Possibility of Obtaining Pion Electromagnetic Form Factor from High Energy Electro-Pion Production

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(Received April 14, 1969)

The possibility of determining pion electromagnetic form factor from high energy electro-pion production experiment is discussed. The amplitude is analyzed in the Regge formalism. If the pion belongs to Class II of $O(3, 1)$, the electromagnetic form factor of the pion can be determined from the knowledge of the $k^2$ dependence of the forward amplitude at high energies. Here $k$ is the four-momentum of the virtual photon exchanged in the reaction. Discrimination between type-II and type-III pions is seen from the polarization dependence of the forward cross section.

§ 1. Introduction

The most reliable experimental method to determine the electromagnetic form factor of the pion would be the elastic scattering of a pion and an electron. This is unpractical at present, and a widely used alternative is the scattering of an electron and a virtual pion, i.e. the electro-pion production off the nucleon.

Extraction of the pion form factor from the experimental data of the electro-pion production suffers from some ambiguity, since it must depend on the theory of strong interaction, in contrast with the case of the nucleon form factors, which can be directly obtained from electron-nucleon scattering. Usually data at rather low energies are analyzed with the help of the dispersion relation for the process

$$\gamma^* + N \rightarrow \pi + N,$$

(1.1)

where $\gamma^*$ stands for the exchanged virtual photon. To solve the dispersion relation, one must use approximations in any way. Furthermore, in the amplitude thus obtained, the part proportional to the pion form factor is numerically smaller than the part proportional to the nucleon form factors. This fact adds uncertainties to the determination of the pion form factor from experiment.

It is therefore desirable to have some other complementary method to determine the pion form factor. In this paper we examine the electro-pion production at high energies, assuming Regge theory for the process (1.1). If the exchange of the pion trajectories dominates the amplitude, one may hope to get some information on the pion form factor from the $k^2$ dependence of the cross section for electro-pion production, where $k_\mu$ is the four-momentum of the virtual photon.
The Regge formalism has thus far been constructed for reactions between particles with time-like momenta. When the momentum $k_\nu$ is time-like ($k_\nu^2 < 0$), the process $(1\cdot 1)$ is the same as the vector-meson production by the pion, for which the Regge theory has been established by many authors.\(^3\) In our case $k_\nu$ is space-like ($k_\nu^2 > 0$), and it is not clear from the beginning that the general theory can be applied also to this case. As is shown in § 3, however, an almost parallel formulation can be made if we define the helicity states of the virtual photon suitably.

Whether or not information on the pion form factor can be extracted from the cross section depends on the $O(3, 1)$ classification of the pion trajectory. This will be discussed in § 4. Our main conclusion is as follows.

1) If the pion belongs to Class III of $O(3, 1)$, the high energy forward amplitude of the process $(1\cdot 1)$ is dominated by $\pi_\nu$, the conspirator of the pion. Then the experiment gives us the $k^2$ dependence of the residue for the $\gamma \pi \pi e$ vertex, which is not connected with the pion form factor directly.

2) If the pion belongs to Class II, the forward amplitude consists of two parts: contributions from pion trajectory and those from non-Regge-pole. The latter is necessary to explain the forward peak of the photo-pion production cross section. From the cross section for high energy electro-pion production in the forward direction, the $k^2$ dependence of the residue function of the $\gamma \pi \pi$ vertex near $t=0$ is obtained, and if a smooth extrapolation to $t=t_i$ is permitted, the pion form factor is determined.

3) If the pion chooses the evasive solution, non-Regge-pole contribution is large in the forward amplitude and the pion form factor is not determined.

The discrimination between Class-II and Class-III cases can be seen from the dependence of the forward cross section on the polarization of the virtual photon.

