Reality of the Eigenvalues of the Bethe-Salpeter Equation

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The reality of the eigenvalues of the unequal-mass Bethe-Salpeter equation is proved in all ladder models of practical interest. Some remarks are made on the analyticity of the eigenvalues as functions of the invariant mass squared and on the Cutkosky-Deo phenomenon of the Regge trajectories.

§ 1. Introduction

As is well known, the Bethe-Salpeter (B-S) equation is a singular integral equation, but this singularity can easily be removed by means of the Wick rotation\(^1\) at least in the bound-state problem. In the equal-mass case of the scalar-scalar scalar-meson-exchange ladder model, the Wick-rotated B-S equation is immediately converted into an integral equation of the Hilbert-Schmidt type, and hence its eigenvalues are real and positive definite. In the unequal-mass case, however, the reality (together with the positive definiteness) of the eigenvalues \(\lambda(s)\) was established only in the unphysical region \(s<0\)\(^2\) and for some very special cases\(^3,4\) in the physical region \(0<s<(m_1+m_2)^2\), where \(s\) denotes the invariant bound-state mass squared, \(m_1\) and \(m_2\) being the masses of the constituent particles. Since Cutkosky and Deo\(^5\) have recently found numerically that Regge trajectory functions \(\alpha(s)\) can become complex in the unequal-mass case, it is of special interest whether \(\lambda(s)\) can also become complex or not.

The purpose of the present paper is to present a general proof of the reality of the eigenvalues \(\lambda(s)\) in the physical region in various ladder models. In § 2, we rigorously prove the reality of \(\lambda(s)\) in the unequal-mass scalar-scalar model. In § 3, the proof is extended to the spinor-spinor model and to the spinor-scalar models. In the final section, we discuss some important consequences of our result; especially, some comments are made on the behavior of Regge trajectory functions.

§ 2. Scalar-scalar model

In the scalar-scalar scalar-meson-exchange ladder model, the Wick-rotated B-S equation for \(s>0\) reads

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\[ K(p, p_4) \phi(p, p_4) = \lambda \int d^3 p' \int d\rho_i I(p, p_i; p', p'_i) \phi(p', p'_i). \]  

(2.1)

Here \( \phi(p, p_4) \) denotes the Wick-rotated B-S amplitude;

\[ K(p, p_4) = F_1(p, p_4) F_2(p, -p_4), \]  

(2.2)

\[ F_j(p, p_4) = m_j^2 + p^2 - (\eta_j \sqrt{s} + i\rho_j)^2, \quad (j = 1, 2) \]  

(2.3)

with \( \eta_j = m_j / (m_1 + m_2) \) and

\[ I(p, p_i; p', p'_i) = \pi^{-3}[\mu^2 + (p - p')^2 + (p_i - p_i')^2]^{-1}, \]  

(2.4)

\( \mu \) being the exchanged-meson mass. From (2.2), (2.3) and (2.4), we immediately have

\[ K^*(p, p_4) = K(p, -p_4), \]  

(2.5)

\[ I^*(p, p_i; p', p'_i) = I(p, p_i; p', p'_i) = I(p, -p_i; p', -p'_i), \]  

(2.6)

where the asterisk stands for the complex conjugation.

Now, from (2.1) we have

\[ \int d^3 p \int d\rho_i \phi^*(p, -p_4) K(p, p_4) \phi(p, p_4) = \lambda \int d^3 p \int d\rho_i \int d^3 p' \int d\rho_i' \phi^*(p, -p_4) I(p, p_i; p', p'_i) \phi(p', p'_i), \]  

(2.7)

where both integrals are convergent.\(^6,4\) On taking the complex conjugate of (2.7) and changing the signs of \( p_4 \) and \( p'_4 \), we find

\[ \lambda(s) = \lambda^*(s), \]  

(2.10)

unless both integrals vanish.

