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A general field-line plotting algorithm

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A new, general algorithm for plotting electrostatic field lines is presented which requires a minimal amount of control to plot two-dimensional field lines in vacuum regions containing conductors. The resulting plot has the property that the density of field lines is linearly related to the strength of the field. The new method is applicable to other fields as well.

INTRODUCTION

Sophisticated graphical techniques exist for plotting many scientifically interesting quantities. However, there seems to be a dearth in the area of plotting field lines. It is all the more surprising since any method developed for plotting electrostatic field lines is equally applicable to many other fields as well—magnetostatic field lines and fluid flow lines being two examples.

Several methods come to mind for developing a field-line plotting algorithm. The first two involve the observation that the field lines can be represented by the level curves (contours) of a scalar function, ψ, usually called the stream function. If this function is known, then the problem is solved since there exist many packages that plot contours of scalar functions in two dimensions. See Ref. 1 for several samples.

In the general case, ψ is not known, but there are at least two methods to compute it. The first is to notice that ψ with the electrostatic potential, ϕ, obey the Cauchy–Riemann equations,[2] i.e.,

$$\frac{\partial\psi}{\partial x}=\frac{\partial\phi}{\partial y},$$

$$\frac{\partial\psi}{\partial y}=-\frac{\partial\phi}{\partial x}.$$

In terms of the electric fields,

$$\frac{\partial\psi}{\partial x}=-E_y,$$

$$\frac{\partial\psi}{\partial y}=E_x.$$

Thus one could numerically integrate the electric field to find ψ and then plot ψ with a standard contour-plotting routine.

Alternatively, one could note that in a vacuum, both ϕ and ψ satisfy Laplace's equation.[3] Thus it would seem that a Laplace solver could be applied to solve for ψ.

Unfortunately, neither of these methods is satisfactory in systems that contain internal conductors. The reason is because ψ contains branch cuts.[4] These branch cuts must be anticipated for each system and the computation of ψ must be significantly modified accordingly. For the first method, integrating the electric fields, this requires troublesome paths of integration to place branch cuts out of the way. The second method, solving Laplace's equation, is even more difficult since the branch cuts become internal boundary conditions on ψ with different values on either side of each cut.

I. THE WHITTAKER METHOD

Fortunately, it is possible to plot field lines without ever computing the stream function. The concept is simple but many minor problems must be overcome to produce the final result. Consider a field line l. By definition, it is parallel to the electric field E, thus

$$\mathbf{l}\times\mathbf{E}=0\Rightarrow l_xE_y-l_yE_x=0\Rightarrow E_y/E_x=l_y/l_x.$$

Thus, starting at any point, a field line can be computed by incrementing the above equation numerically as

$$\Delta_y=(E_y/E_x)\Delta_x, \tag{1}$$

where Δ_y is the vertical displacement to the next point on the field line given an arbitrary x displacement, Δ_x. Let E be the strength of the field. It is convenient to let $\Delta_x=\Delta_l E_x/E$ so that

$$\Delta_y=\Delta_l E_y/E$$

and $\Delta_y^2+\Delta_x^2=\Delta_l^2$. Thus each iteration increases the length of the field line by Δ_l regardless of the strength of the field.

When chosen arbitrarily, a starting point of the integration is usually not the start of the field line. Thus one must integrate Eq. (1) backward to get the rest of a field line. In this case,

$$\Delta_x=-\Delta_l E_x/E,$$

$$\Delta_y=-\Delta_l E_y/E.$$

A method based on this technique was proposed by Whittaker in 1977.[5] It is the method currently employed by the STRMLN routine that is part of the NCAR graphics package supported by the National Center for Atmospheric Research.[6]

This method works to an extent but the density of field lines does not accurately represent the strength of the fields. The problem is related to how the field lines are started. In Whittaker's method, the computational grid is swept in a uniform manner and field lines are started in