\section*{§ 2. Kinematics}

(a) \textbf{Independent variables}

We consider the electro-pion production

$$e(k_1) + N(p_1) \rightarrow e(k_2) + N(p_2) + \pi(q). \quad (2\cdot 1)$$

The four-momentum of each particle is indicated in the parentheses.\(^4\) The four-momentum and the polarization vector of the exchanged virtual photon will be denoted by $k_\nu$ ($k = k_1 - k_2$) and $e_\mu$, respectively. The Mandelstam variables are defined as

$$s = -(p_1 + k)^2, \quad t = -(k - q)^2, \quad u = -(k - p_2)^2, \quad (2\cdot 2)$$

of which the relation

$$s + t + u = 2m^2 + p^2 - k^2 \quad (2\cdot 3)$$

\footnote{\(\hbar=c=1, \ k_\nu=(k; ik_0), \ g_{\mu\nu}=(1, 1, 1, -1).\)}
holds. Here \( m \) and \( \mu \) are the masses of the nucleon and the pion, respectively, and \( \varepsilon = \sqrt{k^2} \). Throughout this paper the electron mass will be neglected.

To describe the \( s \)-channel reaction (1.1), we use three reference frames, i.e. (i) the \( s \)-channel center-of-mass system, where \( \mathbf{k} + \mathbf{p}_1 = 0 \), (ii) the laboratory system, where \( \mathbf{p}_1 = 0 \) and (iii) the Breit system, where \( k_0 = 0 \) and \( \mathbf{k} \) and \( \mathbf{p}_1 \) are parallel. These systems are denoted by the superscripts \( s, L \) and \( B \), respectively. To Reggeize the amplitude we also consider the \( t \)-channel reaction

\[
\gamma^+(k) + \pi(q') \rightarrow N(p'_1) + N(p_s),
\]

for which we use the \( t \)-channel center-of-mass system denoted by the superscript \( t \). The photon momentum \( k_\nu \) is retained in the space-like region throughout this paper.

The production angles of the pion in the \( s \)-channel center-of-mass system are denoted by \( \theta^s \) and \( \phi^s \) (see Fig. 1). The \( t \)-channel center-of-mass scattering angles \( \theta^t \) is defined as the angle between the photon and nucleon:

\[
\cos \theta^t = \hat{k} \cdot \hat{p}_1.
\]

In any frame the photon three-momentum \( \mathbf{k} \) is always fixed in the direction of the third axis.

Five independent variables are needed to describe the electro-pion production (2.1). We choose \( s \), \( \theta^s \) (or \( t \)), \( \mu^2 = k_\nu^2 \), \( \phi^s \) and \( \epsilon \). The variable \( \epsilon \), introduced by Akerlof et al., is defined by

\[
\epsilon^{-1} = 1 + 2\tan^2(\theta_e^s/2) = 1 + 2\frac{\mu^2}{\mu^2} - \tan^2(\theta_e^s/2),
\]

where \( \theta_e \) is the scattering angle of the electron. It takes a value between 0 and 1, and is related to the polarization of the virtual photon. The case \( \epsilon = 0 \) corresponds to an unpolarized transverse photon.

In passing we examine the possible ranges of the variables \( s \), \( \mu^2 \) and \( \epsilon \) with a given value of \( k_\nu^2 \), the energy of the incident electron in the laboratory system. One can easily see that

\[
(m + \mu)^2 \leq s \leq s_{\text{max}} = m(m + 2k_\nu^2),
\]

and that

\[
0 \leq \mu^2 \leq \mu^2_{\text{max}},
\]
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Fig. 2. The range of \( s \) and \( \kappa^2 \) with a given value of \( k_{10}^2 \). The shaded area is the permitted region.

Fig. 3. Relation between \( \epsilon \) and \( k_{10}^2 \) (or \( s_{\text{max}} \)) with a given value of \( s \) and \( \kappa^2 \) (in GeV\(^2\) unit).

\[
\kappa_{\text{max}}^2 (k_{10}^2, s) = \frac{k_{10}^2}{s_{\text{max}} + m/2}. \tag{2.9}
\]

Numerical examples of \( \kappa_{\text{max}}^2 \) are shown in Fig. 2. The variable \( \epsilon \) is a function of \( k_{10}^2 \) (or \( s_{\text{max}} \)), \( \kappa^2 \) and \( s \):

\[
\epsilon = 1 - \frac{1}{2} \frac{(s - m^2)^2 + 2(s + m^2) \kappa^2 + \kappa^4}{s_{\text{max}}^2 - (s + m^2 + \kappa^2) s_{\text{max}} + \frac{1}{2} [s^2 + 2s^2 + (m^2 + \kappa^2)^2]} \tag{2.10}
\]

The relation between \( \epsilon \) and \( k_{10}^2 \) with the given values of \( s \) and \( \kappa^2 \) is shown in Fig. 3.