Since the integral in the left-hand side of (2.7) is analytically dependent on \( m_i^2(>0) \) for \( m_i^2 \) fixed and nonvanishing for \( m_i^2 = m_i^2 \) because then \( K(p, p_i) > 0 \) and \( \phi(p, -p_i) = \pm \phi(p, p_i) \), it can vanish at most at some discrete set of values of \( m_i^2 \). Hence \( \lambda(s) \) is real except for those values of \( m_i^2 \). As shown in § 4, however, \( \lambda(s) \) is continuous in \( m_i^2 \); therefore, it has to be real everywhere.

According to Arafune,\(^4\) we can also prove

\[ \text{Re } \lambda(s) > 0. \]  

(2.11)
Combining it with our result, we obtain
\[ \lambda(s) > 0 \] (2.12)
for all values of \( s < (m_1 + m_2)^2 \).

§ 3. Other models

Since our proof of the reality of the eigenvalues is based on (2.5) and (2.6) together with (2.9), it can easily be extended (without rigor) to almost all ladder models.

First, we consider the spinor-spinor model. For simplicity of notation, we assume that the exchanged meson is spinless. The Wick-rotated B-S equation for both cases of particle-particle and particle-antiparticle\( ^7 \) reads
\[ \bar{F}_1(p, \rho) \hat{\phi}(p, \rho) \bar{F}_2(p, -\rho) \]
\[ = \lambda \int d^3p' I(p, \rho; p', \rho') \Gamma \hat{\phi}(p', \rho') \Gamma, \] (3.1)
where the B-S amplitude \( \hat{\phi}(p, \rho) \) is written in a 4\( \times \)4 matrix form. In (3.1), \( I \) is given by (2.4) and\( ^* \)
\[ \bar{F}_j(p, \rho) = m_j + \gamma p - \tau \eta (\bar{\eta} \sqrt{s} + ip\rho); \quad (j = 1, 2) \] (3.2)
\( \Gamma = 1 \) or \( \tau \),\( ^{**} \) Evidently,
\[ \tau \bar{F}_j(p, \rho) \tau_0 = \bar{F}_j(p, -\rho), \] (3.3)
\[ \tau \Gamma \tau_0 = \Gamma, \] (3.4)
where the dagger stands for the hermitian conjugation.

Now, from (3.1) we have
\[ \int d^3p \int d\rho_1 \text{Tr}[\bar{\phi} \hat{\phi} \bar{F}_1(p, -\rho_1) \hat{\phi}(p, \rho_1) \bar{F}_2(p, -\rho_1)] \]
\[ = \lambda \int d^3p \int d\rho_1 \int d^3p' \int d\rho_1' I(p, \rho_1; p', \rho_1') \]
\[ \times \text{Tr}\left[ \tau_0 \bar{\phi} \hat{\phi} \bar{F}_1(p, -\rho_1) \hat{\phi}(p, \rho_1) \bar{F}_2(p, -\rho_1) \right]. \] (3.5)
As before, on taking the complex conjugate and making \( \rho_1 \rightarrow -\rho_1 \) and \( \rho_1' \rightarrow -\rho_1' \), we find
\[ \int d^3p \int d\rho_1 \text{Tr}[\bar{\phi} \hat{\phi} \bar{F}_1(p, -\rho_1) \hat{\phi}(p, -\rho_1) \bar{F}_1(p, -\rho_1)] \]
\[ = \lambda^* \int d^3p \int d\rho_1 \int d^3p' \int d\rho_1' I(p, -\rho_1; p', -\rho_1') \]
\[ \times \text{Tr}\left[ \Gamma \tau_0 \bar{\phi} \hat{\phi} \bar{F}_1(p', -\rho_1') \hat{\phi}(p, -\rho_1') \bar{F}_1(p, -\rho_1) \right]. \] (3.6)

\( ^* \) For \( \gamma \) matrices, we employ the notation adopted in Schweber's book.\( ^8 \) In particular, \( \tau_0 \) is hermitian, while \( \tau_1, \tau_2, \tau_3 \) and \( \tau_5 \) are antihermitein, and \( (\tau_0)^2 = 1. \)

\( ^{**} \) We may also use a parity-violating coupling.
by means of (3.3) and (3.4). By comparing (3.6) with (3.5) and using (2.9), we obtain the reality of the eigenvalues \( \lambda(s) \).

