On the Stability of a Spherical Gravitating Compressible Liquid Planet without Spin

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Summary

During the past ten years or so there has been considerable discussion in the literature regarding the author's 1963 contention that (neglecting temperature effects and spin) the Earth's liquid core cannot be stable unless the Adams-Williamson condition relating density distribution and compressibility holds there.

The present paper throws light on this question by showing mathematically that a sphere of gravitating compressible liquid cannot be internally stable unless this condition is fulfilled. Physical reasons for the necessity of this condition, which implies that particles of the liquid are in neutral equilibrium, are also discussed. By internal stability is meant stability of the density distribution while the spherical shape is maintained.

The question of shape stability is not treated here, since it may be assumed that the Earth's mantle is sufficiently rigid to keep the core essentially spherical.

The liquid is assumed to be a perfect fluid, elastic, and in the discussion only small strains are considered from an equilibrium configuration of initial hydrostatic stress. Furthermore thermodynamic effects are neglected and there is no spin.

1. Introduction

The question of gravitational stability was originally considered by Jeans (1903), but in more detail by Lord Rayleigh (1906) and Love (1907, 1911). The earlier work of Love contained a defect in the physical theory, which was corrected in his 1911 book, where he explains the necessity for an amendment concerning the increment of stress assumption. However, in order to render the solution of his equations tractable, Love assumed in his book either incompressibility or constant density. The first treatment of the strain of a gravitating sphere of variable density and elasticity, using the amended theory, seems to have been given by Hoskins (1920), and his work was the basis for the method of computation of the periods of free oscillation of the Earth due to Pekeris & Jarosch (1958), Alterman, Jarosch & Pekeris (1959), and Jarosch (1962).

In 1963 the present writer (Longman 1963) noted that while the dynamical equations for the periods of free oscillation of the Earth can be solved, using a model having a liquid core, the corresponding statical equations for deformation of the Earth under surface loads cannot be solved without assuming the Adams-Williamson (1923) condition to hold in the liquid core. It was concluded, therefore, that the liquid core is unstable unless this condition obtains, and in the author's computation of Love numbers and load deformation coefficients (1963, 1966) the Gutenberg
model of the Earth was modified to make the Adams–Williamson condition hold in the core. For a number of years it was not understood by the present writer how it was possible for Takeuchi (1950) and others to calculate Love numbers without this assumption, nor how Alterman et al. (1959) could succeed in calculating periods of free oscillations for an Earth model which was essentially unstable.

The present paper gives some insight into this situation, by showing that for a gravitating compressible liquid sphere with a free surface, the instability is not activated by an externally applied gravitational potential, but that it is activated if the surface is constrained. It is assumed throughout that we are dealing with a perfect fluid which is elastic (compressible), and we consider small strains from a configuration of initial hydrostatic stress due to gravity. All thermodynamic and spin effects are neglected. Two problems are treated in detail:

1. The perturbation due to a small externally applied gravitational potential when the liquid sphere has a free surface (surface tension effects are neglected).

2. The same, but with the liquid sphere contained within a rigid spherical shell of the same radius.

It is shown that while we can solve (1) without needing the Adams–Williamson condition, the same is not true of (2) where the condition is essential.

Finally a physical explanation is given for the necessity of the condition in terms of stability requirements for elements of the liquid, and this is followed by a brief survey of the relevant literature.

2. Theory

In view of the singular nature of the situation which arises when the rigidity $\mu$ of a gravitating elastic sphere is zero, it was thought worth while to adapt the dynamical theory as formulated by Hoskins (1920) to the liquid case ($\mu = 0$) for the statical problem. This is done to ensure and to demonstrate that nothing goes wrong in the derivation of the equations in this particular case. The derivation is given in the Appendix, where, as far as possible, the original notation of Hoskins is used, in order to facilitate comparison with his 1920 paper. As a result it appears that it is perfectly legitimate in describing our present problem, simply to place $\mu = 0$ and omit the dynamic terms in the now classic equations (7), (8), (9) of Alterman et al. (1959). In the main body of the paper, however, the notation of Alterman et al. is used, in order to facilitate comparison with their work and with the author's 1962, 1963 papers on static loading of the Earth.