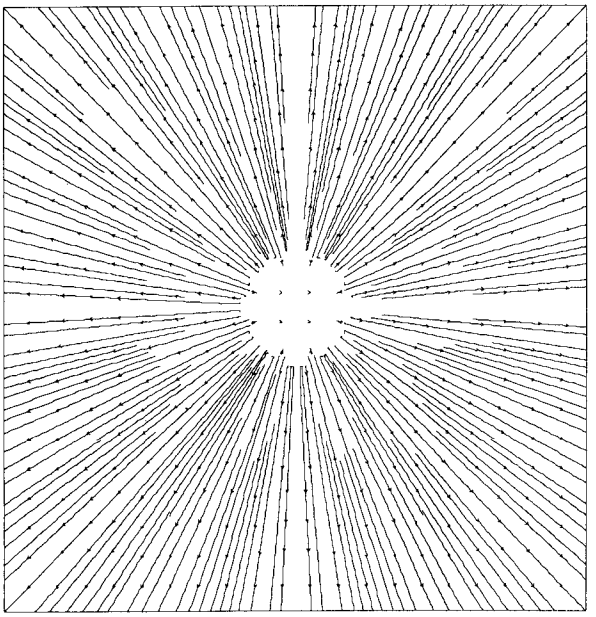

FIG. 1. Electric fields from an infinitely long wire in free space, as plotted by the NCAR graphics routine, STRMLN.

cells that are not occupied by lines started in other cells. Thus a cell in a weak field region that should not have a field line will have one anyway, and a cell in a strong field region, which should have multiple lines will have only one. Figure 1 illustrates this problem with a simple monopole (i.e., a wire of infinite length).

The new method presented here addresses this problem in detail, with the result being a general algorithm for plotting two-dimensional field lines.

II. THEORY

The essential problem is to choose the starting points of the field lines such that when all of the field lines are drawn, their density represents the field strength. In other words, we require any two adjacent field lines to contain a predetermined amount of flux, i.e.,

$$\Phi_0 = \int_a^b \mathbf{E}\cdot\mathbf{n}\, ds,$$

where a is *any* point on one field line and b is *any* point on an adjacent field line, $\mathbf{s}$ is *any* path from a to b, and $\mathbf{n}$ is the normal to $\mathbf{s}$. As long as Φ_0 is the same for any two adjacent field lines, then the density of field lines will represent the strength of the field.

A major simplification is to choose $\mathbf{s}$ to be a level curve (contour) of the potential, ϕ. This accomplishes several things. The first is that since the electric field is always perpendicular to a line of constant potential, the above definition for Φ_0 becomes

$$\Phi_0 = \int_a^b E\, d\mathbf{s}.$$

Now, b and $\mathbf{s}$ are determined by selecting a. Other advantages to using a ϕ contour are (1) it can be specified by one point; (2) it is never parallel to a field line; and (3) it avoids conductors.

III. SIMPLE CASES

It is often the case that all of the field lines pass though a single potential contour. This case is the easiest and illustrates the essentials of the method. The method requires that the potential is given on a grid, from which the electric field can be calculated. In what follows, we assume that the electric field varies linearly between any two points that are reasonably close. This is not a necessary condition for the method. Higher-order interpolations could be used, but our experience shows that linear interpolation is usually sufficient.

Given a point in the (x,y) plane, we can find the potential contour that passes through that point. There are many common methods for doing this so we omit discussion of our particular scheme. The result is a line represented by a series of points, $s_i = \{x_i,y_i\}$; $i = 1,...,I$. The total flux through this line is

$$\Phi_T = \int_{s_1}^{s_I} E\, d\mathbf{s} = \frac{1}{2}\sum_{i=1}^{I-1} (E_i + E_{i+1})|\mathbf{s}_{i+1} - \mathbf{s}_i|.$$

If we want N field lines, we let $\Phi_0 = \Phi_T/(N-1)$. Now N points must be selected along $\mathbf{s}$ such that

$$\int_{s_j}^{s_{j+1}} E\, d\mathbf{s} = \Phi_0$$

for any $j = 1,...,N-1$. The first point is the beginning of $\mathbf{s}$. To find the next point, we integrate from the first point to some point i such that

$$\int_{s_1}^{s_i} E\, d\mathbf{s} < \Phi_0 < \int_{s_1}^{s_{i+1}} E\, d\mathbf{s}.$$

