; p, p_o; k_s, 0)]^{n+2}}, \tag{3.19} \]

where

$$x_{ys}^s = \frac{4A p_s + 2(\beta^2 - 2) k_s}{[s - 4A^2]^{1/2}}. \tag{3.20}$$

We have an alternative expression for the amplitude by putting $\delta = \delta - \pi/2$. The result is

$$\phi_{\text{en}}^{(0-\pi/2)}(p, p_o; k_s, 0) = -iB_en(s) Z_{n-1,1,m}(p_1, p_2, i p_o, y_{ys}^{s-\pi/2})$$

\[ \times \int dz \frac{g_{en}(z, s)}{[\rho(z; p, p_o; k_s, 0)]^{n+2}}, \tag{3.19'} \]

where we can use the relations $y_{ys}^{s-\pi/2} = ix_{ys}^s$ and

$$Z_{n-1,1,m}(p_1, p_2, i p_o, y_{ys}^{s-\pi/2}) = \sum_{\nu = |m|}^{n-1} d_{\nu m}^{(\nu-1,0)} (-\pi/2) Z_{n-1,1,m}(p_1, p_2, x_{ys}^s, p_o). \tag{3.21}$$

The case $k_s = k_o = 0$ ($s = 0$)

The bound state becomes lightlike and its amplitude is given by

$$\phi_{\text{en}}^{(0)}(p, p_o; k_s, k_o) = -iB_en(s_0) Z_{n-1,1,m}(p_1, p_2, x_{ys}^s, y_{ys}^s)$$

\[ \times \int dz \frac{g_{en}(z, s_0)}{[\rho(z; p, p_o; k_s, k_o)]^{n+2}}, \tag{3.22} \]

where $s_0 = -4A^2/(1 - A^2)$ and
On the Solutions of the Bethe-Salpeter Equation

\[ \begin{align*}
\lambda_{\ell}^p &= \frac{\Delta p + \frac{1}{2} (p^2 - 1 - \mathcal{A}) k_0}{[\mathcal{A} + k_0^2 (1 + \mathcal{A})]^{1/2}}, \\
\lambda_{\ell}^- &= \frac{\Delta^p p_0 + \frac{1}{2} (p^2 - 1 - \mathcal{A}) k_0 + (1 + \mathcal{A}) (p_0 - p_1) k_0^2}{\Delta [\mathcal{A} + k_0^2 (1 + \mathcal{A})]^{1/2}}.
\end{align*} \tag{3·23, 3·24}
\]

(iv) The case \( k_3 = k_0 = 0 \) (\( s = 0 \))

The bound state becomes spurionlike and its amplitude is given by

\[ \phi_{\text{spion}}^{(s)}(p, p_0; 0, 0) = -i \mathcal{B}_{\text{sp}}(s_0) \mathcal{Z}_{n-1, l, m}(p_1, p_2, p_3, p_0) \times i \int dz \frac{g_{\text{sp}}(z, s_0)}{[\mathcal{A}(z; p, p_0; 0, 0)]^{n+1}}. \tag{3·25}
\]

It can be shown that each solid harmonic in (3·17), (3·19'), (3·22) and (3·25) contains the solid subharmonic \( \mathcal{Q}_{\text{sp}}^{(r)}(p_1, p_2, p_3) \), \( \mathcal{Q}_{\text{sp}}^{(r)}(p_1, p_2, i p_3) \), \( (p_1 \pm i p_2)^{m_1} \) and \( \mathcal{Z}_{n-1, l, m}(p_1, p_2, p_3, p_0) \), respectively. This corresponds with the fact that the little group of the Poincaré group related to a certain particle becomes \( O(3) \), \( O(2, 1) \), \( E(2) \) and \( O(3, 1) \) according as its momentum becomes timelike, spacelike, lightlike or spurionlike, in respective order.

\section{4. Normalizations and other discussion}

Now, let us calculate the norm of the B-S amplitude following the prescriptions given in reference 1).

Since the B-S amplitude does not necessarily have a positive norm, we normalize the bound state vector with the total momentum \( (0, 0, 2k_3, 2k_0) \), the quantum number \( \kappa, n, l, m \) and the subsidiary parameter \( \delta \) by

\[ \langle (\delta) \kappa, n, l, m (k_3, k_0) | (\delta) \kappa, n, l, m (k_3, k_0) \rangle = \varepsilon^{(\delta)}_{\text{calm}}(s), \tag{4·1}
\]

where \( \varepsilon^{(\delta)}_{\text{calm}}(s) \) takes the value +1 or -1 depending on the values of \( \delta, s \) and various quantum numbers indicated.