We define $\rho_0(r)$ to be the initial density distribution (before perturbation), and $g_0(r)$ to be the downward acceleration due to gravity:

$$g_0(r) = \frac{4\pi G}{r^2} \int_0^r \rho_0(s) s^2 ds,$$

where $G$ is the gravitational constant, and $r$ denotes distance from the centre 0. Using spherical polar co-ordinates, $(u, v, w)$ denote $(r, \theta, \phi)$ components of displacement. The dilatation $\Delta$ is

$$\Delta = \frac{\partial u}{\partial r} + \frac{2u}{r} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (v \sin \theta) + \frac{1}{r \sin \phi} \frac{\partial w}{\partial \phi}. \quad (2)$$

We assume now that the increment in gravitational potential during strain is

$$\Psi + \psi \quad (3)$$
Spherical gravitating compressible liquid planet

where \( \Psi \) is the (small) potential of an externally applied gravity field and satisfies

\[
\nabla^2 \Psi = 0 \tag{4}
\]

throughout the sphere, and \( \psi \) is the perturbation of the gravitational potential inside the sphere caused by the perturbation induced in the density distribution and by the corrugations of the surface. We have

\[
\nabla^2 \psi = 4\pi G(\rho_o \Delta + \rho_o u), \tag{5}
\]

where the dot signifies differentiation with respect to \( r \).

The equations of equilibrium now take the following form:

\[
\rho_o \frac{\partial \Psi}{\partial r} + \rho_o g_o \Delta + \rho_o \frac{\partial \psi}{\partial r} - \rho_o \frac{\partial}{\partial r} (g_o u) + \lambda \frac{\partial \Delta}{\partial r} + \Delta \frac{d \lambda}{dr} = 0 \tag{6}
\]

\[
\frac{\rho_o}{r} \frac{\partial \Psi}{\partial \theta} + \frac{\rho_o}{r} \frac{\partial \psi}{\partial \theta} - \frac{g_o \rho_o}{r} \frac{\partial u}{\partial \theta} + \frac{\lambda}{r} \frac{\partial \Delta}{\partial \theta} = 0 \tag{7}
\]

\[
\frac{\rho_o}{r \sin \theta} \frac{\partial \Psi}{\partial \theta} + \frac{\rho_o}{r \sin \theta} \frac{\partial \psi}{\partial \theta} - \frac{g_o \rho_o}{r \sin \theta} \frac{\partial u}{\partial \phi} + \frac{\lambda}{r \sin \theta} \frac{\partial \Delta}{\partial \phi} = 0, \tag{8}
\]

these being essentially the equations (7), (8), (9) of Alterman et al. for the static case with zero rigidity. \( \lambda = \lambda(r) \) and \( \mu = 0 \) denote the usual Lamé elastic parameters.

In order to find the new state of equilibrium under the influence of the applied potential \( \Psi \), we have to solve the equations (5)–(8) for \( u, v, w \) and \( \psi \) under appropriate boundary conditions. These boundary conditions will be discussed below. \( v \) and \( w \) of course enter the equations via \( \Delta \) as given in (2), and we assume that \( \lambda = \lambda(r) \), \( \rho_o = \rho_o(r) \) are given functions of \( r \), while \( g_o(r) \) of course is given by (1).

We now assume that our external perturbing potential has the form

\[
\Psi = Ar^n S_n(\theta, \phi), \quad n > 0 \tag{9}
\]

where \( S_n(\theta, \phi) \) is a spherical surface harmonic of order \( n \), and \( A \) is a small constant. Following Alterman et al. we write

\[
\begin{aligned}
\begin{bmatrix}
u \\ w \\ \psi
\end{bmatrix} &= \begin{bmatrix}
\frac{\partial S_n}{\partial \theta} \\ \frac{\partial S_n}{\partial \phi} \\
0
\end{bmatrix} \\
\frac{\partial \Psi}{\partial r} &= V(r) \\
\frac{\partial \psi}{\partial r} &= \frac{\partial \psi}{\partial \theta} \\
\end{aligned} \tag{10}
\]

and then we find

\[
\Delta = X(r) S_n(\theta, \phi), \tag{11}
\]

where

\[
X(r) = \bar{U} + \frac{2}{r} U - \frac{n(n+1)}{r} V. \tag{12}
\]
Equation (6) is satisfied provided
\[ nAr^{n-1} \rho_0 + \rho_0 g_0 X + \rho_0 \dot{\rho} - \rho \frac{d}{dr} (g_0 U) + \frac{d}{dr} (\lambda X) = 0, \tag{13} \]
while equations (7), (8) are satisfied provided
\[ Ar^n \rho_0 + \rho_0 P - g_0 \rho_0 U + \lambda X = 0. \tag{14} \]
Also the Poisson equation (5) for \( \psi \) is satisfied if
\[ \ddot{\psi} + \frac{2}{r} \dot{\psi} - \frac{n(n+1)}{r^2} P = 4\pi G (\rho_0 U + \rho_0 X). \tag{15} \]

Equations (13), (14), (15) have to be solved for \( U, V, P \), subject to \( U, V, P \) and \( X \) being regular at \( r = 0 \). The other boundary conditions depend on the conditions obtaining at the surface of the sphere.