We must find the point $\mathbf{s}_j = \{x_j,y_j\}$ on the segment $\mathbf{s}_i,\mathbf{s}_{i+1}$ such that

$$\int_{s_1}^{s_j} E\, d\mathbf{s} = \Phi_0 \Rightarrow \int_{s_i}^{s_j} E\, d\mathbf{s} = \Phi_0 - \int_{s_1}^{s_i} E\, d\mathbf{s}$$

$$= \delta\Phi \Rightarrow \tfrac{1}{2}(E_i + E_j)|\mathbf{s}_j - \mathbf{s}_i| = \delta\Phi.$$

Let $\alpha = |\mathbf{s}_j - \mathbf{s}_i|/|\mathbf{s}_{i+1} - \mathbf{s}_i|$, i.e., the fractional distance from point i to point j. Since E is linear,

$$E_j = (1-\alpha)E_i + \alpha E_{i+1} \Rightarrow \tfrac{1}{2}[E_i + (1-\alpha)E_i + \alpha E_{i+1}]\alpha|\mathbf{s}_{i+1} - \mathbf{s}_i|$$

$$= \delta\Phi \Rightarrow (E_{i+1} - E_i)\alpha^2 + 2E_i\alpha - 2\,\delta\Phi/|\mathbf{s}_{i+1} - \mathbf{s}_i| = 0 \Rightarrow \alpha = \frac{\sqrt{E_i^2 + 2\,\delta\Phi(E_{i+1} - E_i)/|\mathbf{s}_{i+1} - \mathbf{s}_i|} - E_i}{E_{i+1} - E_i}.$$

The positive root of the quadratic equation ensures that $0 < \alpha < 1$ and thus $\mathbf{s}_j$ lies between $\mathbf{s}_i$ and $\mathbf{s}_{i+1}$. If $E_{i+1} = E_i$, then Eq. (1) implies that $E_j = E_i$ and

$$\alpha = \delta\Phi/E_i|\mathbf{s}_{i+1} - \mathbf{s}_i|.$$

In general,

$$\mathbf{s}_j = \{x_j,y_j\}$$

$$= \{(1-\alpha)x_i + \alpha x_{i+1}, (1-\alpha)y_i + \alpha y_{i+1}\}.$$

This point is stored as the start of a field line and the integration proceeds from $\mathbf{s}_j$ to $\mathbf{s}_{i+1}$ then to $\mathbf{s}_{i+2}$, etc., until the next field-line starting point is found. This process continues to the end of $\mathbf{s}$, at which point N points will have been found.

Once the starting points are found, the field lines through each point are drawn by integrating as in the Whittaker method. Figure 2 shows the result of applying our technique. The dotted line is $\mathbf{s}$ and the inner circle is the conducting surface.

As in Whittaker's method, each field line is integrated until one of the following conditions is met:

(1) The field line enters a forbidden region (in our case it is either the conductor or the edge of the computational domain).

(2) The field line reaches a field null.

(3) The field line loops back onto itself.

(4) The field line has too many segments. This is a safety feature in case one of the other checks is not working correctly.

IV. COMPLICATED CASES

In some situations, there does not exist one potential contour through which all of the field lines pass. In these cases, one must choose multiple contours to plot all of the field lines. This commonly occurs when there are many internal conductors. Unfortunately, it is not sufficient to repeat the above procedure for all of the potential contours, the reason being that field lines that start on the first contour could pass through the second. In these regions, one has to assure that superfluous field lines are not plotted.

Consider two infinite wires in free space with opposite line charge ($\pm\lambda$). The potential for this case is

$$\phi(x,y) = \lambda \ln\{[x^2 + (y+d)^2]/[x^2 + (y-d)^2]\},$$

where $d^2 = r^2 - a^2$, r being half the distance between the wires and a being the radius of the wires. The field lines are shown in Fig. 3. The dotted circles, $\mathbf{s}^1$ and $\mathbf{s}^2$, are the two potential contours we chose. (It just so happens that the potential contours for this case are perfect circles, but our code computes the contours in general cases.)