(b) Helicity states of the virtual photon

We define the helicity states of the virtual photon by the polarization vectors in the Breit system:
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\[ e^{B(\pm)} = \frac{1}{\sqrt{2}}(\mp 1, -i, 0; 0), \]
\[ e^{B(0)} = (0, 0, 0; i), \quad (2.11) \]
\[ e^{B(s)} = (0, 0, 1; 0). \]

In the \( s \)-channel center-of-mass system these definitions correspond to

\[ e^{s(\pm)} = \frac{1}{\sqrt{2}}(\mp 1, -i, 0; 0), \]
\[ e^{s(0)} = (0, 0, k^s/\varepsilon; ik^s/\varepsilon), \quad (2.12) \]
\[ e^{s(s)} = (0, 0, k^s/\varepsilon; ik^s/\varepsilon), \]

where \( k^s = |k^s| \). In the \( t \)-channel center-of-mass system photon helicities are defined by analogous expressions. It turns out that in the following formulation the helicity states \( (\pm) \) and \( (0) \) may be regarded as if they were the helicity states of a spin \( 1 \) particle with a time-like momentum. The helicity state \( (s) \) does not contribute to the amplitude due to gauge invariance.

(c) **Differential cross section**

The matrix element \( T \) of the electropion production (2.1) is expressed as

\[ T = \frac{e^3}{k^2} j_\mu j^{\mu}, \quad (2.13) \]

where

\[ j_\mu = \bar{u}_e(k_2) i \gamma_\mu u_e(k_1), \]
\[ J^{\mu} = \langle \pi N; \text{out} | j_\mu | N \rangle. \quad (2.14) \]

Here \( J^\mu \) is the electromagnetic current of hadrons. The differential cross section for the electropion production is given by

\[ \frac{d^3\sigma}{dk^2d\Omega_e d\Omega_\pi} = N \frac{d\sigma^n}{d\Omega_e}(s, k^2, \epsilon, \theta^\pi, \phi^\pi), \]
\[ N = \frac{1}{(2\pi)^3} \frac{e^3}{k^2} \left| k_2^L \right| \left| k^L \right| \frac{1}{1-\epsilon}, \quad (2.15) \]
\[ \frac{d\sigma^n}{d\Omega_e} = \frac{1}{8(2\pi)^3} \frac{m e^2}{s} \frac{|q^L|}{|k^L|^2} \frac{1}{k^2} \sum_{\text{spin}} |J^s j_\mu|^2, \]

where \( d\Omega_e \) and \( d\Omega_\pi \) are differential solid angle of the final electron in the laboratory system and that of the pion in the \( s \)-channel center-of-mass system, respectively. \( d\sigma^n/d\Omega_e \) is the cross section for the pion production by the virtual photon and is equal to the cross section for the photo-pion production when \( \epsilon = k^2 = 0 \). \( N \) is the number of virtual photons per electron scattered into \( dk^2d\Omega_e \).

As is shown in reference 4), the cross section \( d\sigma^n/d\Omega_e \) has the following
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The general form is:

$$\frac{d\sigma^e}{d\Omega^e} = \frac{1}{8(2\pi)^3} \frac{me^2 |q|^2 M}{|k^e| \sqrt{s}},$$

$$M = \frac{1 - \epsilon}{\epsilon^2} \sum_{\text{pions}} |J_{\mu} J_{\mu}'|^2$$  \hspace{1cm} (2·16)

$$= A + \epsilon B + \epsilon C \sin^2 \theta^e \cos 2\varphi^e + \left[ \epsilon (1 + \epsilon) \right]^{1/2} D \sin \theta^e \cos \varphi^e.$$