Before considering these, however, an important point should be noted. Equation (13) can be written in the form
\[ nAr^{n-1} \rho_0 + \frac{d}{dr} [\rho_0 (P - g_0 U) + \lambda X] + \rho_0 g_0 X - (P - g_0 U) \frac{\dot{\rho}}{\rho_0} = 0. \tag{13'} \]

But by (14) the expression in square brackets
\[ \rho_0 (P - g_0 U) + \lambda X = -\rho_0 Ar^n. \]
Thus
\[ \rho_0 g_0 X - (P - g_0 U) \frac{\dot{\rho}}{\rho_0} = A \frac{\dot{\rho}_0}{r^n} \]
so that
\[ P - g_0 U = \frac{\rho_0 g_0}{\dot{\rho}_0} X - Ar^n. \]

However from (14) we also have
\[ P - g_0 U = -\frac{\lambda X}{\rho_0} - Ar^n \]
and so by subtraction
\[ \left( \frac{\rho_0 g_0}{\dot{\rho}_0} + \frac{\lambda}{\rho_0} \right) X = 0. \tag{16} \]

This relation, which was derived in the author's 1963 paper, shows that we must either have \( X = 0 \) (zero dilation) or
\[ \frac{\rho_0 g_0}{\lambda} + \frac{\dot{\rho}_0}{\rho_0} = 0, \tag{17} \]
which is the Adams–Williamson condition as applied to a liquid. Adopting (16), equations (13) and (14) become equivalent, and therefore we have now to solve (14) and (15) with the condition (16). From (16) we deduce as stated above that either \( X = 0 \) or the Adams–Williamson condition must hold. In order to clarify this question we consider now our deformation problem under two specific surface conditions.
3. Deformation of the gravitating liquid sphere

We consider then our sphere of radius \( a \), to be subjected to the (small) externally applied gravitational potential \( \Psi \) of the form (9). As a first case we consider the surface of the sphere to be free. Then in addition to the requirement of regularity at the origin, it is necessary that the stress on the deformed surface vanish, and also at the deformed surface we have continuity of the gravitational potential and of its normal gradient. It is easily shown (Pekeris & Jarosch 1958) that we then require

\[
\lambda A = 0
\]

\[
\frac{\partial \psi}{\partial r} + \frac{n+1}{a} \psi = 4\pi G \rho_0 u \quad \text{at } r = a.
\] (18)

In terms of \( U, V, P, X \) we thus have to solve the ordinary differential equations (14), (15) with the condition (16), regularity at the origin, and

\[
X(a) = 0
\] (19)

\[
\dot{P}(a) + \frac{n+1}{a} P(a) = 4\pi G \rho_0(a) U(a)
\] (20)

since we assume \( \lambda(a) \neq 0 \).

From (16) let us now suppose that we choose the condition \( X = 0 \), i.e. zero dilatation. Then equations (14), (15) become

\[
A r^n + P - g_0 U = 0
\] (14')

and

\[
\ddot{P} + \frac{2}{r} \dot{P} - \frac{n(n+1)}{r^2} P = 4\pi G \rho_0 U.
\] (15')

We can thus eliminate \( U \) and obtain for \( P(r) \) the differential equation

\[
\ddot{P} + \frac{2}{r} \dot{P} - \left[ \frac{n(n+1)}{r^2} + \frac{4\pi G \rho_0}{g_0} \right] P = \frac{4\pi G \rho_0}{g_0} A r^n.
\] (21)

Now it is important to consider the nature of this differential equation near the origin \( r = 0 \). Let us suppose that sufficiently near the origin we have the expansion

\[
\rho_0(r) = a_0 + a_1 r + a_2 r^2 + \ldots
\]

Then

\[
\dot{\rho}_0(r) = a_1 + 2a_2 r + 3a_3 r^2 + \ldots
\]

while from (1) we have for small \( r \)

\[
g_0(r) = 4\pi G \left( \frac{a_0}{3} r + \frac{a_1}{4} r^2 + \frac{a_2}{5} r^3 + \ldots \right).
\]

Evidently then the term

\[
\frac{4\pi G \rho_0}{g_0} = \frac{1}{r} \left( \frac{a_0}{3} \right) + \frac{a_1}{4} r + \frac{a_2}{5} r^2 + \ldots
\]

\[
= \frac{1}{r} \left( \frac{a_0 + 2a_1 r + 3a_2 r^2 + \ldots}{3} \right) + \frac{a_1}{4} r + \frac{a_2}{5} r^2 + \ldots
\]
has at most a simple pole at the origin which is thus a regular point (Whittaker & Watson 1940, p. 97) for the homogeneous differential equation obtained from (21) by replacing its right-hand side by zero. Also its indicial equation is
\[ \alpha^2 + \alpha - n(n+1) = 0 \]

having one positive root \( \alpha = n \) and one negative root \( \alpha = -n - 1 \). The general solution of (21) is thus of the form

\[ P(r) = cf(r) + c_1 f_1(r) + Ah(r) \]

