To obtain reasonable accuracy with a straightforward use of finite-difference approximations to derivatives, it is usually found that rather more mesh points are desirable than have been used by the author.

Again, when using collocation methods for satisfaction of boundary conditions, i.e., satisfaction of such conditions at a small finite number of points on the boundary, it is desirable to examine, in some way or another, the results given by such solutions at boundary points intermediate to the "collocation" boundary points. It occasionally happens that in between the collocation points the solutions oscillate quite wildly.

Author's Closure

The author would like to thank Dr. Shaw for his interest in this application of numerical methods to the problem of the rectangular cantilever plate. As emphasized in the paper, the results hold only for a plate subject to uniform normal pressure and having a span-to-chord ratio of 1:2; other cases would have to be dealt with individually and it is probable that more mesh points would be necessary in some cases, particularly those involving unsymmetric loadings. In the numerical solution of the problem discussed in the paper the values of the bending moment and edge reaction at points intermediate to the mesh points shown were computed and found to be negligibly small.

On the Axisymmetric Problem of the Theory of Elasticity for an Infinite Region Containing Two Spherical Cavities[1]

H. Poritsky.[2] The authors have added one further interesting case to the impressive list of stress problems solved by them; in this case the solution, though exact, is not in finite form.

The writer would like to ask what general criteria the authors could suggest for separability of the type suggested by Equation [17] of the authors' paper for the Laplace or the repeated Laplace equation, namely, where solutions exist which are of the form $\mu A(\alpha)B(\beta)C(\gamma)$, that is, a product of a fixed function μ by functions of the independent variables, $A(\alpha)$, $B(\beta)$, $C(\gamma)$, satisfying proper ordinary differential equations.

In view of the extreme complexity of the solution found and the large amount of algebraic manipulation required in applying it, it is proper to ask whether it would not be simpler to apply an alternative procedure, similar to a method utilized by A. E. Green for plates with circular holes.[3] For the region external to two spheres S, S_1 with centers at O, O_1 on the z-axis, this method, when applied to harmonic functions which are axially symmetric about the z-axis, proceeds as follows:

Let the center O of S be at the origin $z = 0$, the center O_1 of S_1 at $z = a$. One starts with the two series

$$V = \Sigma A_n P_n(\cos\theta)/R^{n+1} + \Sigma B_n P_n(\cos\theta_1)/R_1^{n+1}$$

where R, R_1 are the distances from O, O_1 and θ, θ_1 the colatitude angles based on spherical co-ordinates with O, O_1 as poles. To apply proper boundary conditions on S one expands the terms of the second series in positive powers of R, utilizing the Neumann-type expansion

$$\frac{P_n(\cos\theta_1)}{R_1^{n+1}} = \frac{1}{a^{n+1}} + \frac{(n+1)RP_1(\cos\theta)}{a^{n+2}} + \frac{(n+1)(n+2)}{2!}\frac{R^2P_2(\cos\theta)}{a^{n+3}} + \ldots$$

The second series is then rearranged so that terms with like Legendre polynomials in $\cos\theta$ are grouped together, and the boundary conditions over S are applied. Similarly, one applies the boundary conditions over S_1 by expanding the A_n-terms in positive powers of R_1 and rearranging in Legendre polynomials in $\cos\theta_1$. In this manner one is led to an infinite number of equations in the two sets of coefficients A_n, B_n. These equations are solved by a method of successive approximations, based on the fact that the coefficients in the diagonal terms are much larger than the remaining coefficients.

Quite similar methods are applicable to the repeated Laplace equation and to regions with more than two spherical cavities.

The writer has not had any experience with the foregoing method as applied to the region exterior to several spheres and to elastic problems, but he once applied an analogous method to a harmonic problem for the region exterior to a cylinder and between parallel planes, and found the convergence very gratifying.

Authors' Closure

The authors greatly appreciate Dr. Poritsky's interesting comments. In connection with his first question, reference is made to Bôcher's book[4] which contains a pertinent discussion. Further references are cited in a more recent paper on the same subject.[5]

With regard to the second question, it is certainly correct that the scheme outlined by Dr. Poritsky provides an alternative approach to the problem under consideration. Whether this approach is simpler is difficult to foresee. In the present problem it would involve four, rather than two, infinite sets of initially unknown coefficients, and would lead to a doubly infinite system of equations; in contrast, the method which was adopted yields a single infinite system which has a maximum of ten unknowns in each equation. Moreover, it would seem questionable whether expectations as to the speed of convergence derived from experience with a Dirichlet problem, are justified. An obvious important advantage of Dr. Poritsky's suggestion is the fact that the proposed method remains applicable to any number of cavities, whereas the approach based on spherical dipolar co-ordinates is confined to the case of two cavities.

It perhaps should be pointed out that the algebraic complexity of the solution given stems from various rearrangements of the initial aggregates of component solutions. These rearrangements were undertaken in order to accelerate the convergence of the solution. It would have been possible to leave the solution in a form whose simpler appearance is maintained at the sacrifice of a more favorable convergence. Incidentally, the authors believe that the algebraic work could now be shortened considerably by means of an alternative stress-function approach.[6]

[1] By E. Sternberg and M. A. Sadowsky, published in the March, 1952, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 74, pp. 19–27.

[2] Consulting Engineer, General Electric Company, Schenectady, N. Y. Mem. ASME.

[3] "General Bi-Harmonic Analysis for a Plate Containing Circular Holes," by A. E. Green, Proceedings of the Royal Society, series A, vol. 176, 1940, pp. 121–139.

[4] "Ueber die Reihenentwicklungen der Potentialtheorie," by Maxime Bôcher, B. G. Teubner, Leipzig, Germany, 1894.

[5] "Separation of Laplace's Equation," by N. Levinson, B. Bogert, and R. M. Redheffer, *Quarterly of Applied Mathematics*, vol. 7, 1949, pp. 241–262.

[6] "Pure Bending of an Incomplete Torus," by M. A. Sadowsky and E. Sternberg; now in preparation.