From $\mathbf{s}^1$ we generate a set of 30 field lines, $\mathbf{l}^1_{n1}$; $n1 = 1,...,30$. Equivalently, from $\mathbf{s}^2$ we could generate another set of 30 field lines, $\mathbf{l}^2_{n2}$. However, we want to exclude those field lines starting from $\mathbf{s}^2$ that enter regions where field lines from $\mathbf{s}^1$ have already been drawn. To do this, we must keep track of the intersection of any $\mathbf{l}^1_{n1}$ line with $\mathbf{s}^2$.

Any given field line $\mathbf{l}^1_{n1}$ will consist of several points $\mathbf{l}^1_{n1,k}$. A segment of $\mathbf{l}^1_{n1}$ from point k to $k+1$ is checked to see if it intersects any segment of $\mathbf{s}^2$. If so, the intersecting point is added to $\mathbf{s}^2$ and identified as containing a field line. Thus, after all of the $\mathbf{l}^1_{n1}$ lines have been drawn, $\mathbf{s}^2$ will contain several additional points that are the intersections of all of the $\mathbf{l}^1_{n1}$ lines with $\mathbf{s}^2$.

Now field lines can be generated from $\mathbf{s}^2$ similar to the way they were generated from $\mathbf{s}^1$ with a few changes. The steps are

(1) Look for the first intersection point on $\mathbf{s}^2$.

(2) Integrate from that point until either (a) another intersection point is reached; or (b) the integral exceeds Φ_0 so that a starting point must be found.

(3) Redo step 2 until the end is reached and then... .

(4) Integrate from the first intersecting point backward to find the remaining starting points.

The field lines that were plotted from $\mathbf{s}^2$ are shown in Fig. 3 as dashed lines.

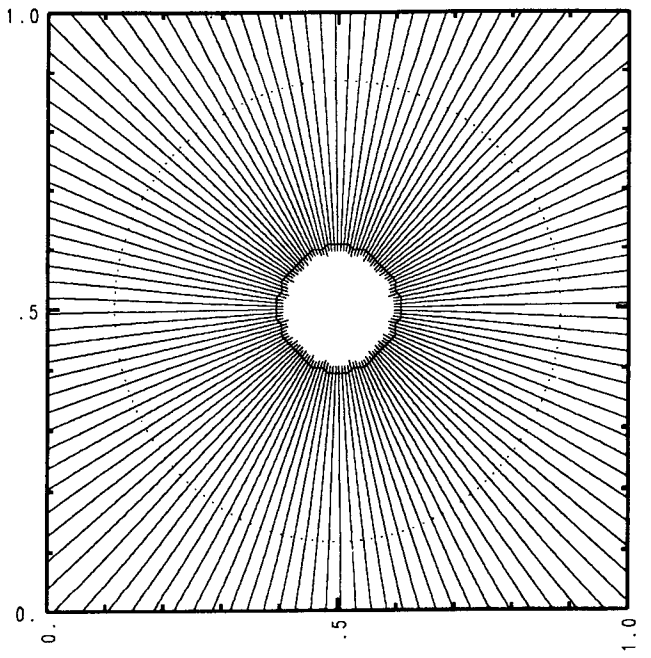

FIG. 2. The same fields used in Fig. 1 are plotted using the new method. The dotted line is the potential contour used to find the field-line starting points.