where \( f(r) \) is regular and \( f_1(r) \) is singular at the origin, while \( h(r) \) (regular at \( r = 0 \)) is a particular solution of (21) with \( A \) replaced by unity. Of the two arbitrary constants \( c, c_1 \), we must therefore take \( c_1 = 0 \), and our general solution (regular at \( r = 0 \))

\[ P(r) = cf(r) + Ah(r) \] (22)

contains thus only one arbitrary constant \( c \). We now have to satisfy the two boundary conditions (19) and (20), but evidently (19) is satisfied automatically by our assumption \( X = 0 \). Using (14'), (20) can now be expressed in the form

\[ \dot{P}(a) + \left[ \frac{n+1}{a} - \frac{4\pi G \rho_0(a)}{g_0(a)} \right] P(a) = \frac{4\pi G \rho_0(a)}{g_0(a)} A a^n. \] (20')

Thus our constant \( c \) is determined by substitution from (22) into (20'), and our problem is solved, since there is no reason in general why this should not be determinate. What we have achieved, then, is to solve the deformation problem for the case of the free surface. Having determined \( P(r) \) we immediately obtain \( U(r) \) from (14'), and then since \( X(r) = 0 \) we can obtain \( V(r) \) from

\[ \frac{n(n+1)}{r} V = \dot{U} + \frac{2}{r} U, \]

and thus all the deformation parameters of the sphere have been uniquely determined. But it should be noted that although we have here only one free parameter \( c \) at our disposal, we were able to satisfy the two boundary conditions (19), (20) because (19) was satisfied automatically. This will not in general happen with other surface conditions, and in particular not when we go on to consider the case (2) where we confine our liquid sphere within a rigid spherical shell.

Before considering case (2), however, let us consider what happens in case (1) if we take the Adams–Williamson condition to hold instead of \( X = 0 \). Then from (17) and (14) we have evidently

\[ \dot{\rho}_0 U + \rho_0 X = - \frac{\rho_0^2}{\lambda} P - \frac{\rho_0^2}{\lambda} Ar^n \]

so that our equations of equilibrium (14), (15) now take the form

\[ \rho_0 P - g_0 \rho_0 U + \lambda X = - A \rho_0 r^n \] (14)

\[ \dot{\dot{P}} + \frac{2}{r} \dot{P} + \left[ \frac{4\pi G \rho_0^2}{\lambda} - \frac{n(n+1)}{r^2} \right] P = - \frac{4\pi G \rho_0^2}{\lambda} A r^n. \] (23)
Once again the differential equation for \( P(r) \) has a regular point at the origin, and the solution regular at \( r = 0 \) has only one arbitrary constant:

\[
P(r) = c_2 f_2(r) + Ah_1(r),
\]

where \( h_1(r) \) (regular at \( r = 0 \)) is a particular solution of (23) when \( A \) is replaced by unity, and \( f_2(r) \) satisfies the homogeneous equation. We can determine the constant \( c_2 \) as before by the boundary conditions (19), (20), which used together with (14) yield

\[
\dot{P} + \frac{n+1}{a} P = \frac{4\pi G \rho_0(a)}{g_0(a)} P, \quad P(a) = 4\pi G \rho_0(a) \frac{a}{g_0(a)} A a^n.
\]

However, this time it should be noted that \( U, V \) are not uniquely determined, but are merely limited by the condition (14), \( X \) being given by (12), with \( X(a) = 0 \). This lack of uniqueness of \( U, V \) is related to the fact that the Adams–Williamson condition implies that elements of the fluid are in neutral equilibrium. This point will be discussed further later.

We now turn to case (2), where our sphere is contained within a rigid spherical shell of the same radius \( a \). We can consider the shell to be massless, or of uniform surface density, since in either case there will be no internal gravitational field due to the shell. Now we apply our external gravitational potential \( \Psi \) as given by (9). The difference here from the previous problem lies in the boundary conditions, which apart from the condition of regularity at \( r = 0 \), now take the form

\[
U(a) = 0, \quad \dot{P}(a) = 0. \tag{24}
\]

Let us try again to solve the problem first of all by assuming \( X = 0 \). We have to solve (21) again, but with the new boundary conditions (24), (25). In view of (14') we can replace (24) by

\[
P(a) = -A a^n, \tag{26}
\]

so that we can say, then, that our solution (22), which has only the one parameter \( c \) at our disposal, has to satisfy the two conditions

\[
P(a) = -A a^n, \quad \dot{P}(a) = -(n+1) A a^{n-1}, \tag{27}
\]

and this must in general be impossible. For example if \( \rho_0 \) is constant the general solution of (21), which now takes the form

\[
\dot{P} + \frac{2}{r} \dot{P} - \frac{n(n+1)}{r^2} P = 0, \tag{21'}
\]

is

\[
P(r) = c r^n, \tag{22'}
\]

assuming regularity at the origin. The conditions (27) then give

\[
c = -A, \quad nc = -(n+1)A
\]

which is impossible for \( A \neq 0 \).
We may note that the condition (24) must obtain, since our shell is rigid and impenetrable, and on the other hand cavitation cannot occur since we have assumed the dilatation everywhere zero. We see then that in this case (2) our assumption \( X = 0 \) leads to a contradiction. This result is hardly surprising since under these conditions of zero dilatation and rigid boundary our liquid is essentially 'stuck' apart from rotations which cannot accommodate the applied potential. The compressibility has no chance to manifest itself.

