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The Rational Formula Interpreted Using a Physically-Based Mathematical Model

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A new interpretation is given for the rational formula in view of a mathematical model founded on the kinematic overland flow and Green and Ampt infiltration equations. The potential variability of the runoff coefficient with various rainfall and basin characteristics is demonstrated. However, using the concept of hydrologic similarity, it is possible to predict the runoff coefficient in terms of several physically-based non dimensional parameters for homogeneous planar, rectangular basins.

Introduction

Conventionally, the rational formula is used in conjunction with the so called »rational method« to design storm drainage facilities in small basins. For a specified design return period, a design rainfall intensity is chosen from the available intensity-duration relationships so that the rainfall duration is equal to the time of concentration of the basin. Then, the rational formula

$$Q_p = C\bar{i}A \qquad (1)$$

is applied to estimate the peak flow rate, that is the design discharge for the drainage facility considered. In Eq. (1), Q_p is the peak runoff rate, C is the runoff coefficient, $\bar{i}$ is the time-averaged rainfall intensity, and A is the basin area.

There are at least three ways in which the rational formula/method may be viewed as a relationship between rainfall and peak runoff rates. Eq. (1) can be taken in a statistical sense where the runoff coefficient, C, links peak runoff rates of given frequencies with rainfall rates of the same frequencies (Schaake *et al.* 1967; French *et al.* 1974). Secondly, the rational method can be considered as a parametric method in which the run-off coefficient and the time of concentration are obtained on the basis of parameter optimization (Dooge 1973). Finally, the rational formula can be regarded as a deterministic mathematical model of the rainfall-runoff process. Although, much of the criticism of the rational formula has been based on this interpretation (French *et al.* 1974), the formula can be most useful as a deterministic method in hydrologic analyses of ungaged basins. Therefore, the rational formula is regarded essentially as a deterministic model here.

The limitations of the rational formula/method are well understood (McPherson 1969). Briefly the assumptions involved are (a) the rainfall intensity is constant over the duration, (b) the rainfall duration is equal to the time of concentration of the basin under analysis, (c) the peak runoff rate is proportional to the rainfall intensity, and (d) the return period of the computed discharge is the same as that of the rainfall which produces it. Even with these restrictive assumptions, the success of the rational formula rests heavily upon the selection of an appropriate runoff coefficient. Traditionally, the runoff coefficient is chosen mainly on the basis of the characteristics of the basin surface (ASCE 1972). To account for the greater runoff produced by rarer storms, Wright-McLaughlin Engineers (1969) suggest that the runoff coefficient be modified by use of correction factors depending upon the rainfall frequency. Meanwhile, Hromadka (1984) expresses the runoff coefficient in terms of several parameters of the Soil Conservation Service curve number technique. However, the runoff coefficient remains a key factor that is evaluated subjectively for the most part.

Previous theoretical studies of overland flow on pervious surfaces indicate that surface runoff rates depend on subsurface basin properties and the soil moisture content in addition to the surface characteristics of a basin (Smith and Woolhiser 1971; Akan and Yen 1981; Yen and Akan 1983). Also, the distribution of the rainfall intensity over the duration affects the overland flow rates (Akan and Yen 1984). Reliable estimates of the runoff coefficients can be obtained only if all the influencial factors are taken into account simultaneously and quantitatively. The objective of this paper is to re-evaluate the rational formula using a mathematical model founded on the kinematic overland flow and the Green and Ampt infiltration equations (Akan 1985). First the kinematic model is summarized herein. Then, using the concept of hydraulic similarity, the results of the kinematic model are generalized to form a physical basis for the runoff coefficient. Finally, a set of physically-based charts are developed to evaluate this coefficient for an idealized rectangular basin. It is demonstrated that with the use of the charts given, some of the assumptions of the conventional rational method can be released.

