Economic modelling for groundwater resources management

A. Xepapadeas

Department of Economics, University of Crete, University Campus, 74100 Rethymno, Greece
(E-mail: xepapad@econ.soc.uoc.gr)

Abstract Conceptual models related to the management of groundwater resources are presented. Socially optimal and privately optimal, or atomistic, management rules and outcomes are compared, and decentralized policy schemes, in the form of water taxes, that can be used to sustain socially optimal use of groundwater resources are explored.

Keywords Groundwater resources; open loop and feedback Nash equilibrium; regulation; socially optimal management

Introduction

Historically, surface water has been the main source of water for human consumption, as it was easy and cost effective to access. However, increased rainfall shortages, especially in areas such as the Mediterranean islands, have resulted in increased use of groundwater to satisfy the ever increasing domestic, agricultural and environmental/ecosystem preservation water demands. Hence groundwater withdrawals have substantially increased, up to the point where they now constitute one third of the world’s freshwater consumption. This extensive use of groundwater, which is basically a common access resource, in many parts of the world has resulted in depletion of groundwater resources, biodiversity loss due to adverse effects on wetlands, pollution of groundwater resources because of percolation of pollutants associated with agricultural activities, or seawater intrusion in coastal aquifers. For an extensive analysis of some of these problems see Water Resources Research (2004).

Therefore, groundwater resources in areas with significant surface water shortages are of paramount importance in sustaining production and consumption patterns, as well as sustaining the flows of ecosystem services (provisioning and supporting services), and sustaining biodiversity and ecosystems resilience. Combining these needs with the acute scarcity of groundwater in many parts of the world gives rise to the necessity of making choices about how this resource should be allocated among competing uses and over time. This poses a very interesting management problem.

The purpose of the present paper is to present some conceptual models related to the management of groundwater resources, to compare socially optimal and privately optimal, or atomistic, management rules and outcomes using game theoretic solutions (see for example Provencher and Burt, 1982), and to explore decentralized policy schemes, in the form of water taxes, that can be used to sustain socially optimal use of groundwater resources.

Groundwater management: socially optimal and game theoretic solutions

We consider the management of a confined groundwater aquifer. Let $S(t)$ denote the groundwater stock level at time $t$ and $R(S(t))$ the natural recharge rate (net water inflow excluding extraction). This recharge rate is assumed to be a concave function of the
groundwater stock with \( R(S^C) = 0 \) where \( S^C \) is the aquifer’s capacity. Let \( x_i(t) \) represent groundwater extraction by agent \( i = 1, \ldots, n \), and let \( x(t) = (x_1(t), \ldots, x_n(t)) \) be the vector of extractions at time \( t \). An agent could be, for example, a farmer or any other decision-making unit that can extract water from the aquifer. Then the aquifer’s stock evolves as\(^1\):

\[
\frac{dS}{dt} = R(S) - x, \quad x = \sum_{i=1}^{n} x_i, \quad S(0) = S^0
\]

Let \( y_i(x_i(t)) \) denote benefits (e.g. agricultural production) accruing to economic agent \( i \), by extracting \( x_i(t) \) water at time \( t \). Then total benefits are defined as

\[
Y(t) = \sum_{i=1}^{n} y_i(x_i(t)), \quad \dot{y}_i(x_i) > 0, \quad y''_i(x_i) < 0
\]

The total cost of extracted water by agent \( i \) when the stock is \( S \) is given by \( C(S)x_i \), where the common unit cost \( C(S) \) is nonincreasing and convex\(^2\). Assume that a water authority manages the aquifer. The aim of the authority is to choose time paths for water extraction which will then be assigned to the individual agents, such that total benefits accruing from the use of the aquifer’s water are maximized. This is a formal optimal control problem that determines a socially optimal solution for the aquifer, and which can be stated as

\[
\max_{\{x(t)\}} \int_0^\infty e^{-\rho t} \sum_{i=1}^{n} [y_i(x_i(t)) - C(S)x_i(t)]dt
\]

subject to

\[
\frac{dS}{dt} = \dot{S} = R(S) - x, \quad x = \sum_{i=1}^{n} x_i, \quad S(0) = S^0
\]