Next, we consider the spinor-scalar models. If the exchanged particle is spinor as in the pion-nucleon system, the Wick-rotated B-S equation reads

\[
\hat{F}_3(p, p_i) \phi(p, p_i) F_3(p, -p_i) = \lambda \int dp' \int dp'_i \Gamma \hat{I}(p, p_i; p', p'_i) \Gamma \phi(p', p'_i),
\]

where the B-S amplitude \( \phi(p, p_i) \) is a four-component spinor and

\[
\hat{I}(p, p_i; p', p'_i) = \pi^{-1} [\mu + \tau(p + p') - i\gamma_0(p_i + p'_i) - \gamma_0(\gamma_1 - \gamma_2) P_i]^{-1}.
\]

Evidently,

\[
\gamma_0 \hat{I}(p, p_i; p', p'_i) \gamma_0 = \hat{I}(p, -p_i; p', -p'_i),
\]

\[
\hat{I}(p, p_i; p', p'_i) = \hat{I}(p', p'_i; p, p_i).
\]

From (3.7) we have

\[
\int d^3p \int d^3p_i \hat{I}(p, p_i) \phi(p, p_i) \phi(p, p_i) \cdot F_3(p, -p_i)
= \lambda \int d^3p \int d^3p_i \int d^3p' \int d^3p'_i \hat{I}(p, p_i; p', p'_i) \phi(p', p'_i).
\]

Then by the same reasoning as before, we can prove the reality of \( \lambda(s) \). If the exchanged particle is spinless, the proof is easier.

Finally, we remark that the exchanged particle in the above argument can be replaced by a spin-one particle. If it couples with spinor particles, we note that the coupling \( \gamma_s \) or \( i\gamma_s \gamma_r \) satisfies the property (3.4). If it couples with scalar particles as in § 2, the coupling \( p_s + p'_s \) has the symmetry between \( p_s \) and \( p'_s \). Furthermore, since the numerator of the spin-one propagator of the exchanged particle is

\[
-g_{ss} + a(p_s - p'_s)(p_s - p'_s),
\]

where \( a \) is a constant, the property (2.9) is not violated. Thus our proof of the reality of \( \lambda(s) \) holds for all ladder models of practical interest.

\[\textit{*) Of course, we can artificially construct unphysical ladder models whose eigenvalues are not real. For example, in the scalar-scalar model, if the exchanged-meson propagator is multiplied by a factor \( P_m(p_m - p'_m) \), i.e. by \( i\sqrt{s}(p_m - p'_m) \), then the eigenvalues are all purely imaginary because the integral kernel is odd under the interchange of \( p_m \) and \( p'_m \). This counterexample shows that the reality of the eigenvalues is not a consequence of the fact\textit{[9]} that the Feynman \(-i\delta\) term in each (unrotated) propagator does not contribute to the scattering amplitude in the bound-state energy region because of the 4-momentum conservation.}\]
§ 4. Discussion

In this paper, we have proved the reality of the eigenvalues \( \lambda(s) \) in various ladder models. In this section, we discuss its important consequences by restricting ourselves to the scalar-scalar scalar-meson-exchange model for definiteness.

First, we note that the Wick-rotated form of the conjugate (i.e. time-reversed) B-S amplitude is given by \(-\tilde{\phi}^*(p, -p')\). From (2·1) together with (2·5), (2·6) and (2·9), it satisfies

\[
[ -\tilde{\phi}^*(p, -p) ] K(p, p_a) = \lambda^* \int d^3p' \int d\rho' [ -\tilde{\phi}^*(p', -\rho') ] I(p', \rho'; p, p_a),
\]

an equation which is the Wick-rotated form of the time-reversed B-S equation. Since (2·8) directly follows from (4·1), the reality of \( \lambda(s) \) is closely related to the time-reversal invariance of the ladder model.