Summary of the Mathematical Model Used

This study addresses a rather simple overland flow situation occurring in a rectangular basin with uniform physical properties. It is assumed that infiltration is the dominant type of abstraction. Overland flow in such an elementary basin can be modeled by using the kinematic surface runoff equations coupled with the Green and Ampt formulation of the infiltration process. No attempt is made here to test the accuracy of this approach. It has already been shown elsewhere that kinematic solutions of overland flow give very accurate results for most hydrologically significant cases (Woolhiser and Liggett 1967). Also, the Green and Ampt infiltration model has a precise physical basis and its results agree with those of the Richards equation (Mein and Larson 1971).

Governing Equations

Overland flow in a rectangular basin having a flow length L is considered. The basin is subject to a rainfall which has a duration, t_d, and intensity is i that varies with time t. The average rainfall intensity is T. The overland flow occurs in the x-direction. The variable rate of infiltration is f. The overland flow discharge per unit width of the basin is q, and it changes with x and t.

In non-dimensional form, the kinematic wave equation for overlandflow can be written as

$$\frac{\partial q^*}{\partial x^*} + \alpha^* \frac{\partial q^{*\,1/m}}{\partial t^*} = i^* - f^* \quad (2)$$

where

$$q^* = \frac{q}{\bar{i} L} \quad (3)$$

$$x^* = \frac{x}{L} \quad (4)$$

$$\alpha^* \equiv \left(\frac{L \bar{i}}{\alpha} \right)^{1/m} \simeq \frac{1}{\bar{i}\, t_d} \quad (5)$$

$$t^* = \frac{t}{t_d} \quad (6)$$

$$i^* = \frac{i}{\bar{i}} \quad (7)$$

$$f^* \equiv \frac{f}{\bar{i}} \quad (8)$$

In Eq. (5), α is a factor which relates the discharge per unit width, q, to flow depth y as

$$q = \alpha y^m \quad (9)$$

where α and m are evaluated differently depending upon the flow resistance formula used. In this study the Manning formula is employed. Therefore $m \equiv 5/3$ and

$$\alpha = \frac{k_0}{n} \sqrt{S_0} \tag{10}$$

where n is the roughness factor, S_o is the basin slope and $k_o = 1.0\ \mathrm{m}^{1/3}/\mathrm{s}$. For Eq. (9) to be dimensionally homogeneous, α has the dimension of $(\mathrm{length})^{m-2}/\mathrm{time}$.

For an initially dry surface, the initial condition for Eq. (2) is $q^* = 0$ for all x^* at $t^* = 0$. The upstream boundary condition is $q^* = 0$ at $x^* = 0$ for all t^*. No downstream boundary condition is needed for the kinematic-wave equation.

The rate of infiltration depends upon the subsurface conditions. Let ϕ, K and S_i denote the porosity, saturated hydraulic conductivity, and antecedent degree of saturation of the underlying soil, respectively. The depth of the wetting front below the soil surface at any time t is Z_f. The suction head at the wetting front is P_f. The potential infiltration rate at any time t is f_p.

The Green and Ampt equations of infiltration can be written in non-dimensional form as

$$f_p^* \equiv \frac{K^*(Z^*+P^*)}{Z^*} \tag{11}$$

and

$$Z^* = \int_0^{t^*} f^*\, dt^* \tag{12}$$

where

$$f_p^* = \frac{f_p}{\bar{i}} \tag{13}$$

$$K^* = \frac{K}{\bar{i}} \tag{14}$$

$$Z^* = \frac{Z_f\, \phi(1-S_i)}{\bar{i}\, t_d} \tag{15}$$

$$P^* \equiv \frac{P_f\, \phi(1-S_i)}{\bar{i}\, t_d} \tag{16}$$

Before the commencement of overland flow $f^* = i^*$ if $i^* < f_p^*$, and $f^* = f_p$ otherwise. After the overland flow is generated $f^* = f_p^*$ until the surface runoff ceases. Initially, f_p^* is assumed to be very large, and accordingly f^* is set equal to i^* for the first time step of computation.