Let

\[
H = \sum_{i=1}^{n} [y_i(x_i(t)) - C(S)x_i(t)] + \lambda(t)[R(S) - x] \tag{5}
\]

be the current value Hamiltonian of the problem. The optimality conditions derived by the maximum principle imply that \( x^*_i \) maximizes the Hamiltonian function, or

\[
\dot{y}_i(x^*_i) = C(S) + \lambda = x^*_i = x^*_i(S, \lambda) \tag{6}
\]

\[
\dot{\lambda} = \rho \lambda - \frac{\partial H}{\partial x} = (\rho - R'(S)) \lambda + C'(S) \sum_{i=1}^{n} x^*_i(S, \lambda) \tag{7}
\]

\[
\dot{S} = R(S) - \sum_{i=1}^{n} x^*_i(S, \lambda) \tag{8}
\]

Condition (6) is the short-run socially optimal extraction rule, stating that extraction should take place up to the point where marginal benefits equal marginal (and average) extraction cost plus scarcity rents (or shadow value) of the groundwater stock \( \lambda(t) \).

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\(^1\) To simplify things we assume that no return takes place from the water applied by the farmers back to the aquifer.

\(^2\) It is usually assumed that \( \lim_{S \to 0} C'(S) = +\infty \).
By totally differentiating (6) we obtain
\[
\frac{\partial x_i(t)}{\partial S} = \frac{C'(S)}{y_i'(x_i)} > 0, \quad \frac{\partial x_i(t)}{\partial \lambda} = \frac{1}{y_i'(x_i)} < 0
\] (6a)

Which implies that an increase in the groundwater stock, \(S\), will increase the short run application of groundwater, while an increase in groundwater scarcity rent, \(\lambda\), will reduce it. Solution of the system of differential equations (7), (8), assuming it exists, will determine the socially optimal time paths \((S^*(t), \lambda^*(t))\) and the socially optimal steady-state equilibrium \((S^\infty, \lambda^\infty)\), defined as the limit of \((S^*(t), \lambda^*(t))\) as \(t \to \infty\), for the water stock \(S\) and its shadow value \(\lambda\) as well as the corresponding socially optimal extraction paths \(x_i(S^*(t), \lambda^*(t))\). We can further examine the properties of a steady state.

**Proposition.** Assume that a steady state, \((S^\infty, \lambda^\infty)\), exists and assume that \(R(S^\infty) < 0\) and that \(-R'(S^\infty)\lambda^\infty + C''(S^\infty)\sum_{i} x_i(S^\infty, \lambda^\infty) + C'(S^\infty)\sum_{i} \frac{\partial x_i(S^\infty, \lambda^\infty)}{\partial S} > 0\), then the steady state is a local saddle point.

**Proof.** The Jacobian matrix of the system of (8) and (7) evaluated at the steady state is:
\[
\begin{pmatrix}
R'(S^\infty) - \sum_{i} \frac{\partial x_i(S^\infty, \lambda^\infty)}{\partial S} & 0 \\
-R'(S^\infty)\lambda^\infty + C''(S^\infty)\sum_{i} x_i(S^\infty, \lambda^\infty) + C'(S^\infty)\sum_{i} \frac{\partial x_i(S^\infty, \lambda^\infty)}{\partial S} & 0
\end{pmatrix}
\]

Under the assumptions made above its determinant is negative and therefore the steady state is a saddle point. If we consider that extraction decisions are taken individually where each economic agent has open access to the aquifer, that is, the aquifer is a common pool resource, there are three possible behavioural rules:

- **Myopic equilibrium.** The economic agent maximizes current profits and treats the groundwater stock level as fixed at a level \(\bar{S}\) without taking into account the evolution of the water stock defined by (1). This myopic extraction rule determines extraction as:
\[
x_i^0(t) : y_i'(x_i^0) = C(\bar{S})
\] (9)

It is clear that by ignoring groundwater scarcity rents, extraction is higher than the socially optimal, \(x_i^0(t) > x_i^*(t)\) and the resource tends to be depleted faster. Since this is basically an open access resource harvesting problem, this solution indicates tragedy of the commons. By substituting \(x_i^0(t)\) in (1) we can obtain the steady-state water stock \(S^0\) under myopic open access behavior as the limit for \(t \to \infty\).