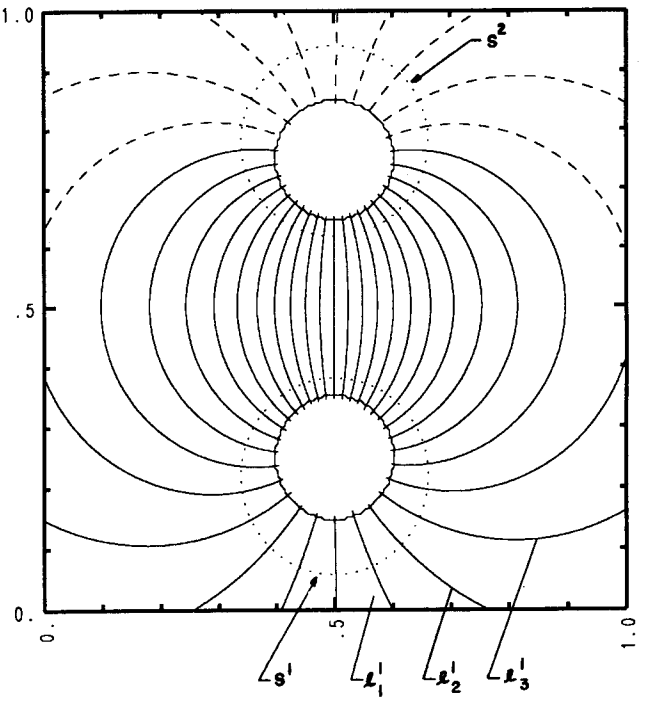

FIG. 3. Electric field lines from two infinitely long wires of opposite charge in free space. $\mathbf{s}^1$ and $\mathbf{s}^2$ are potential contours. The solid field lines were drawn from $\mathbf{s}^1$. The dashed field lines were drawn from $\mathbf{s}^2$.

V. FINER POINTS

The above method will work to an extent, but to get perfect pictures a minor problem must be resolved. Consider the situation in Fig. 4. The left and right boundaries are conducting surfaces held at 0 and 400 V, respectively. The line of circles near the center represents infinitely long wires held at 100 V. The upper and lower boundaries are periodic. The potential was obtained from a computational solution to Laplace's equation.

Field lines were first drawn from $\mathbf{s}^1$. Many of them intersect $\mathbf{s}^2$, and $\mathbf{s}^2$ contains regions with no intersection points separated by regions with intersection points. One such region is the section between $\mathbf{l}_6^1$ and $\mathbf{l}_9^1$. If we start from the intersection of $\mathbf{l}_6^1$ and $\mathbf{s}^2$ and integrate along $\mathbf{s}^2$ to find starting points for the $\mathbf{l}^2$ lines, we may find a point very close to, but before, the intersection of $\mathbf{l}_9^1$ with $\mathbf{s}^2$. This will result in a field line being drawn too close to $\mathbf{l}_9^1$. To avoid this, the flux between an intersection point and the previous starting point must be checked to determine if the previous starting point is too close. Some parameter must be specified for this check. In all of the plots here, a starting point would be removed if the flux between it and the next intersection point was less than $0.6\Phi_0$.

Note that in this example only two potential contours were needed even though the system contains many conductors. This is not always true. Figure 5 shows four wires. The left and right were held at 100 V, the top and bottom were held at $-$ 100 V, and all of the boundaries were held at 0 V. Here also, the potential was obtained computationally.

In this case, four potential contours were needed and all of the intersections had to be checked to avoid duplicate field lines. As the $\mathbf{l}^1$ lines were being drawn, they were checked for intersections with $\mathbf{s}^2$, $\mathbf{s}^3$, and $\mathbf{s}^4$. Then the $\mathbf{l}^2$ lines were checked for intersections with $\mathbf{s}^3$ and $\mathbf{s}^4$. And finally, the $\mathbf{l}^3$ lines were checked for intersections with $\mathbf{s}^4$. This method will work with any number of contours but the computational effort required increases as the square of the number of contours.

VI. FLUID DYNAMICS EXAMPLES

The methods described herein are also applicable to fluid dynamics. A couple of examples from Ref. 1 with known solutions are presented. All page references are to Ref. 1. For example, consider a log of radius a lying on the ground in a fluid of speed U (p. 49). The complex potential for this case is

$$W = \phi + i\psi = a\pi U \coth(a\pi/z),$$

where $z = x + iy$. Given the potential and two starting points, our method produced Fig. 6 (cf. p. 50).

An interesting example is the Karman vortex street (p. 74). This is a configuration consisting of two infinite rows of vortices. One row rotates clockwise and the other counterclockwise. The complex potential for this case is

$$W = i\kappa \log\left(\sin\frac{\pi}{a}z - i\frac{b}{a}\right) - i\kappa \log\left(\sin\frac{\pi}{a}z - \frac{a}{2} + i\frac{b}{a}\right),$$

where a is the distance between consecutive vortices in a row and b is the spacing between the rows. In this case, the stream function, ψ, is an analytic function and thus the stream lines can be plotted with a contour-plotting routine as was done in Fig. 7. However, the potential ϕ has a branch cut for every vortex. Here, our method was used to produce ϕ from ψ by switching their roles. The result is shown in Fig. 8.