It should be noted that the suction head P_f is treated as a constant characteristic of the soil mantle. This is a valid assumption, and P_f can be evaluated from the pressure head-degree of saturation-relative permeability relationships of a soil (Mein and Larson 1971).

Method of Solution

An implicit finite difference method is employed to solve Eqs. (2), (11) and (12) since no closed-form analytical solutions are available at present. Omitting the superscript * for the purpose of clarity, Eqs. (2) and (12) are written in finite difference form as

$$\frac{1}{2\Delta x}\left(q_{k-1}^{j+1} - q_{k-1}^{j+1} + q_{k}^{j} - q_{k-1}^{j}\right) + \left(\frac{2\alpha}{m\Delta t}\right) \frac{q_{k}^{j+1} = q_{k}^{j\,''}}{\left(q_{k}^{j+1}\right)^{1-1/m} + \left(q_{k}^{j}\right)^{1-1/m}} = i^{j+1} - f^{j+1} \tag{17}$$

and

$$z^{j+1} = \Delta t \sum_{n=1}^{j} f_n \tag{18}$$

respectively where Δx^* and Δt^* are the non-dimensional space and time increments, k indicates the grid number, and j denotes the time step of computation.

The non-dimensional rate of infiltration at the $(j+1)$-th time step is obtained using Eqs. (11) and (18). As well, all the terms with superscript j are known from either the initial conditions or the previous time step computations. Therefore, the unknowns in Eq. (17) are q_k^{j+1} for $k = 1$ to N, when the solutions at the $(j+1)$-th time step are sought. Eq. (17) can be written for $k = 2, 3, \ldots N$ to yield at set of $(N-1)$ algebraic equations which contain N unknowns. The additional equation for the system to be complete is provided from the upstream boundary condition $q_1^{j+1} = 0$. The set of N non-linear algebraic equations thus obtained are solved using Newton's iterative technique.

Sample Results

The input parameters for the model include the time distribution of rainfall intensity, the nondimensional similarity parameters α^*, K^* and P^* and the non-dimensional finite difference parameters Δt^* and Δx^*. The model output includes the ordinates of non-dimensional discharge hydrographs obtained at selected sections of the overland flow plane.

Three different patterns of rainfall intensity distribution considered in this paper are shown in Fig. 1. The uniform distribution is employed in conjunction with conventional rational formula. The Soil Conservation Service (SCS) 24-hr Type II hyetograph (Brakensiek and Rawls 1982) forms the basis for SCS methods commonly used to predict runoff from ungaged basins in practice. The Huff first quartile distribution is used extensively in urban storm drainage studies, particularly since it is built into the Illinois urban drainage area simulator (ILLUDAS) and the urban runoff and design model for optional use (Terstriep and Stall 1974).

The overland flow hydrographs calculated using the mathematical model for

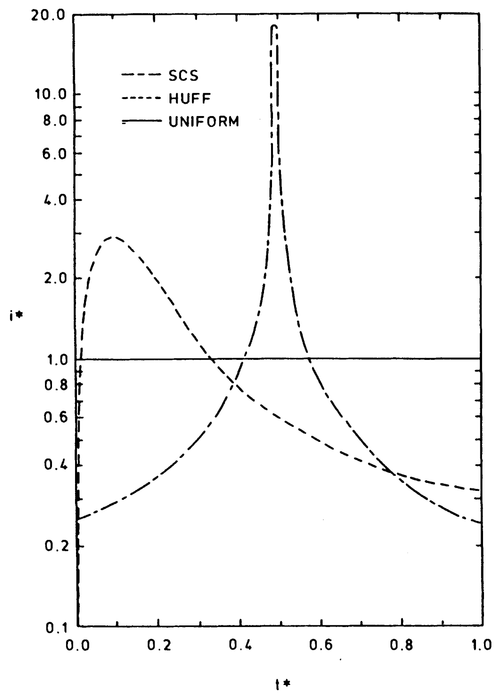

Fig. 1. Dimensionless Rainfall Hyetographs Considered.