- **Open Loop Nash Equilibrium.** The economic agent takes into account the evolution of the water stock defined by (1), but maximizes the present values of his/her net benefits, by choosing his/her extraction path and by treating the extraction paths of the other agents as fixed at a best response level. The problem can be set up as an n player noncooperative differential game, where extraction paths \{\(x_i(t)\)\} are each agent’s strategies. The strategy space is determined by the information structure of the game. In an open loop information structure individual extractions are defined as\(^3\):
\[
OL : x_i(t) = h_i(S^0, t) \quad i = 1, \ldots, n
\] (10)

\(^3\) For a detailed analysis of information structures, see Basar and Olsder (1982). For applications in environmental and resource economics, see Xepapadeas (1997).
Since each agent’s strategy depends only on the initial water stock \( S^0 \), the problem can be written as

\[
\max_{\{x(t)\}} \int_0^\infty e^{-pt} [y_i(x_i(t)) - C(S)x_i(t)]dt
\]  

subject to

\[
\frac{dS}{dt} = S = R(S) - x_i - \sum_{j \neq i} x_j, \; S(0) = S^0, \; x_j = h_j(S^0, t)
\]

The solution of the problem (11)–(13) corresponds to an Open Loop Nash Equilibrium (OLNE), which is characterized, assuming symmetry, by the following optimality conditions:

\( x_i^{\text{OL}} \) maximizes the Hamiltonian function,

\[
H_i = [y_i(x_i(t)) - C(S)x_i(t)] + \lambda_i^{\text{OL}}(t) \left[ R(S) - x_i - \sum_{j \neq i} x_j \right]
\]

or

\[
x_i^*(S) = C(S) + \lambda_i^{\text{OL}} \Rightarrow x_i^{*\text{OL}} = x_i^{*\text{OL}}(S, \lambda_i^{\text{OL}})\]

\[
\dot{\lambda}_i^{\text{OL}} = - \frac{\partial H}{\partial x_i} - (\rho - R'(S))\lambda_i^{\text{OL}} + C'(S)x_i^{*\text{OL}}(S, \lambda_i^{\text{OL}})
\]

\[
\dot{S} = R(S) - \sum_{i=1}^n x_i^{*\text{OL}}(S, \lambda^{\text{OL}})
\]

where, due to symmetry, \( \lambda_i^{\text{OL}} \) is the same for all \( i \). By comparing the socially optimal solution with the OLNE it is clear that \( \lambda^{\text{OL}} < \lambda \) so that the OLNE values resource stocks less than the social optimum and therefore extraction is higher, or \( x_i^{*\text{OL}}(S) > x_i^*(S) \). It also holds that \( x_i^0(S) > x_i^{*\text{OL}}(S) > x_i^*(S) \). This is because the individual extraction effects on costs are partially internalized in the OLNE, through the term \( C'(S)x_i^{*\text{OL}}(S, \lambda_i^{\text{OL}}) \) in (16), so that extractions are less than the myopic rule. But internalization is not full as in the social optimum, where full internalization is obtained through the term \( C'(S)\sum_{i=1}^n x_i^*(S, \lambda) \) in (8).

- **Feedback Nash Equilibrium.** Under a feedback (FB) information structure the strategy depends on the current state of the system, that is the current water stock \( S(t) \) and time. Therefore with an FB information structure individual extractions are defined as:

\[
FB : x_i^{FB}(t) = h_j(S(t), t) \; i = 1, \ldots, n
\]

The FB strategy described by (17) is often referred to as a *Markov perfect strategy* in which the water stock is a “sufficient statistic” for the history of the game. Given the strategy spaces determined by OL or FB information structures, the OLNE described above and the Feedback Nash Equilibrium (FBNE) are defined. An important feature of these two Nash equilibrium solutions relates to the concept of time consistency. A dynamic policy is time consistent if, at any point in time, a decision-maker looking into the future should have no reason to revise the future portion of his/her policy, provided that the truncated version of the policy is subject to the same criteria as the original version (Basar, 1989). Both OLNE and FBNE are time consistent in this sense. The OLNE however corresponds to an infinite period of commitment. Players,
that is polluting firms, commit themselves to a particular emission path at the outset of the game and do not respond to observed variation in the stock of the pollution. Thus if there is a small deviation from the optimal path, the policy that was optimal at the outset of the game will not necessarily be optimal for the remaining part of the game. In this sense OLNE is referred to as weakly time consistent. On the other hand the FBNE is a strongly time consistent solution in the sense of possessing the property of subgame perfectness (Fershtman, 1987; Basar, 1989) which implies that the equilibrium strategies constitute an equilibrium for all subgames along the equilibrium path. Thus a strong time consistent solution corresponds to extraction depending on the current water stock. Under the FB information structure the problem can be written as:

$$\max_{\{x(t)\}} \int_0^\infty e^{-\rho t}[y_i(x_i(t)) - C(S)x_i(t)]dt$$

subject to

$$\frac{dS}{dt} = \dot{S} = R(S) - x_i - \sum_{j \neq i} h_i(S(t)), \quad S(0) = S^0$$