Here $A$, $B$, $C$ and $D$ are functions of $s$, $t$ and $\epsilon^2$, and are expressed in terms of the $s$-channel helicity amplitudes as follows:

$$A = |f_{1/2}(1/2;0) + f_{1/2}(1/2;1)|^2 + |f_{1/2}(1/2;0) - f_{1/2}(1/2;1)|^2,$$

$$B = 2|f_{1/2}(1/2;0) + f_{1/2}(1/2;0)|^2,$$

$$C = -\frac{2}{\sin^2 \theta^e} \text{Re}[f_{1/2}(1/2;0) f_{1/2}(1/2;0) + f_{1/2}(1/2;1) f_{1/2}(1/2;1)],$$  \hspace{1cm} (2·17)

$$D = \frac{2}{\sin \theta^e} \text{Re}[f_{1/2}(1/2;0) f_{1/2}(1/2;0) + f_{1/2}(1/2;0) f_{1/2}(1/2;1) f_{1/2}(1/2;0) f_{1/2}(1/2;0)],$$

where $f_{a,b,c}^e$ are the $s$-channel helicity amplitudes with $a$, $b$ and $c$ the helicity of the final nucleon, the initial nucleon and the virtual photon, respectively.

Next we have to express the cross section in terms of the $t$-channel helicity amplitudes. Expressions for general directions are rather complicated, and will be derived in Appendix I. In the forward direction ($\theta^e = 0$), however, many of the amplitudes vanish owing to $\sin \theta^e = 0$ and $\sin \varphi^e = 0$, and furthermore the crossing matrix between the $s$- and $t$-channel amplitudes become diagonal. Relations among nonvanishing independent amplitudes are (at $\theta^e = 0$)

$$|f'_{1/2}(0/1;1)| = |f'_{1/2}(0/1;0)|,$$

$$|f'_{1/2}(0/1;0)| = |f'_{1/2}(0/1;0)|,$$

where $f'_{a,b,c}^e$ are the $t$-channel amplitudes with $a$, $b$ and $c$ the helicity of the nucleon, anti-nucleon and virtual photon, respectively. A proof of (2·18) is given in Appendix II. From (2·17) and (2·18), the forward cross section is given by

$$A = |f'_{1/2}(0/1;0)|^2, \quad B = 2|f'_{1/2}(0/1;0)|^2.$$  \hspace{1cm} (2·19)

§ 3. Regge formalism

There have been no attempts, to the author's knowledge, to analyze the electro-pion production in terms of the Regge theory. Ball and Jacob have investigated, however, the Regge theory for the pion production induced by a

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*) The definitions of $A$--$D$ in (2·16) are different from those in reference 4) by the factor before $M$ on the r. h. s. of (2·16).
virtual photon (1.1) in order to clarify the role of gauge invariance in the photo-pion production. Although they assumed implicitly that the momentum of the virtual photon is time-like, their procedure can be applied, almost word to word, to our case. Some differences between Ball and Jacob's expressions and ours come from different definitions of photon helicities. 

Eight invariant amplitudes are introduced by the following decomposition of the amplitude for the process (1.1): 

$$e_{\mu}J_\mu = \sum_{i=1}^{8} B_i(s, t, \kappa) \bar{u}(p_2) N_i u(p_1),$$  \hfill (3.1)  

where 

$$N_i = i\gamma_\mu \cdot e \cdot k, \quad N_5 = i\gamma_\mu \cdot e,$$ 

$$N_6 = \frac{1}{2} i\gamma_\mu \cdot k(p_1 + p_2) \cdot e, \quad N_7 = \frac{1}{2} i\gamma_\mu \cdot k \cdot e,$$  \hfill (3.2)  

$$N_8 = 2i\gamma_\mu \cdot k \cdot e, \quad N_9 = i\gamma_\mu \cdot k \cdot q \cdot e.$$

Gauge invariance requires that 

$$\kappa^2 B_1 + (p_1 + p_2) \cdot k B_2 + 2q \cdot k B_3 + 2\kappa^2 B_4 = 0,$$

$$2B_2 + (p_1 + p_2) \cdot k B_3 + 2\kappa^2 B_4 + 2q \cdot k B_5 = 0.$$  \hfill (3.3)  

Since $B_i$ and $B_t$ do not contribute to the amplitude of electro-pion production, gauge invariance does not play a significant role as long as $\kappa^2 \approx 0$.