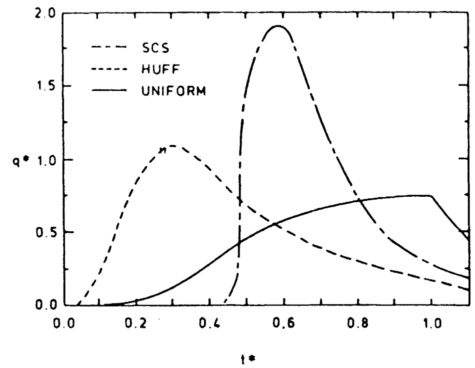

Fig. 2. Sample Dimensionless Overland Flow Hydrographs Calculated.

different intensity distributions are displayed in Fig. 2. The similarity parameters are $\alpha^* = 0.40$, $K^* = 0.10$ and $P^* = 0.50$ in all three cases. The non-dimensional finite difference parameters employed are $\Delta x^* = 0.1$ and $\Delta t^* = 0.01$ for all cases except for the SCS Type II distribution where $\Delta t^* = 1/96$ is used. Fig. 2 clearly illustrates the effect of time distribution of rainfall intensity on overland flow hydrographs. This effect may be pronounced to a greater or lesser extent for other combinations of α^*, K^* and P^*.

Similarity Criterion for Overland Flow-Infiltration Process

The conjunctive process of overland flow and infiltration is described by Eqs. (2), (11) and (12). A close examination of these equations reveals that for a constant m and a given time distribution of rainfall, which specifies i^* as a function of t^*, there are only three non-dimensional parameters; namely, α^*, K^* and P^* that govern the flow process. Consequently, different rainfall infiltration situations in different basins will be hydraulically similar if they have the same α^*, K^*, P^*, and rainfall intensity pattern.

This similarity criterion can be used to generalize the results of the mathematical model presented. For instance, let q_p denote the peak overland flow discharge at the downstream end of a rectangular basin. The corresponding non-dimensional variable is

$$q_p^* = \frac{q_p}{\bar{i}L} \tag{19}$$

In accordance with the similarity criterion, $q_p{}^*$ is a function of only α^*, K^*, P^* and the time distribution of rainfall, or

$$q_p^* = F[i^*(t^*), \alpha^*, K^*, p^*] \tag{20}$$

where F represents the functional relationship among the non-dimensional parameters. The form of this function cannot be determined analytically, but it can be evaluated based on the simultaneous numerical solutions of Eqs. (2), (11), and (12).

The Rational Formula Runoff Coefficient

The rational formula, Eq. (1), can be rearranged for a rectangular basin as

$$C = \frac{q_p}{\bar{i}L} \tag{21}$$

A comparison of Eq. (21) to Eq. (19) indicates that the runoff coefficient C is mathematically equivalent to the non-dimensional peak discharge $q_p{}^*$. Therefore, within the context of the mathematical model employed, the runoff coefficient is a function of the similarity parameters α^*, K^* and P^* as well as the time distribution of the rainfall intensity. That is

$$C = F[i^*(t^*), \alpha^*, K^*, P^*] \tag{22}$$

Physical Basis for the Runoff Coefficient

All the non-dimensional quantities which are shown to affect the runoff coefficient in Eq. (22) have a physical basis. The function i^* $(t)^*$ represents the time distribution of the rainfall intensity considered. As well, the non-dimensional parameters α^*, K^* and P^* can be described as

$$\alpha^* = \frac{\text{Time to equilibrium at a rainfall excess rate of } \bar{i}}{\text{Storm duration}}$$

$$K^* = \frac{\text{Soil hydraulic conductivity}}{\text{Average rainfall intensity}}$$

$$P^* = \frac{\text{Initial available storage volume to depth } P_f \text{ in the soil}}{\text{Volume of rain}}$$