The FBNE is characterized, assuming symmetry, by the following optimality conditions: $$x^*_{i,FB}$$ maximizes the Hamiltonian function,

$$H_i = [y_i(x_i(t)) - C(S)x_i(t)] + \lambda_i^{FB}(t)[R(S) - x_i - \sum_{j \neq i} h_i(S(t))]$$

or

$$y'_i(x^*_i) = C(S) + \lambda_i^{FB} \Rightarrow x^*_i(S, \lambda_i^{FB})$$

$$\dot{\lambda}_i^{FB} = -\frac{\partial H_i}{\partial x} = (\rho - R'(S) + (n - 1)h'_i(S))\lambda_i^{FB} + C'(S)x_i^{FB}(S, \lambda_i^{FB})$$

$$\dot{S} = R(S) - \sum_{i=1}^n x_i^{FB}(S, \lambda_i^{FB})$$

If $$h'_i(S) > 0$$, that is each agent expects that the response of the other agents to a decrease in the water stock will be a reduction in the water extraction, then it can be shown that under certain assumptions regarding the production and the unit extraction cost function, the ranking of extraction paths and steady state water stock is:

$$x^*_0(t) > x^*_{i,FB}(t) > x^*_{OL}(t) > x^*_i(t)$$

$$S^0 < S^{FB} < S^{OL} < S^\infty$$

Thus management under the water authority leads to greater water conservation relative to atomistic equilibria.

Although the theoretical results suggest that the socially optimal extraction path implies water conservation relative to the non-cooperative atomistic paths, there is an empirical example known as the Gisser Sanchez Effect that suggests that the numerical deviation between solutions is small. The result however is sensitive to the slope of the demand curve, and this sensitivity – along observed depletion of groundwater resources – provides a strong case for regulation.

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4 For a survey on the issue, see Koundouri (2004).
Regulation

The water authority can regulate the system by using as decentralized instruments water taxes or water quotas. Water quotas can be defined in a straightforward way as a path of quotas such that individual water extractions do not exceed \(x^*_i(t)\). On the other hand, the use of water taxes depends on what assumptions are made about the behavior of individual agents. The target of regulation through water taxes is to make the agents conditions for determining the optimal amount of water used equal to the socially optimal condition (6). Under the tax scheme the tax is imposed per unit of extracted water, and individual benefits are defined as \(y_i(x_i(t)) - C(S)x_i(t) - \pi(t)x_i(t)\), where \(\pi(t)\) is a time flexible water tax. Thus water taxes can be defined, under symmetry in the following way:

- Myopic Behavior: \(\tau_1(t) = \lambda(t)\)  \(\text{(26)}\)
- OL Behavior: \(\tau_2(t) = \lambda(t) - \lambda^{OL}(t)\)  \(\text{(27)}\)
- FB Behavior: \(\tau_3(t) = \lambda(t) - \lambda^{FB}(t)\)  \(\text{(28)}\)

It should be noticed that (26)–(28) imply a time flexible water tax. This type of regulation might be difficult to implement because of possible transaction costs associated with continuous tax adjustments. An alternative way would be to consider time invariant water taxes determined by the corresponding steady state values. In this case we have:

- Myopic Behavior: \(\tau_1 = \lambda^\infty\)  \(\text{(29)}\)
- OL Behavior: \(\tau_2 = \lambda^\infty - \lambda^{OL^\infty}\)  \(\text{(30)}\)
- FB Behavior: \(\tau_3 = \lambda^\infty - \lambda^{FB^\infty}\)  \(\text{(31)}\)

Under this scheme, the regulated system converges to the social optimum in the long run, but the regulated time paths are not identical to the socially optimal time paths.