Our next remark is related to the analyticity of \( \lambda(s) \) in \( s < (m_1 + m_2)^2 \). Since the Fredholm theory is applicable to the partial-wave B-S equation, the eigenvalues are determined by the Fredholm determinant

\[
D_z(s, \lambda) = 0,
\]

where \( D_z(s, \lambda) \) is holomorphic in a topological product of the \( s \) plane with a cut \( s \geq (m_1 + m_2)^2 \) and the \( \lambda \) plane without a point at infinity. According to a theorem for the implicit function, if \( f(z, w) \) is holomorphic near a point \( (z, w) = (z_0, w_0) \), the functions \( w = w_0(z) \) which satisfy \( f(z, w) = 0 \) have at worst an algebraic singularity at that point. Therefore, if we assume that the eigenvalues \( \lambda_{kl}(s) \) are bounded in a bounded domain of the \( s \) plane [e.g. a neighborhood of the interval \(-a < s < (m_1 + m_2)^2\), where \( a \) is an arbitrarily large positive number], they have only algebraic singularities in that domain. In particular, they are continuous functions of \( s \).

Now, we state an important consequence of the reality of \( \lambda_{kl}(s) \): The eigenvalues \( \lambda_{kl}(s) \) are holomorphic in a neighborhood of the real axis for \( s < (m_1 + m_2)^2 \). This is shown as follows. An eigenvalue \( \lambda_{kl}(s) \) can have only branch points there as shown above. If it has a branch point at \( s = s_0 < (m_1 + m_2)^2 \), then we take its branch cut along the real axis. Since \( \lambda_{kl}(s) \) is real, Schwarz's principle of reflection implies that the jump at the branch cut is identically zero.

\[\text{(4·8)}\]

In the same way, we can show that the eigenvalues are continuous in \( m_2^2 \) (cf. §2).
That means that \( s = s_0 \) is not a branch point.

Finally, we consider the Reggeization of our result. By continuing analytically (4·2) in \( l \), we can prove that \( \lambda_{st}(s) \) is still real for \( l \) real and \( l > -3/2 \). Though the eigenvalues \( \lambda_{st}(s) \) are, therefore, holomorphic in \( s \) near \( s < (m_1 + m_2)^2 \) even for non-integral values of \( l \), they are not necessarily holomorphic in \( l \). That is to say, the Regge trajectories\(^{*)} \) \( l = \alpha_k(s) \) can have branch points on the real axis of the \( s \) plane, as was observed by Cutkosky and Deo.\(^5) \) Since we see from (4·2) that the Fredholm determinant is real analytic in \( s \) and \( l \) for \( \text{Re} \ l > -3/2 \), complex trajectories can appear only in pairs of complex conjugate ones. We should also note that since the norm of the B-S amplitude\(^7) \) is determined before it is Reggeized, the Cutkosky-Deo phenomenon cannot change the norm of a Regge trajectory \( \alpha_k(s) \). Thus, as was pointed out previously,\(^11) \) it can change only at a multiple pole of the scattering Green's function.

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References

3) S. Naitô, “Breakdown of \( O(4) \) symmetry and the Bethe-Salpeter equation” (1967), unpublished.

Note added in proof:

If \( \lambda \) has a branch point in \( m_2^2 \), \( \phi^*(p, -p_0) \phi(p, p_0) \) changes non-analytically at such a point, and consequently we cannot exclude the possibility that the integrals in (2·7) may vanish identically in an interval of \( m_2^2 \). The authors thank Prof. M. Ida for pointing out the incompleteness of our proof.

\(^{*)} \) The Regge trajectory is expected to be unique for each \( k \) if we confine ourselves to \( \text{Re} \ l > -3/2 \).