The normalization condition presented by Nakanishi reads\(^5\)

\[ I_{\kappa n}(s) = \frac{\langle \kappa_n(\delta) | d\phi_{\text{calm}}(\delta) \rangle}{d\delta} \tag{4·2}
\]

for the amplitude (3·9), where

\[ I_{\kappa n}(s) = i\varepsilon^{(\delta)}_{\text{calm}}(s) (1 - \mathcal{A})^{-1} \int dp \psi^{(\delta)}_{\text{calm}}(p, p_0; k_3, k_0) [(1 + \mathcal{A}) + (p + k)^2] \times [(1 - \mathcal{A}) + (p - k)^2] \psi^{(\delta)}_{\text{calm}}(p, p_0; k_3, k_0). \tag{4·3}
\]

Here the time reversed amplitude \( \bar{\phi} \) is given simply by replacing \( B_{\text{sp}}(s) \) and \( \mathcal{Z}_{n-1, l, m}(p_1, p_2, x_3, y_3) \) by their complex conjugates in \( \phi \).

First we investigate the normalization of the solutions (3·19), (3·22) and (3·25) and the solution (3·17) with restriction in energy \( s_0 < s < 0 (0 < s < 4\mathcal{A}) \).
In these solutions, \( \delta ( = \delta' ) \) and negative \( s \) satisfy the condition (A). Changing the variable from \((p, p_0)\) to \((\hat{p}, \hat{p}_0)\) in Eq. (4.3), we have

\[
I_{en}(s) = i\varepsilon_{enlm}^{(\theta)}(s)(1 - \hat{J})^{-1}M^s\int d^4\hat{p} \, \hat{\phi}_{enlm}(\hat{p}, \hat{p}_0) \left[ (1 + \hat{J})^3 + \hat{p}^2 \right] \\
\times \left[ (1 - \hat{J})^2 + \hat{p}^2 \right] \phi_{enlm}(\hat{p}, \hat{p}_0).
\]

The right-hand side of (4.4) can be estimated, by substituting (3.1), to be

\[
\frac{I_{en}(s)}{|B_{en}(s)|^2} = \frac{(-1)^{n-1-1}\varepsilon_{enlm}^{(\theta)}(s)}{2(n+1)^2(1 - J^2)^{2n+4}} J_{en}(s),
\]

where

\[
J_{en}(s) = \int dz x^{n+1}(1 - x)^{n+1} \int dz' g_{en}(z, s) \int dz' g_{en}(z', s) \\
\times \left[ \partial^2 - (\alpha + \beta)^2 \right]_{\beta = 1 - \alpha} \left[ (\alpha + \beta)^2 - \alpha \beta s \right]^{-n+1}
\]

with \( \alpha = \frac{1}{2}[(1 + z)x + (1 + z')(1 - x)] \), and we use the relation

\[
B_{en}(s) = M^s(1 - \frac{1}{2}s)^{n+2}B_{en}(s)
\]

between the normalization constants appearing in (3.1) and (3.10). \( J_{en}(s) \) has already been calculated by Nakanishi in the lowest order of \( s \)

\[
J_{en}(s) = c_{en}(-s)^s + 0(s^{s+1})
\]

with the positive constant \( c_{en} \) which is explicitly given in reference 8).

The right-hand side of (4.2) is expected to be negative in general, as is indicated by (3.8) for the vanishing \( s \). From this property and Eq. (4.5) through (4.8), the normalization factor \( \varepsilon_{enlm}^{(\theta)}(s) \) is determined to

\[
\varepsilon_{enlm}^{(\theta)}(s) = (-1)^{n-1-1}
\]

for the solutions with \( \delta \) and negative \( s \) which satisfy the condition (A), (see also the Appendix).