In this case, the circle is $\mathbf{s}^1$ and the other dotted line is $\mathbf{s}^2$. In truth, we could have used only $\mathbf{s}^2$ since it passes

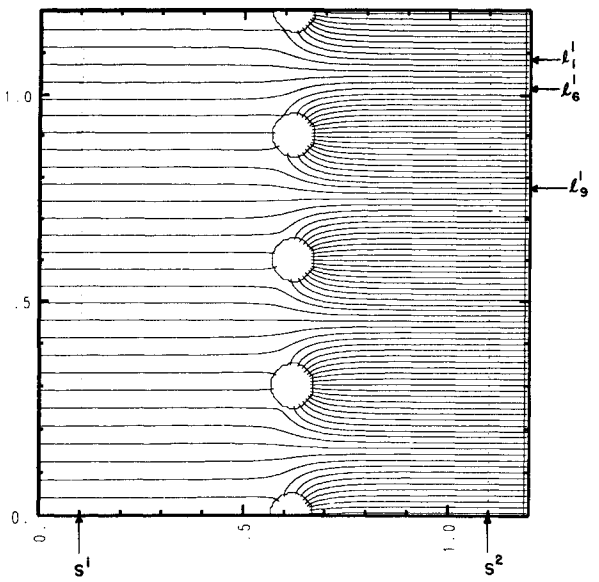

FIG. 4. A row of wires at 100 V. The left boundary is a conductor at 0 V and the right is a conductor at 400 V. The top and bottom boundaries are periodic.

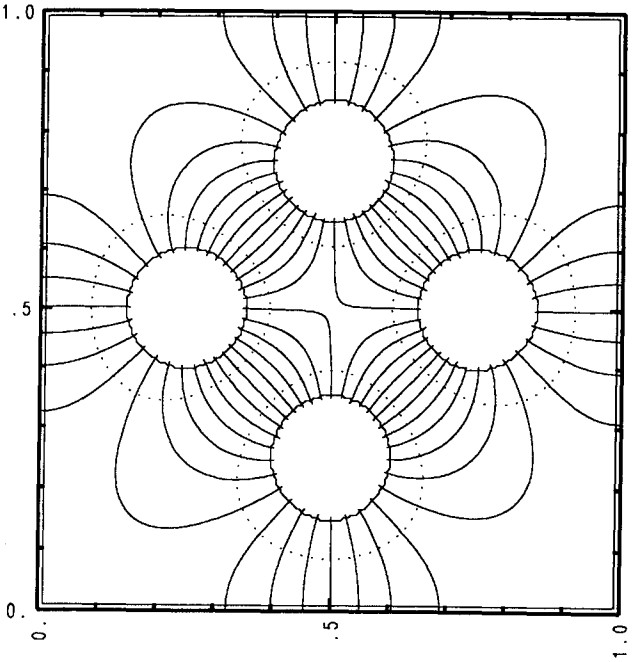

FIG. 5. Electrostatic quadrupole. The left and right wires are at 100 V and the top and bottom wires are at $-$ 100 V. All boundaries are conductors at 0 V.

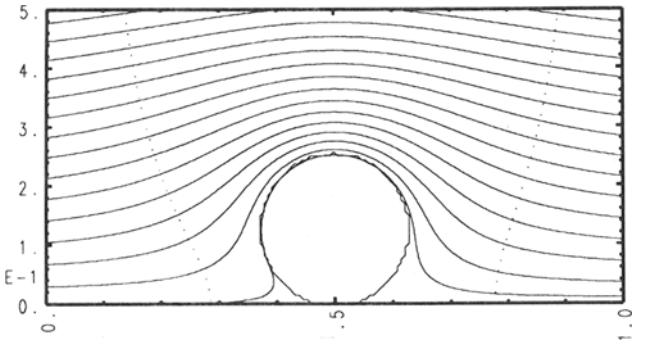

FIG. 6. Fluid motion over a log on the ground.

through all of the potential lines, but this "mistake" does not affect the performance of our algorithm which checks for superfluous field lines as explained above.