A numerical example

To illustrate the above ideas we proceed with a numerical example. Let

\[ y(x_i) = \ln x_i \]  \(\text{(32)}\)
\[ C(S) = 1/S \]  \(\text{(33)}\)
\[ R(S) = F - bS \]  \(\text{(34)}\)

where,

- \(F\): reflects the inflow rate and \(b\) is loss from the aquifer.

From (6) optimal extraction is determined as

\[ x_i^* = \frac{1}{1/S + \lambda} \]  \(\text{(35)}\)

Assuming \(F = 100\), \(b = 0.1\), \(\rho = 0.01\), \(n = 10\) and using the conditions for the social optimum (6)–(8), we obtain the steady-state levels of the water stock, its shadow value, and steady-state extractions as

\[ S^\infty = 91.3 \quad \lambda^\infty = 0.099 \quad x_i^\infty = \frac{1}{(1/S^\infty + \lambda^\infty)} \]  \(\text{(36)}\)

As suggested in the proposition above, this steady state is a local saddle point since the eigenvalues of the corresponding modified Hamiltonian dynamic system
(MHDS) defined by (7)-(8), are (0.483528, 0.473528). Solution for the OLNE provides a steady state

\[ S_{OL}^\infty = 34.3, \quad \lambda_{OL}^\infty = 0.074, \quad x_{OL}^\infty = \frac{1}{(1/S_{OL}^\infty + \lambda_{OL}^\infty)} \]

which has also the local saddle point property since the eigenvalues are \((-1.17155, 0.469934)\). It is clear that the OLNE leads to overexploitation of the aquifer relative to the social optimum. Thus the steady state taxes are defined as:

\[ \tau_1 = 0.099 \]
\[ \tau_2 = 0.099 - 0.074 = 0.025 \]

Another issue related to regulation can be analyzed using the policy function of the socially optimal solution. Since the socially optimal solution has the local saddle point property, this means that there exists a one-dimensional stable manifold such that for any initial condition \( S_0 \) in the neighbourhood of this steady state, there exists an initial value for the costate variable \( \lambda^0 \) such that the system converges to the steady state along the stable manifold. This stable manifold provides the optimal policy function that determines the optimal water extraction \( x^*_i(t) \) for each value of the water stock \( S(t) \). Thus, the policy function can be written as \( x^*_i(t) = \phi(S(t)) \), with \( x^*_i(t) \) the same for all \( i \) under symmetry. The optimal policy function can be used to design water quotas which depend on the measured water stock. This function can be approximated by the tangent line to the stable manifold at the steady state. The tangent line is defined as:

\[ \lambda = \lambda^\infty + \gamma(S - S^\infty) \]

where,

\[ \gamma \]: is the slope of the eigenvector that corresponds to the negative eigenvalue, which in our case is \((-1.0, 0.0, 0.00332404)\).

Furthermore, since with the chosen functional forms \( \lambda = 1/x - 1/S \), by substitution we obtain the linearized policy function as:

\[ x^*_i(t) = \frac{1}{1/S(t) + \lambda^\infty + \gamma(S(t) - S^\infty)}, \quad \gamma = -0.000332404 \]

A water quota \( x^*_i \) can then be set so that \( x^*_i(t) \leq x^*_i(t) \).

### Groundwater management: quantity–quality problems

In a general quantity–quality (q-q) problem, the deterioration of the quality of a resource due to pollution results in the reduction of the effective use of the resource\(^5\). Thus the management of a resource should account for both its use and the emission of the pollutants that influence the effectiveness of its use. In the case of groundwater management, water which is pumped by agents (farmers) from a common access aquifer for irrigation purposes results in deep percolation that causes accumulation of pollutants in the aquifer. Pollution negatively affects the production of the agricultural output through the deterioration of the irrigation water quality.

A q-q problem can be set as follows. Let \( P \) be the stock of pollutants (e.g. salinity) accumulated in an aquifer. The stock of pollutants is a negative externality in the

production process. Thus the benefit function can be written as:

\[ y_i(t) = y_i(x_i(t), P(t)), \quad \frac{\partial y_i}{\partial P} \leq 0, \quad \frac{\partial^2 y_i}{\partial P^2} \leq 0, \quad \frac{\partial^2 y_i}{\partial P \partial x_i} \leq 0 \] (42)

The evolution of the pollution stock is given by

\[ P = g(x(t)) - bP, \quad P(0) = P^0, \quad x(t) = \sum_{i=1}^{n} x_i(t) \] (43)

In this model \( g(x(t)) \) is an increasing convex function which can be regarded as reflecting an emission function associated with pollution accumulation, while \( b \geq 0 \) reflects the aquifer’s self cleaning capacity.