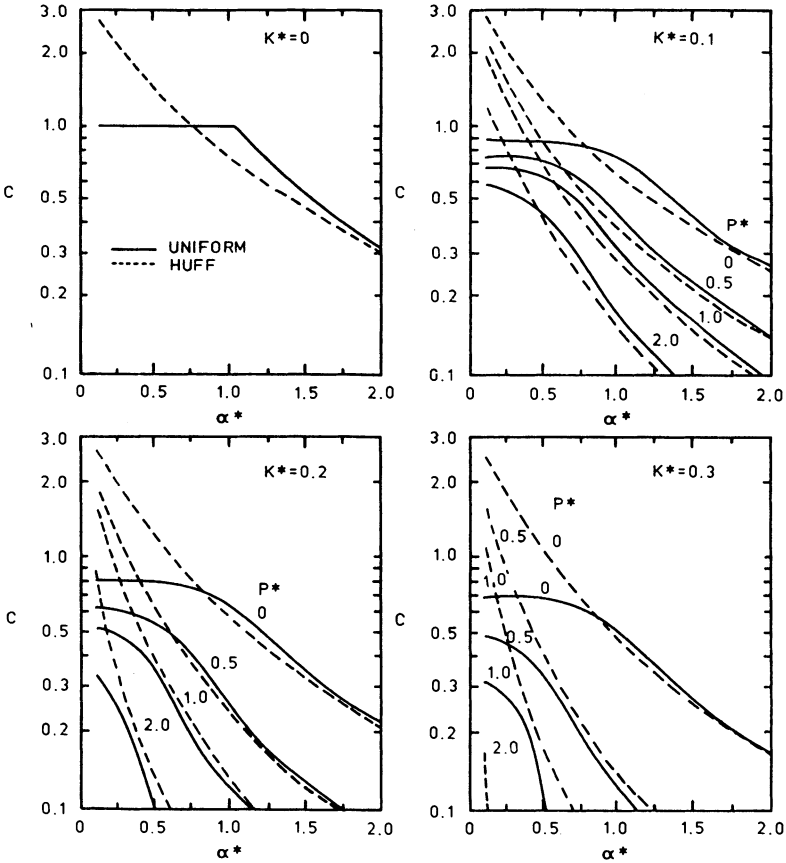

Fig. 3. Runoff Coefficients for Uniform and Huff Distributions.

Evaluation of the Runoff Coefficient

As mentioned earlier, the form of the function F in Eq. (22) is not known, and therefore C cannot be determined directly from such an expression. However, the mathematical model presented in this paper can be utilized to develop a set of physically-based charts to evaluate the runoff coefficient. The charts displayed in Figs. 3 and 4 have been obtained for the Huff first quartile, SCS Type II, and uniform intensity distributions. The uniform intensity is included in both figures for the purpose of comparison.

These charts were developed by using the results of the mathematical model for a large number of systematically chosen combinations of α^*, K^* and P^* and the specified intensity distribution. The calculated values of $q_p{}^*$, which is equivalent to C, were plotted in terms of the physically-based parameters α^*, K^* and P^*.

The curves given for $K^* = 0$ in Figs. 3 and 4 represent overland flow on imper-