We now consider the alternative which is the Adams–Williamson condition (17). Then we have to solve the differential equation (23), subject to the boundary conditions (24), (25). However (24) does not now impose any requirement on \( P(r) \) since now (14), as distinct from (14'), merely imposes a condition on \( X \):

\[
\lambda(a) X(a) = -\rho_0(a) P(a) - A\rho_0(a) a^n
\]

and this is related to the surface stress applied by our rigid shell on the liquid.

Thus in this case we can solve (23) subject to the single boundary condition (25), and once again \( U, V \) are not uniquely determined but are merely related by (14) and (12).

It thus appears that when the liquid is contained within a rigid shell, we can only obtain small internal deformations if we apply a small external potential \( \Psi \) of the form (9) provided the Adams–Williamson condition holds. In other words our liquid sphere is only then stable internally.

It is not difficult to see that we reach the same conclusion if our gravitating compressible liquid is contained between a rigid spherical shell and an internal massless rigid sphere fixed to be concentric with the shell. In this case equation (21) has indeed two arbitrary constants in its solution, but now we have, in addition to the two conditions (27), the extra condition \( U(b) = 0 \), where \( b < a \) is the radius of the internal rigid sphere. By (14') this condition is equivalent to

\[
P(b) = -Ab^n,
\]

so that now \( P(r) \) has to satisfy three boundary conditions. Thus once again the assumption \( X = 0 \) leads to the impossibility of solving the perturbation problem, and the Adams–Williamson condition has to hold.

4. Physical explanation

In the above a mathematical reason has been given for the necessity that the Adams–Williamson condition must hold in a self-gravitating compressible liquid sphere, and it has been shown otherwise to be unstable, in that it cannot respond by small internal perturbations to a small externally applied gravitational potential, when it is confined within a rigid spherical shell. It is proposed in this section to give a physical explanation of this fact. First of all we may note that a gravitating sphere of compressible liquid cannot possibly be stable if it has uniform density

\[
\rho_0(r) = \text{const}.
\]

(and this violates the Adams–Williamson condition). This conclusion contradicts that of Pekeris & Accad (1972), but the reason for the invalidity of their conclusion is now evident from the present paper, and is explained in detail in the next section. It is quite evident that if in such a sphere we move a small element of fluid slightly outwards, say, in the radial direction, the hydrostatic pressure on it will decrease, and thus it will expand and acquire a lower density. Since we are assuming \( \rho_0(r) \) to be constant in the sphere, our element will now be less dense than its new environment, and therefore will continue to 'float upwards', and decrease in density. If, on the other hand, we push our element slightly 'downwards' towards the centre, the
pressure on it will increase, and it will become denser than its new environment, and will continue to ‘sink’, and increase in density. Thus $\rho_0(r)$ constant is certainly not a stable configuration, and we would expect such a sphere to start to condense towards the centre, until a certain density distribution is achieved. The same general conclusion was reached by Love (1911, p. 124) where he remarks that ‘A sphere of homogeneous fluid subject to its own gravitation, but free from surface tension, would be gravitationally unstable, and some degree of rigidity would be necessary to stability if the sphere is to be homogeneous’.

Now it is not difficult to see that the Adams-Williamson condition (17) which was originally conceived for the solid earth as a condition of chemical homogeneity, is for a liquid sphere the condition for neutral equilibrium. If we displace radially an element of fluid, (17) is just the condition that the displaced element will take up the density of its new environment. It might therefore be thought that a sphere in which $\frac{g_0 \rho_0}{\lambda} + \frac{\dot{\rho}_0}{\rho_0} < 0$ (28) would be stable, since then an element displaced radially would then have a tendency to return to its original position. (28) does give us stability in this sense, but this is only one type of stability. Evidently (28) implies that $\rho_0$ increases towards the centre sufficiently rapidly. However, too massive a centre would be subject to another type of instability. In order to make matters definite—and simple—let us consider the following extreme case. Suppose all the mass of the sphere be concentrated at the centre, so that we have a liquid sphere of zero density containing a particle of mass $m$ at the centre $r = 0$. If we displace this particle from the centre it will obviously show no tendency to return. But if we suppose our liquid to be placed in a field having gravitational potential $\Psi$ as given by (9), then however small $A$ may be, if we displace our central particle in a direction in which $S_\nu(\theta, \phi)$ is positive, then it will tend to move further outward. Thus this extreme state of concentration of mass at the centre is unstable, since we may take $A$ as small as we please. It is the author’s belief that this argument may be generalized to any density distribution $\rho_0(r)$ which increases towards the centre faster than that dictated by the Adams-Williamson condition. If we displace a central part of the sphere, say $0 < r < b < a$, in a direction in which $S_\nu(\theta, \phi)$ is positive, for sufficiently large $b$ so that the sphere $r = b$ contains a sufficient proportion of the mass of the sphere, then it will continue to move outwards since the rest of the sphere will have insufficient mass to cause it to return. By a continuation of this process motions will continue until the Adams-Williamson condition holds throughout the sphere.