In this q-q problem, the social optimum is defined as the solution of the following problem:

\[
\max_{\{x(t)\}} \int_0^\infty e^{-P} \sum_{i=1}^{n} [y_i(x_i(t), P(t)) - C(S)x_i(t)] dt
\] (44)

subject to

\[ \frac{dS}{dt} = S = R(S) - x, \quad x = \sum_{i=1}^{n} x_i, S(0) = S^0 \] (45)

\[ \dot{P} = g(x(t)) - bP, \quad P(0) = P^0 \] (46)

The current value Hamiltonian for the problem is

\[ H = \sum_{i=1}^{n} [y_i(x_i, P) - C(S)x_i] + \lambda[R(S) - x] + \mu[g(x(t)) - bP] \] (47)

and the optimality conditions derived by the maximum principle imply that \( x_i^* \) maximizes the Hamiltonian function, or

\[ \frac{\partial y_i(x_i^*, P)}{\partial x_i} = C(S) + \lambda - \mu \frac{\partial g}{\partial x_i} \Rightarrow x_i^* = x_i^*(S, \lambda, \mu) \] (48)

\[ \dot{\lambda} = \rho \lambda - \frac{\partial H}{\partial x_i} = (\rho - \dot{R}(S)) \lambda + C^0(S) \sum_{i=1}^{n} x_i(S, \lambda) \] (49)

\[ \dot{S} = R(S) - \sum_{i=1}^{n} x_i^*(S, \lambda) \] (50)

\[ \mu = (\rho + b) \mu - \sum_{i=1}^{n} \frac{\partial y_i(x_i^*, P)}{\partial P} \] (51)

\[ \dot{P} = g(x^*(t)) - bP \] (52)

It is interesting to note from (48) that optimal water extraction is determined at the point where marginal water benefit equals extraction costs plus the shadow value of water

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6 If pollution creates further environmental damages, e.g. contamination of freshwater sources, or destruction of wetlands, then an increasing and convex damage function \( D(P) \) should be incorporated into the benefit flow, which will become \( \sum_{i=1}^{n} [y_i(x_i(t), P(t)) - C(S)x_i(t)] - D(P) \).

7 The analysis of the socially optimal steady state is similar to the analysis in the previous section, although more complicated since we are dealing with a control problem with two state equations.
stock, plus the shadow cost of the pollutant reflected by \( \mu < 0 \). Comparing this condition with (6), it is clear that extractions are smaller in the q-q problem. The analysis of atomistic equilibria in the q-q model can be conducted along the lines described above.

- **Myopic equilibrium.** The agents take as given and exogenous not only water stock but pollution stock as well. Thus their optimal extractions are determined as:

\[
x_i^0 : \frac{\partial y_i(x_i^0, P)}{\partial x_i} = C(\hat{S})
\]

(53)

- **Open Loop Nash Equilibrium.** The optimality conditions with open loop information structure and symmetry are:

\[
x_i^{*\text{OL}} \text{maximizes the Hamiltonian function}, \quad H_i = [y_i(x_i, P) - C(S)x_i(t)] + \lambda_i^{\text{OL}}(t) \left[ R(S) - x_i - \sum_{j \neq i} x_j \right] + \mu_i^{\text{OL}} \left[ g \left( x_i \sum_{j \neq i} x_j \right) - bP \right]
\]

or

\[
y_i'(x_i) = C(S) + \lambda_i^{\text{OL}} - \mu_i^{\text{OL}} \frac{\partial g}{\partial x_i}
\]

(54)

\[
\Rightarrow x_i^{*\text{OL}} = x_i^{*\text{OL}}(S, \lambda_i^{\text{OL}}, \mu_i^{\text{OL}})
\]

(55)

\[
\lambda_i^{\text{OL}} = \rho \lambda_i^{\text{OL}} - \frac{\partial H_i}{\partial x} = (\rho - R(S)\lambda_i^{\text{OL}} + C(S)x_i^{*\text{OL}}(S, \lambda_i^{\text{OL}}, \mu_i^{\text{OL}})
\]