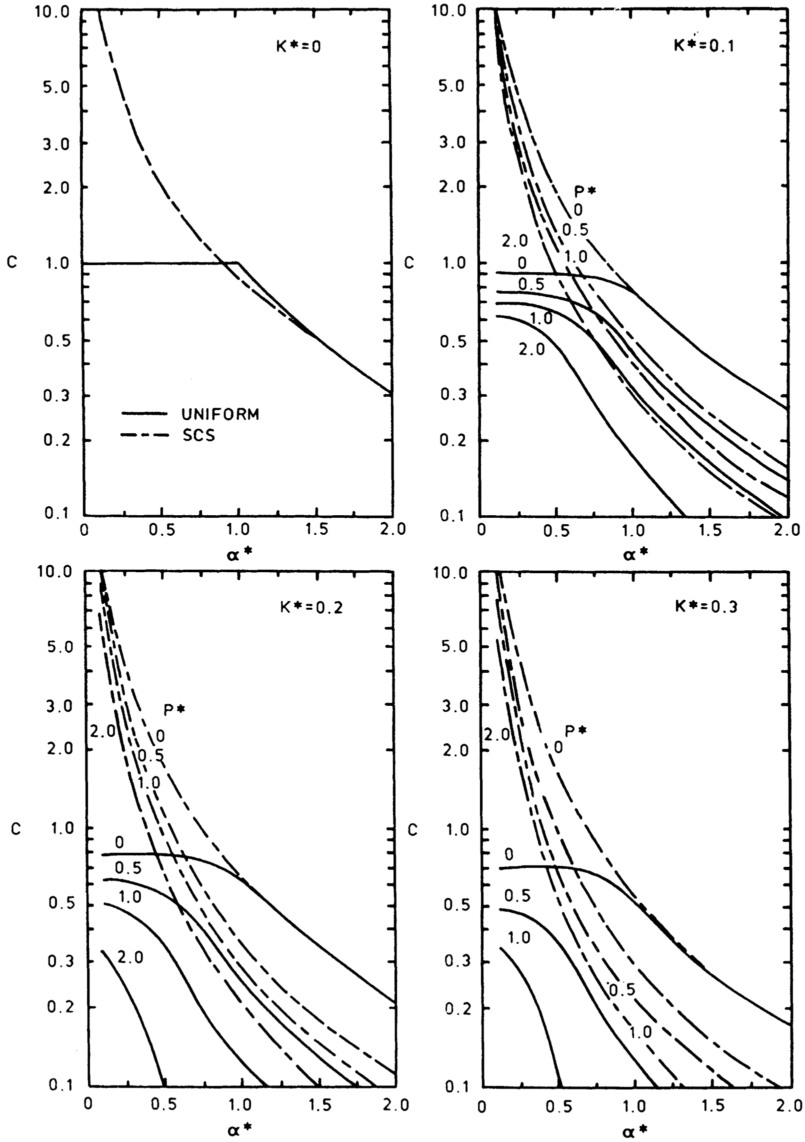

Fig. 4. Runoff Coefficients for Uniform and SCS Type II Distributions.

vious surfaces. Meanwhile, the curves for $P^* = 0$ are for initially saturated soils. For an impervious surface under uniform rainfall intensity, $C = 1$ for $\alpha^* < 1$. This accords with the kinematic wave theory.

Figs. 3 and 4 indicate that the maximum value of the runoff coefficient is unity under uniform intensity. However, for the Huff and the SCS Type II distributions, the value of the coefficient may be considerably greater. Generally, the SCS Type II hyetograph produces greater runoff coefficients than the other two distributions considered. Comparing the Huff first quartile and uniform rainfalls, the former produces substantially larger coefficients for small values of α^*, which represent short, steep, basins. For large values of α^*, the differences are less pronounced. As expected, the runoff coefficient increases with decreasing K^* and P^* in all cases. In other words, for all other parameters constant, the runoff coefficient will increase with decreasing soil conductivity and increasing antecedent moisture content.

Usability of the Runoff Coefficient Charts

In order to use the charts given in Figs. 3 and 4 to determine the runoff coefficient for a specified rainfall hyetograph, one needs to estimate a Manning's n value and the soil characteristics ϕ, K and P_f. Since overland flow resistance is affected by many factors such as raindrop impact, channelization, and obstacles like litter and rocks, various authors interpret the Manning roughness factor as a fitting parameter into which all these effects are lumped (Woolhiser 1975; Engman 1983). Based on observed runoff data and the kinematic-wave theory, Woolhiser (1975), and Engman (1983) obtained the so-called »effective n« values for a variety of agricultural and natural surfaces. Their results can be used conveniently in conjunction with the model presented in this paper. As well, one can easily obtain the soil characteristics using the physically-based charts developed by Rawls and Brakensiek (1983). These charts express the Green and Ampt soil parameters in terms of the soil texture.