5. A brief survey of the literature

We give here various excerpts from the literature, which support and contend the author’s 1963 contention. The first criticism seems to have appeared in a paper by Jeffreys & Vicente (1966) concerning the deformation of the Earth. They acknowledge the existence of the difficulty, but conclude that there must be a definite relation between external potential and normal stress at the core–mantle boundary, since they are unwilling to accept the Adams-Williamson condition in the liquid core unless it is ‘Chemically homogeneous and has an adiabatic distribution of temperature’.

Smylie & Mansinha (1971) also decline to accept the need for the Adams-Williamson condition in the liquid core of the Earth, and attempt to create the extra adjustable constant that is then necessary, by assuming discontinuity in radial displacement at the core–mantle boundary. They claim that ‘The mantle may be
allowed to project into the liquid core, by use of an argument connected with equipotential surfaces. Dahlen (1971a), like the present writer, finds this argument to be invalid, and also in a later paper (Dahlen 1971b) agrees with the present writer’s 1963 contention. However recently Dahlen (1974) has revised his previous conviction, and claims to solve the problem of the static deformation of an Earth model with a fluid core, without requiring the Adams-Williamson condition to hold in the core. The basic philosophy of his paper is that the Lagrangian formulation of the equations in the liquid core is not appropriate, since the displacements of individual fluid particles may be arbitrarily large, thus invalidating the small strain assumption of elasticity. His method is to use an Eulerian description of the core motion combined with a Lagrangian description of the solid mantle deformation. The present writer feels that the Lagrangian formulation may be used in the liquid core, since small particle displacements should not be unable to accommodate a small perturbation in the gravitational potential.

Another attempt to get over the difficulty, i.e. to solve the equations of static deformation of the Earth having a liquid core, without the Adams-Williamson condition, was made by Pekeris & Accad (1972). Unable to accept the Smylie and Mansinha claim of discontinuity in the radial component of displacement at the core-mantle boundary, Pekeris and Accad introduce a discontinuity in the normal stress component and have a ‘boundary layer’ near the surface of the liquid core. The present writer is not aware that boundary layers exist in hydrostatic problems, and does not agree that a statical problem has to be solved as the limit of a dynamical oscillatory problem as the frequency tends to zero.

Now Pekeris & Accad deny the present writer’s corollary, that a uniform liquid core (of the Earth) is physically impossible, and ‘demonstrate’ this by computing the tidal yielding of an Earth model consisting of a liquid sphere of uniform density. It is evident from Section 3 of the present paper, why they can succeed in this. In fact, as we have seen earlier, if $\rho_0$ is constant, equation (21) takes the form (21'), and the general solution which is regular at the origin, is as given in (22'), and contains the one arbitrary constant $c$. Since the Adams-Williamson condition does not hold if $\rho_0$ is constant, we must take $X = 0$ (zero dilatation). Our boundary conditions are (19) and (20'). However in this case of uniform density we have

$$g_0(r) = \frac{4}{3} \pi G \rho_0 r$$

so that (20') reduces to

$$\dot{P}(a) + \frac{n-2}{a} P(a) = 3A a^{n-1}. \quad (20'')$$

Substituting from (22') into (20') we can determine $c$:

$$c = \frac{3A}{2(n-1)},$$

and so for $n = 2$ we find the Love number

$$k = c/A = \frac{3}{2},$$

agreeing with the result of Pekeris & Accad when their frequency $\sigma$ tends to zero. However, as we have seen, the application of an external gravitational potential, under free surface conditions, does not cause the instability to be activated. As an analogy, we may consider a pin balanced on its point. It is in unstable equilibrium, but this instability does not show itself if we merely raise the point of the pin vertically, or spin the pin about its vertical axis. Pekeris and Accad have indeed a compressible sphere, but by choosing the $X = 0$ condition, they do not allow it to be compressed.
It is hardly surprising then, that they obtain Lord Kelvin's result (Lamb 1945, p. 451) for the frequency of oscillation of an incompressible fluid sphere.