(56)

\[
\hat{S} = R(S) - \sum_{i=1}^n x_i^{*\text{OL}}(S, \lambda_i^{\text{OL}})
\]

(57)

\[
\dot{\lambda}_i = (\rho + b) \mu_i^{\text{OL}} - \frac{\partial y_i(x_i^{*\text{OL}}, P)}{\partial P}
\]

(58)

\[
\dot{P} = g(x_i^{*\text{OL}}(t)) - bP
\]

(59)

- **Feedback Nash Equilibrium.** Assuming a Markov perfect strategy \( h_i(S, P) \), with \( \partial h_i/\partial S > 0 \) and \( \partial h_i/\partial P < 0 \), the optimality conditions under symmetry, can be written as:

\[
x_i^{*\text{FB}} \text{maximizes the Hamiltonian function},
\]

\[
H_i = [y_i(x_i, P) - C(S)x_i(t)] + \lambda_i^{\text{FB}}(t) \left[ R(S) - x_i - \sum_{j \neq i} \bar{x}_j \right] + \mu_i^{\text{FB}} \left[ g \left( x_i \sum_{j \neq i} \bar{x}_j \right) - bP \right]
\]

(60)

\[
\dot{S} = R(S) - \sum_{i=1}^n x_i^{*\text{FB}}(S, \lambda_i^{\text{OL}}, \mu_i^{\text{OL}})
\]

(61)
or

\[ y_i'(x_i) = C(S) + \lambda_i^F - \mu_i^F \frac{\partial g}{\partial x_i} \]  

(62)

\[ \Rightarrow \lambda_i^{*F} = x_i^{*F} (S, \lambda_i^{*F}, \mu_i^{*F}) \]  

(63)

\[ \hat{\lambda}_i^F = \rho \lambda_i^F - \frac{\partial H_i}{\partial x_i} = (\rho - R'(S)) \lambda_i^F + C'(S) x_i^{*F} (S, \lambda_i^{*F}, \mu_i^{*F}) \]  

(64)

\[ \hat{s} = R(S) - \sum_{i=1}^n x_i^{*F} (S, \lambda_i^{*F}) \]  

(65)

\[ \mu_i^F = (\rho + b) \mu_i^F - \frac{\partial y_i(\lambda_i^F, x^F)}{\partial P} \]  

(66)

\[ P = g(\lambda_i^F) - bP \]  

(67)

It can be shown under certain assumptions regarding the emission function that the ranking, with respect to water extraction and steady-state water stock, is the same as that implied by (24)–(25), while the ranking with respect to the steady-state pollution accumulation is:

\[ p^0 > p^{*FB} > p^{*OL} > p^\infty \]  

(68)

Thus pollution is smaller and water conservation is greater at the social optimum.

Regulation

Regulation in this case has to take into account the pollution externality, so that the water tax should be adjusted accordingly. Since, however, there is a second externality – pollution – the water tax should have two parts. One will account for the water over extraction, while the second will account for excess emissions indicated by (68). The time flexible water taxes are defined, under symmetry, as 8:

**Myopic Behavior:**  
\[ \tau_1(t) = \lambda(t) - \mu(t) \frac{\partial g}{\partial x_i} \]  

(69)

**OL Behavior:**  
\[ \tau_2(t) = (\lambda(t) - \lambda^{OL}(t)) - (\mu(t) - \mu^{OL}(t)) \frac{\partial g}{\partial x_i} \]  

(70)

**FB Behavior:**  
\[ \tau_3(t) = (\lambda(t) - \lambda^{FB}(t)) - (\mu(t) - \mu^{FB}(t)) \frac{\partial g}{\partial x_i} \]  

(71)

Furthermore, the corresponding steady-state taxes are:

**Myopic Behavior:**  
\[ \tau_1 = \lambda^\infty - \mu^\infty \frac{\partial g}{\partial x_i} \]  

(72)

**OL Behavior:**  
\[ \tau_2 = (\lambda^\infty - \lambda^{\infty OL}) - (\mu^\infty - \mu^{\infty OL}) \frac{\partial g}{\partial x_i} \]  

(73)

**FB Behavior:**  
\[ \tau_3 = (\lambda^\infty - \lambda^{\infty FB}) - (\mu^\infty - \mu^{\infty FB}) \frac{\partial g}{\partial x_i} \]  

(74)

8 Note that the shadow cost of pollution is \( \mu^\infty < 0 \) in all cases.
A numerical example

These ideas are illustrated by extending the numerical example developed above. We assume that damages due to the pollution externality are given by a damage function \((1/2)P^2\), so net benefits are defined as:

\[
\ln x_i - (1/S)x_i - (1/2)P^2
\]

We assume that the stock of pollution in the aquifer evolves according to

\[
\dot{P} = v(nx_i) - \delta P, \quad x_i \text{ the same for all } i
\]

where,

- \(v\) is a fixed unit emission coefficient and \(\delta \geq 0\) reflects the aquifer’s self cleaning capacity.