Sample Application

The use of the runoff coefficient charts may be illustrated by a simple example. Let a rectangular plot be 100 m long, with a slope of 0.0089 and Manning roughness factor of 0.10. The underlying soil has a porosity of 0.50, hydraulic conductivity of 1.0 cm/hr = 2.78×10^{-6} m/s and an average suction head of 6.6 cm = 0.066 m. The initial degree of saturation is 0.50. The peak overland flow produced by a rainfall of depth of 3.3 cm = 0.033 m and duration of 1 hr = 3,600 s is to be determined.

The average rainfall intensity is obtained as $\bar{i} = 3.3/1 = 3.3$ cm/hr $= 9.17 \times 10^{-6}$ m/s. Also, using Eq. (10), one obtains $\alpha = 0.94\ m^{1/3}/s$. Then the similarity parameters α^*, K^* and P^* are evaluated from Eqs. (5), (14) and (16) respectively with the values $\alpha^* = 0.47$, $K^* = 0.30$ and $P^* = 0.50$ being obtained.

If the time distribution of rainfall intensity is uniform, Fig. 3 gives $C = 0.37$. If the Huff intensity distribution is used, the runoff coefficient becomes $C = 0.46$. Then using Eq. (21), the peak overland flow discharge per unit width of the basin is calculated as $q_p = (0.37)\ (9.17 \times 10^{-6})\ (100) = 3.30 \times 10^{-4}\ m^2/s$ for the uniform rainfall, and $q_p = (0.46)\ (9.17 \times 10^{-6})\ (100) = 4.22 \times 10^{-4}\ m^2/s$ for the Huff distribution. It should be noted that the SCS Type II distribution is not used in this example deliberately since that distribution requires a rainfall duration of 24 hours.

Conclusions

The rainfall-runoff situation addressed in this paper is simple, with only the overland flow and infiltration processes being considered. It should be understood that the quantitative results of this paper can be used directly only in hydrologic studies of small watersheds having uniform properties. However, the conclusions given below are general at least qualitatively.

1) The runoff coefficient of the rational formula may be viewed as a variable for a given basin. It can be treated as a non-dimensional parameter which should and can be evaluated in terms of several other physically-based non-dimensional parameters representing the basin and rainfall conditions.
2) The rational formula, as interpreted in this paper, is not restricted to uniform rainfall intensity. It can be used in conjunction with any temporal distribution of rainfall intensity provided that the runoff coefficient is selected properly. The upper limit of the runoff coefficient is then unity only under uniform intensity. It may take greater values if the distribution of intensity is non-uniform over the rainfall duration.
3) The rational formula, as interpreted in this paper, is not limited to rainfalls which have a duration equal to the time of concentration of a basin. It can be used for any rainfall duration if the runoff coefficient is evaluated properly. Therefore, the users of the rational formula do not have to estimate the time of concentration which is a very loosely defined basin parameter. Furthermore, the rational formula can be used not only for design purposes, but also for analyses for existing drainage structures.
4) One does not have to assume that the peak discharge calculated by the rational formula has the same return period as the rainfall used. The interpretation of the rational formula given in this paper allows assigning frequencies not only to rainfall but also to basin characteristics such as the antecedent soil moisture content. Therefore, it should be possible, at least theoretically, to obtain a frequency distribution for the runoff coefficient from specified frequency distributions of rainfall, soil moisture content and other variable basin properties. This point, however, needs further investigation.