Next we investigate the normalization of the solution (3.17) with restriction \( 0 < s < 4(4J^2 < s < 4) \) where \( \delta \) does not satisfy the condition (A). Although we have no expression corresponding to (4.4) in this case, we can have a relation

\[
\frac{I_{en}(s)}{|B_{en}(s)|^2} = \frac{(-1)^{n-1-1}\varepsilon_{enlm}^{(\theta)}(s)}{2(n+1)^2(1 - J^2)^{2n+4}} J_{en}(s)
\]

similar to (4.5). Equation (4.10) comes from the fact that the solution (3.17) with restriction \( 0 < s < 4 \) is analytically continued to the one in the range \( s < 0 \), so that Eq. (4.3) in this case should be continued to (4.5) except the factor \((-1)^{n-1-1}\) coming from the non-analytical agency

\[
[\mathcal{Z}_{n-1, l, m}(p_1, p_2, p_3, y'_{\alpha})]^* = (-1)^{n-1-1}\mathcal{Z}^*_{n-1, l, m}(p_1, p_2, p_3, y'_{\alpha}).
\]


Equation (4·10) gives
\[ \varepsilon_{exlm}^{(\delta)}(\hat{s}) = (-1)^s \] (4·12)
because \( J_{en}(\hat{s}) \) has the extra sign \((-1)^s\) in this case.

Finally, we investigate the case of (3·19'). In this case, \( \delta = \beta'' - \pi/2 \) does not satisfy the condition (A), too. Equation (3·19') can be expanded in the linear combination of (3·19) by
\[ \phi_{exlm}^{(\beta'' - \pi/2)}(p, p_0; k_3, 0) = \sum_{\nu=-|\nu|m}^{n+l} d_{\nu lm}^{(n-l,0)} \left( -\frac{\pi}{2} \right)^\nu \phi_{en\nu m}^{(\beta)}(p, p_0; k_3, 0) \] (4·13)
in a way similar to (3·12), and its time reversed one is represented by
\[ \phi_{exlm}^{(\beta'' - \pi/2)}(p, p_0; k_3, 0) = \sum_{\nu=-|\nu|m}^{n+l} \left[ d_{\nu lm}^{(n-l,0)} \left( -\frac{\pi}{2} \right)^\nu \right] \phi_{en\nu m}^{(\beta)}(p, p_0; k_3, 0). \] (4·13')

By substituting (4·13) and (4·13') into (4·3) and comparing with (4·5), we get
\[ \frac{I_{en}(\hat{s})}{|B_{en}(\hat{s})|^2} = \frac{-\varepsilon_{exlm}^{(\beta'' - \pi/2)}(\hat{s}) J_{en}(\hat{s})}{2(n+1)^2(1 - d_0^{2n+4})} \sum_{\nu=-|\nu|m}^{n+l} (-1)^{n-l-\nu} \left| d_{\nu lm}^{(n-l,0)} \left( -\frac{\pi}{2} \right)^\nu \right|^2. \] (4·14)
Since \( d_{\nu lm}^{(n-l,0)}(-\pi/2) \) vanishes when \( n - l - 1 + m \) is an odd integer, and since
\[ \sum_{\nu=-|\nu|m}^{n+l} \left| d_{\nu lm}^{(n-l,0)}(\alpha) \right|^2 = 1 \] for real \( \alpha \) (4·15)
we have
\[ \varepsilon_{exlm}^{(\beta'' - \pi/2)}(\hat{s}) = (-1)^{l-m} \] (4·16)
for the solution (3·19').

It is worthwhile to note that the normalization factor \( \varepsilon_{exlm}^{(\beta)}(\hat{s}) \) depends on the value of \( \delta \), even when \( \hat{s} \) is held fixed. This circumstance is quite natural because to choose a different \( \delta \) corresponds to make a different superposition of the bound states belonging to various \( l \) values. The normalization factor \( \varepsilon_{exlm}^{(\beta)}(\hat{s}) \) does not, however, depend on \( \delta \) analytically.

It is also clear from the discussions made above that if the normalization factor \( \varepsilon_{exlm}^{(\beta)}(\hat{s}) \) is appropriately chosen, the normalization constant \( B_{en}(\hat{s}) \) in general satisfies the relation
\[ \frac{I_{en}(\hat{s})}{|B_{en}(\hat{s})|^2} = \frac{-1}{2(n+1)^2(1 - d_0^{2n+4}) |J_{en}(\hat{s})|}. \] (4·17)

We will add a brief discussion on the singular behaviors of the B-S amplitude. The normalized B-S amplitude of (3·17) behaves like \( \hat{s}^{-(\kappa + n - l - 1)/2} \) near \( \hat{s} = 0 \) on account of the singular behaviors of \( B_{en}(\hat{s}) \) given in (4·17) with (4·8) and that of the solid harmonic in (3·17). This circumstance causes the multiple poles of the corresponding scattering Green's function of order \( \kappa + n - l \) at \( \hat{s} = 0 \), i.e. \( s = 4d' \).