The present writer has not, at the time of writing, investigated the question whether the instability of a non Adams–Williamson liquid core would show itself in the deformation of an Earth model, subject to an externally applied gravitational potential (calculation of Love numbers), or in the free oscillation of the Earth (both under free surface conditions), but presumably it does not, and if not, this is why such calculations were able to be carried out by various authors.

It is interesting to note that Pekeris & Accad define a parameter \( \beta(r) \) in the Earth's liquid core by

\[
g_0 + \lambda \rho_0 / \rho_0^2 = \beta(r) g_0
\]

and state explicitly that a model in which \( \beta(r) > 0 \) is statically unstable. How then can they contemplate the existence of a uniform compressible self-gravitating liquid sphere in which, since \( \rho_0 \) is constant, \( \beta = 1 > 0 \)? Further, with regard to their comments that the Adams–Williamson condition does not hold in the atmosphere, it occurs to the author that the reason may be connected with temperature effects and spin of the Earth, but this is merely a conjecture.

It seems also appropriate to mention here Farrell's (1972) review article 'The deformation of the Earth by surface loads'. Here it is taken for granted that for the statical problem without spin the Adams–Williamson condition must hold in the fluid core. And finally mention may be made of the paper of Israel, Ben-Menahem & Singh (1973) who adopt the boundary layer theory of Pekeris & Accad to describe the situation at the core–mantle boundary.

6. Conclusion

A mathematical demonstration has been given, and a physical explanation suggested, as to why the Adams–Williamson condition must hold in a stable self-gravitating compressible liquid sphere. A brief survey of the pros and cons given in the literature has also been presented.

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References

Appendix

In this Appendix we adapt the dynamical theory as formulated by Hoskins (1920) to the liquid case ($\mu = 0$) for the statical problem. With Hoskins we use the notation as given in the classical treatise by Love (1944) on the mathematical theory of elasticity. The equations of equilibrium in Cartesian co-ordinates ($x, y, z$) for a solid (independent of elasticity theory) are

$$\rho X + \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = 0$$

$$\rho Y + \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} = 0$$

$$\rho Z + \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} = 0$$

(A1)
where \((X, Y, Z)\) are components of body force per unit mass, \(\rho = \rho(x, y, z)\) denotes density, and the \(X_x\) etc. are the stress tensor components.

The stress-strain relations for an isotropic liquid with Lamé constants \(\mu = 0\) and \(\lambda\) reduce to

\[
\begin{align*}
X_x &= Y_y = Z_z = \lambda \Delta \\
X_y &= \text{etc.} = 0
\end{align*}
\]

(A2)

where

\[
\Delta = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}
\]

(A3)

is the dilatation, \((u_x, u_y, u_z)\) being the displacement components of a particle of liquid. Thus for a liquid, equations (A1) reduce to

\[
\begin{align*}
\rho x + \frac{\partial T}{\partial x} &= 0 \\
\rho y + \frac{\partial T}{\partial y} &= 0 \\
\rho z + \frac{\partial T}{\partial z} &= 0
\end{align*}
\]

(A1')

where \(T = T(x, y, z)\) denotes the common value of \(X_x, Y_y, Z_z\). Also of course we have

\[
T = \lambda \Delta.
\]

(A2')

Now equation (A2') applies when the liquid is strained from a condition of zero stress. Following Love (1911) and Hoskins (1920), we now suppose that the liquid is initially in equilibrium under body forces (gravity), and that \((u_x, u_y, u_z)\) denote the displacement components from this initial configuration. We assume with them that an equation like (A2') now holds when each stress component is replaced by the increment of the stress component, due to the strain. Let \(\delta T = \delta X_x = \delta Y_y = \delta Z_z\) denote the increment of the stress components for an element of liquid initially at \((x, y, z)\). Then our assumption is expressed by the equation

\[
\delta T = \lambda \Delta
\]

(A4)

with \(\delta X_y\) etc. all zero.

By differentiation we obtain

\[
\begin{align*}
\frac{\partial}{\partial x} (\delta T) &= \lambda \frac{\partial \Delta}{\partial x} + \frac{\partial \lambda}{\partial x} \Delta \\
\frac{\partial}{\partial y} (\delta T) &= \lambda \frac{\partial \Delta}{\partial y} + \frac{\partial \lambda}{\partial y} \Delta \\
\frac{\partial}{\partial z} (\delta T) &= \lambda \frac{\partial \Delta}{\partial z} + \frac{\partial \lambda}{\partial z} \Delta
\end{align*}
\]

(A5)
Now let $T$ represent the (isotropic) stress at $(x, y, z)$ before strain, and suppose that $T'$ represents stress at the same point $(x, y, z)$ during strain. Then the increment of stress for an individual element of liquid initially at $(x, y, z)$ is given by

$$\delta T = T' - T + u_x \frac{\delta T}{\delta x} + u_y \frac{\delta T}{\delta y} + u_z \frac{\delta T}{\delta z}. \quad (A6)$$

Substituting in (A5) we get

$$\frac{\partial T'}{\partial x} - \frac{\partial T}{\partial x} + \frac{\partial}{\partial x} \left( u_x \frac{\delta T}{\delta x} + u_y \frac{\delta T}{\delta y} + u_z \frac{\delta T}{\delta z} \right) = \lambda \frac{\partial \Delta}{\partial x} + \frac{\partial \lambda}{\partial x} \Delta, \quad (A7)$$

and two similar equations.