References

Akan, A.O. (1985) Similarity Solution of Overland Flow on Pervious Surface, *Journal of Hydraulic Engineering, ASCE, Vol. 111,* No. 7, pp. 1057-1067.

Akan, A.O., and Yen, B.C. (1981) Mathematical Model of Shallow Water Flow over Porous Media, *Journal of the Hydraulics Division, ASCE, Vol. 107,* No. HY4, pp. 479-494.

Akan, A.O., and Yen, B.C. (1984) Effect of Time Distribution of Rainfall on Overland Runoff. Analysis and Design of Stormwater Systems, Proceedings of the Third International Conference on Urban Storm Drainage, Chalmers University of Technology, Göteborg, Sweden, pp. 193-202.

American Society of Civil Engineers (1969) *Design and Construction of Sanitary and Storm Sewers,* Washington, D.C.

Brakensiek, D.L., and Rawls, W.J. (1982) An Infiltration Based Rainfall-runoff Model for a SCS Type II Distribution, *Transaction of the American Society of Agricultural Engineers, Vol. 25,* No. 6, pp. 1607-1611.

Dooge, J.C.I. (1973) Linear Theory of Hydrologic Systems, U.S. Department of Agriculture, Agricultural Research Service, Technical Bulletin No. 1468, Washington, D.C.

Engman, E.T. (1983) Roughness Coefficients for Routing Surface Runoff. Proceedings of the Conference on Frontiers in Hydraulic Engineering, ASCE, pp. 560-565.

French, R., Pilgrim, D.H., and Laurenson, E.M. (1974) Experimental Examination of the Rational Method for Small Rural Catchments, *Civil Engineering Transactions,* The Institution of Engineers, Australia, *Vol. CE16,* pp. 95-102.

Hromadka, II, T.V. (1984) *Computer Methods in Urban Hydrology,* Lighthouse Publications, Mission Visjo, Ca.

Mein, R.G., and Larson, C.L. (1971) Modeling the Infiltration Component of the Rainfall-runoff Process, Bulletin 43, Water Resources Research Center, University of Minnesota, Minneapolis, MN.

McPherson, M.B. (1969) Some Notes on the Rational Method of Storm Drainage Design, ASCE, Urban Water program TM No. 6, New York.

Rawls, W.J., and Brakensiek, D.L. (1983) A Procedure to Predict Gren and Ampt Infiltration Parameters, Proceedings of the National Conference on Advances in Infiltration, American Society of Agricultural Engineers, pp. 102-112.

Schaake, J.G., Geyer, J.C., and Knapp, J.W. (1967) Experimental Examination of the Rational Method, *Journal of the Hydraulics Division, ASCE, Vol. 93,* No. HY6, pp. 353-370.

Smith, R.E., and Woolhiser, D.A. (1971) Mathematical Simulation of Infiltrating Watersheds, Colorado State University, Hydrology Paper No. 47, Fort Collins, Colorado.

Terstriep, M.L., and Stall, J.B. (1974) The Illinois Urban Drainage Area Simulator, ILLUDAS. Bulletin 58, Illinois State Water Survey, Urbana.

Woolhiser, D.A. (1975) Simulation of Unsteady Overland Flow. In: *Unsteady Flow in Open Channels,* Vol. II, K. Mahmood and V. Yevjevich (editors). Water Resources Publications, Fort Collins Colorado, pp. 485-508.

Woolhiser, D.A., and Liggett, J.A. (1967) Unsteady One-dimensional Flow over a Plane – the Rising Hydrograph, *Water Resources Research, Vol. 3,* No. 3, pp. 753-771.

Wright-McLaughlin Engineers (1969) Urban Storm Drainage Criteria, Urban Drainage and Flood Control District, Denver, CO.

Yen, B.C., and Akan, A.O. (1983) Effects of Soil Properties on Overland Flow and Infiltration, *Journal of Hydraulic Research, IAHR, Vol. 21,* No. 2, pp. 153-173.

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