As has recently been discussed by Nakanishi,\(^9\) the singular behavior in the
residue function of the scattering Green's function does not necessarily imply the multiple poles of the Green's function itself. In the case mentioned above, in order for the Green's function to have multiple poles of order $\kappa + n - l$ at $s = 0$, the following condition is necessary and sufficient: the $(\kappa + n - l)$ eigenvalues $\hat{\lambda}_m(s)$ with the constant $\kappa + n$ become coincident at $s = 0$, as is seen from (3·7), provided that the respective $(d/ds)\hat{\lambda}_m(s)\mid_{s=0}$ are different from each other. We can see from (3·8) that the latter condition is satisfied.

For a counter example, we can present the solution (3·9) with vanishing $\partial$ which is singular at $s = 0$ from (2·19) and (2·20). This singularity is, however, extinguishable by the linear combination of the degenerate solutions as was showed in the previous section.

When $\partial = 0$, the singularity at $s = 4\partial(\delta = 0)$ of the solution (3·17) is shifted to that of the lightlike solution (3·22). It is important for detailed investigation of these singular behaviors to solve the generalized B-S amplitude. $^{9,10}$ Unfortunately, the generalized B-S amplitude has not yet been revealed even for the Wick-Cutkosky model.

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Appendix

Solid harmonic and boost matrix

The $L$-th order four-dimensional solid harmonic is defined by

$$\mathcal{H}_{Llm}(p, p_\nu) = \lvert p \rvert^L H_{Llm}(\alpha, \theta, \varphi), \quad (A·1)$$

where the spherical harmonic $H_{Llm}(\alpha, \theta, \varphi)$ is

$$H_{Llm}(\alpha, \theta, \varphi) = A_{Ll}(\sin \alpha)^L C^{(1)}_{L-l}(\cos \alpha) Y_{lm}(\theta, \varphi) \quad (A·2)$$

with polar angles $(\alpha, \theta, \varphi)$ of $(p, p_\nu)$ and a Gegenbauer polynomial denoted by $C_j^n(x)$. The normalization constant $A_{Ll}$ is given by

$$A_{Ll} = (i)^{L} [2^{L+1}(L+1)! / (L+1)!]^{1/2} \quad (A·3)$$

by the requirement

$$\int_0^\pi \sin^2 \alpha d\alpha \int_0^{2\pi} \sin \theta d\theta \int_0^{2\pi} d\varphi |H_{Llm}(\alpha, \theta, \varphi)|^2 = 1. \quad (A·4)$$

The boost matrix $d_{Llm}^{(L \nu)}(\delta)$ is defined by

$$d_{Llm}^{(L \nu)}(\delta) = \sum_{\mu} C(L/2, L/2, \nu, \mu, m-\mu) C(L/2, L/2, \nu, \mu, m-\mu) e^{-i(3\nu-3\mu)\delta} \quad (A·5)$$
with Clebsch-Gordan coefficients. The solid harmonic boosted to the direction of third-axis is related to the original one through the boost matrix by

\[
\mathcal{H}_{Llm}(p_1, p_2, \cos \delta p_3 + \sin \delta p_4, -\sin \delta p_3 + \cos \delta p_4)
= \sum_{l' = |m|}^L d_{l'm}^{(L,0)}(\delta) \mathcal{H}_{L'l'm}(p_1, p_2, p_3, p_4)
\]  

(A·6)

which has been used in the derivation of (3·13).

While we have the orthonormality relations of the boost matrix from the known properties of the Clebsch-Gordan coefficient

\[
\sum_{l' = |m|}^L [d_{l'm}^{(L,0)}(\alpha)]^* d_{l'm}^{(L,0)}(\alpha) = \delta_{l l'}
\]  

(A·7)

and

\[
\sum_{l' = |m|}^L (-1)^l [d_{l'm}^{(L,0)}(n\pi + i\beta)]^* d_{l'm}^{(L,0)}(n\pi + i\beta) = (-1)^l \delta_{ll'}
\]  

(A·8)

for arbitrary real \(\alpha, \beta\) and integer \(n\). Equation (A·8) assures the validity of the expression (4·5) for all \(\phi_{vulm}^{(0)}\) with \(\delta\) which satisfies the condition (A) and is indefinite by \(n\pi + i\beta\).

References

   See also N. Nakanishi, Phys. Rev. 138 (1965), B1182; 147 (1966), 1153.
8) N. Nakanishi, Phys. Rev. 139 (1965), B1401.