Now from (A1')

$$\frac{\partial T}{\partial x} = -\rho X,$$

e tc., and we have the corresponding relations

$$\frac{\partial T'}{\partial x} = -\rho' X', \quad (A8)$$

e tc. We use accented letters to denote values during strain, and unaccented letters to denote corresponding values before strain at the same point $(x, y, z)$.

Also

$$\rho' = \rho (1 - \Delta) - \left( u_x \frac{\partial \rho}{\partial x} + u_y \frac{\partial \rho}{\partial y} + u_z \frac{\partial \rho}{\partial z} \right), \quad (A9)$$

and so in (A7) we get, since $\rho$, $X$ may replace $\rho'$, $X'$ in terms containing $u_x$, $u_y$ or $u_z$ (small strain first-order theory),

$$\rho(X - X' + X\Delta) + x \left( u_x \frac{\partial \rho}{\partial x} + u_y \frac{\partial \rho}{\partial y} + u_z \frac{\partial \rho}{\partial z} \right.$$

$$\left. - \frac{\partial}{\partial x} (\rho u_x X + \rho u_y Y + \rho u_z Z) = \lambda \frac{\partial \Delta}{\partial x} + \frac{\partial \lambda}{\partial x} \Delta \right), \quad (A10)$$

and two similar equations.

We now specialize our equations for a gravitating sphere $\theta < r \leq a$ in which $V(r)$ denotes the gravitational potential before strain. Then

$$X = \frac{x}{r} \frac{dV}{dr}, \quad Y = \frac{y}{r} \frac{dV}{dr}, \quad Z = \frac{z}{r} \frac{dV}{dr} \quad (A11)$$

and, using $u$ to denote radial displacement, we have

$$u_x X + u_y Y + u_z Z = u \frac{dV}{dr} \quad (A12)$$

and

$$X \left( u_x \frac{\partial \rho}{\partial x} + u_y \frac{\partial \rho}{\partial y} + u_z \frac{\partial \rho}{\partial z} \right) = u \frac{x}{r} \frac{d\rho}{dr} \frac{dV}{dr}. \quad (A13)$$
Thus equation (A10) may be written in the form

\[
\rho \left[ \frac{x}{r} \frac{dV}{dr} + \frac{x}{r} \frac{dV}{dr} \Delta - X' - \frac{\partial}{\partial x} \left( u \frac{dV}{dr} \right) \right] = \lambda \frac{\partial \Delta}{\partial x} + \frac{\partial \lambda}{\partial x} \Delta, \quad (A14)
\]

and we have two similar equations.

We now change to polar co-ordinates \((r, \theta, \phi)\) with origin at the centre of the sphere, and assuming that the body force \((X', Y', Z')\) is derived from a potential \(V'\), we obtain the three equations

\[
\lambda \frac{\partial \Delta}{\partial r} + \Delta \frac{d\lambda}{dr} = \rho \left[ \frac{dV}{dr} \Delta - \frac{\partial}{\partial r} \left( V' - V + u \frac{dV}{dr} \right) \right] \quad (A15)
\]

\[
\frac{\lambda}{r} \frac{\partial \Delta}{\partial \theta} = \rho \left[ - \frac{1}{r} \frac{\partial}{\partial \theta} \left( V' - V + u \frac{dV}{dr} \right) \right] \quad (A16)
\]

\[
\frac{\lambda}{r \sin \theta} \frac{\partial \Delta}{\partial \phi} = \rho \left[ - \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left( V' - V + u \frac{dV}{dr} \right) \right], \quad (A17)
\]

where it is now assumed that \(\lambda\) is a function of \(r\) only.

We have in \((A15), (A16), (A17)\) three equations of equilibrium for a self-gravitating liquid sphere subjected to a small perturbation potential \(V'\). These equations can in fact be obtained from the equations (7), (8), (9) of Alterman et al. (1959) by omitting the dynamic terms and putting \(\mu = 0\). Their independent verification here shows that nothing goes wrong in this special case.

With the further assumptions made in the main body of the paper, and in the notation there adopted, equations (6), (7) and (8) are then obtained.