VII. SUMMARY

The new algorithm will plot two-dimensional field lines given the potential on a Cartesian grid. In comparison to NCAR's STRMLN routine, the only additional information needed is a small group of starting points from which potential contours can be computed. As in the case of STRMLN, several control variables can be changed, but usually default values suffice. In our case the control variables are

(1) Number of field lines to pass through first potential contour. This sets the amount of flux enclosed between any two field lines. 15 is a good number.

(2) Minimum field strength. This determines when field nulls are reached. I use $10^{-4}\times$ the maximum field strength.

(3) Segment length. This is Δ_l mentioned above. I use $0.3\times$ the diagonal of a grid cell.

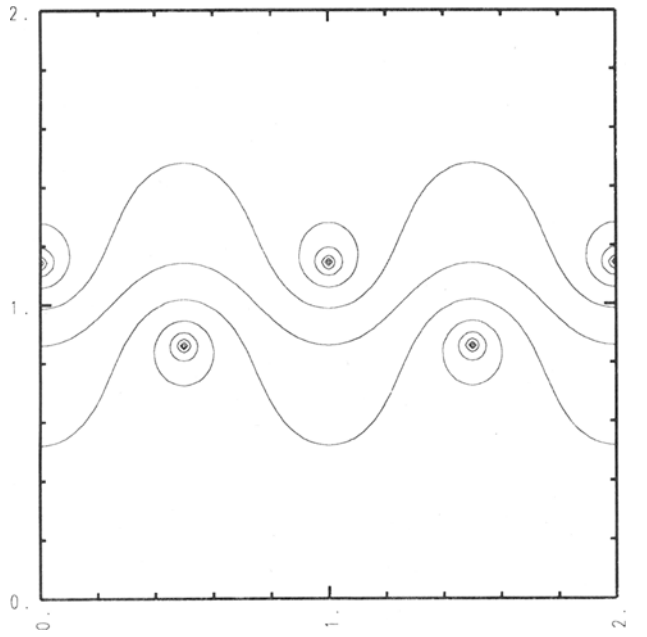

FIG. 7. Fluid motion of the Karman vortex street.

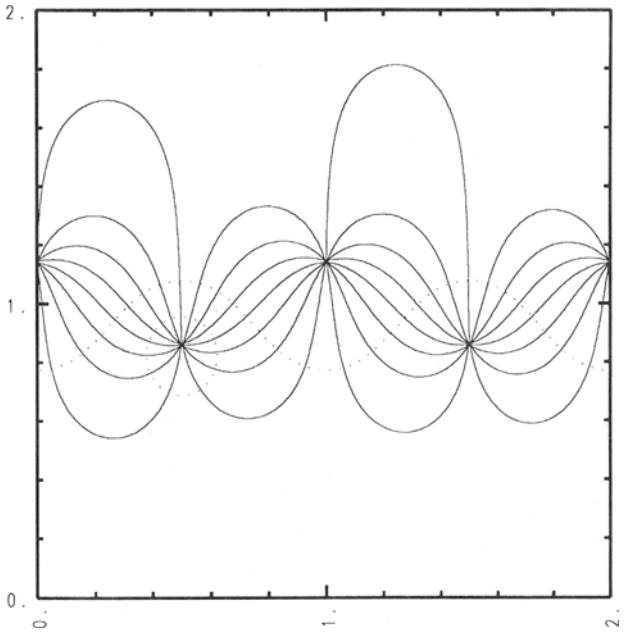

FIG. 8. Potential lines of the Karman vortex street. In this case, ψ was used to get ϕ. The circle is $\mathbf{s}^1$ and the other dotted line is $\mathbf{s}^2$.

(4) Fuzz factor. As mentioned in Sec. V, this determines when to remove field lines from one potential contour if they are too close to field lines originating from another contour. I use $0.6\Phi_0$.

In addition, there are two safety variables that avoid infinite loops in uncommon situations. One specifies the maximum number of points for any given field line. The other specifies a maximum number of field lines.

This method works well as is, but several improvements are conceivable. Some projects for future work are

(1) Automate the selection of starting points.

(2) Put the algorithm in a package for inclusion in standard graphics libraries.

(3) Allow other grid geometries.

(4) Allow higher-order interpolation for finding fields between grid points.

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