Using optimality conditions (48), socially optimal extraction is determined as:

\[
x_i^* = \frac{1}{1/S + \lambda - v\mu}
\]

Assuming furthermore that \(v = 0.2\) and \(\delta = 0.05\) and using the conditions for the social optimum (48)–(52), we obtain the steady-state levels of the water stock, its shadow value, the pollution stock and its shadow cost, and steady-state extractions as:

\[
S^\infty = 997.262, \quad \lambda^\infty = 0.00907257
\]

\[
P^\infty = 1.09529, \quad \mu^\infty = -182.549, \quad x_i^\infty = \frac{1}{(1/S^\infty + \lambda^\infty - v\mu^\infty)}
\]

This steady state is a local saddle point since the eigenvalues of the corresponding modified Hamiltonian dynamic system (MHDS) are: \((0.209052, -0.1, 0.0826101, -0.07261)\). Thus a two-dimensional stable manifold exists, and for any initial values for \(S\) and \(P\) in the neighbourhood of the steady state, initial values for \(\lambda\) and \(\mu\) and consequently \(x_i\) can be chosen so that the system converges to the steady state.

It should be noticed that under the pollution externality water conservation and the steady-state water stock are larger relative to the previous case where the application of the extracted water (irrigation) did not create pollution in the aquifer. Solution for the OLNE provides a steady state

\[
S^{OL} = 991.341, \quad \lambda^{OL} = 0.002578
\]

\[
P^{OL} = 3.46356, \quad \mu^{OL} = -57.7261
\]

\[
x_i^{OL} = \frac{1}{(1/S^{OL} + \lambda^{OL} - v\mu^{OL})}
\]

which also has the local saddle point property since the eigenvalues of the corresponding modified Hamiltonian dynamic system (MHDS) are: \((0.189069, -0.100001, 0.0826086, -0.0726081)\). It is clear that the OLNE leads to water overexploitation and excess pollution relative to the social optimum. Thus, the steady-state taxes are defined, using (72) and (73), as:

\[
\tau_1 = 0.00907257 + (0.2)(182.549)
\]

\[
\tau_2 = (0.00907257 - 0.002578) + (0.2) (-182.549 + 57.7261)
\]

It is clear that the water tax is higher relative to the no pollution case since it is adjusted to take into account the production externality. It should be noticed that because there are two degrees of freedom on the stable manifold the policy function cannot be constructed...
in the previous way. The policy function could be approximated using time elimination methods, but this is beyond the scope of the present paper.

**Concluding remarks**

The purpose of this paper was to present models related to groundwater management. Two models were developed: one in which only the quantity of the water in an aquifer was the management objective, and a second in which both the quantity and the quality of the aquifer’s water were managed. In this second model, quality was negatively affected by the water used for agricultural activities and the consequent deep percolation of agricultural run off. In each model alternative management regimes were examined: the socially optimal management problem, where the purpose was to manage water and pollution stock by maximizing total benefits in a given region, and the non-cooperative or atomistic problems, where the objective of each individual agent was to maximize own benefits. When individual agents did not take into account the dynamic constraints imposed by the evolution of the water stock and the pollution stock, a myopic atomistic equilibrium was obtained; when these constraints were taken into account, atomistic equilibrium was obtained either as an open loop or feedback Nash equilibrium. We manage to define decentralized water taxes both for the pure quantity and the quality – quantity problem capable of regulating the system so that the socially optimal outcome is achieved. Numerical estimates confirming the theoretical analysis were obtained for the cases of myopic and open loop equilibrium. The complete characterization of feedback solutions and the corresponding water taxes for general (non linear-quadratic) problems is a current open research problem and its analysis is beyond the scope of the present work.

**References**