The expressions for the $t$-channel helicity amplitudes in terms of the invariant amplitudes are given in Appendix II. Then it is straightforward to construct the following parity conserving amplitudes $F_j (j=1\sim 6)$ which are free of both $s$- and $t$-channel kinematical singularities: 

$$F_j(t, s, \theta_t) = K_t^{-1}(t) F_j(t, s, \theta_t), \quad (j=1\sim 6)$$  \hfill (3.4)  

where 

$$F_1 = \frac{1}{\sin \theta_t} \left( f_{(1/2)(1/2);1} + f_{-(1/2)-(1/2);1} \right),$$  

$$F_2 = \frac{1}{\sin \theta_t} \left( f_{(1/2)(1/2);1} - f_{-(1/2)-(1/2);1} \right),$$  

$$F_3 = \frac{1}{1+ \cos \theta_t} f_{(1/2)-(1/2);1} + \frac{1}{1- \cos \theta_t} f_{-(1/2)-(1/2);1},$$  \hfill (3.5)  

$$F_4 = \frac{1}{1+ \cos \theta_t} f_{(1/2)-(1/2);1} - \frac{1}{1- \cos \theta_t} f_{-(1/2)-(1/2);1},$$  

$$F_5 = f_{(1/2)(1/2);0},$$  

$$F_6 = \frac{1}{\sin \theta_t} f_{(1/2)-(1/2);0}.$$
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and where
\[ K_1 = 4k'p'_s = \left[ (t - \mu^2 - \kappa^2)^2 + 4\kappa^2t \right]^{1/2}, \]
\[ K_2 = \frac{p^2}{p_0^2} = \left( t - 4m^2 \right)^{1/2} t^{-3/2}, \]
\[ K_3 = 2k' = \left[ (t - \mu^2 - \kappa^2)^2 + 4\kappa^2t \right]^{1/2} t^{-1/2}, \]
\[ K_4 = 2p^2 = \left( t - 4m^2 \right)^{1/2}, \]
\[ K_5 = 1/8k'\left( p_0^2 \right)^3 = \left[ (t - \mu^2 - \kappa^2)^2 + 4\kappa^2t \right]^{-1/2} t^{-3/2}, \]
\[ K_6 = p^4/p_0^4 = \left( t - 4m^2 \right)^{1/2} t^{-1/2}. \]

Relations between the amplitudes \( F \) and \( B \) are

\[ F_1 = -\frac{\sqrt{2}i}{4m} B_1, \]
\[ F_2 = \frac{\sqrt{2}i}{4m} \left\{ (t - \mu^2 - \kappa^2) (B_1 - mB_0) + 2tB_3 \right\}, \]
\[ F_3 = \frac{\sqrt{2}i}{8m} \left\{ 4mB_1 + (t - 4m^2) B_0 \right\}, \]
\[ F_4 = \frac{\sqrt{2}i}{8m} \left\{ (s - u) B_0 - 4B_3 \right\}, \]
\[ F_5 = \frac{i}{4m\kappa} \left\{ (s - u) \left[ 2\kappa^2B_1 + t (t - \mu^2 - \kappa^2) B_2 - \frac{m}{2} (t - \mu^2 - \kappa^2)^2 B_0 \right] + \left[ (t - \mu^2 - \kappa^2)^2 + 4\kappa^2t \right] \left[ -2tB_0 + 2mB_3 + m (t - \mu^2 - \kappa^2) B_0 \right] \right\}, \]
\[ F_6 = \frac{-i}{16m\kappa} \left\{ (t - \mu^2 - \kappa^2) \left[ 4B_0 - (s - u) B_0 \right] + 2 \left[ (t - \mu^2 - \kappa^2)^2 + 4\kappa^2t \right] B_0 \right\}. \]

From (3·7) we get the following constraint equations:

\[ 2mF_1(4m^2) + F_6(4m^2) = 0, \]
\[ 2\sqrt{2}\kappa F_5 - (t - \mu^2 - \kappa^2) F_4 = 0, \]
\[ 2\sqrt{2}\kappa F_6 - (s - u) (t - \mu^2 - \kappa^2) F_3 = 0 \] at \( t = (\mu \pm i\kappa)^2 \),

and

\[ 2mF_1(0) + (\mu^2 + \kappa^2) F_3(0) = 0, \]
\[ F_2(0) - 2m (\mu^2 + \kappa^2) F_3(0) = 0, \]

where the arguments denote the values of \( t \). Three relations (3·8) are threshold conditions. The consequences of (3·9) and (3·10) will be discussed in § 4.

Finally we get the Regge asymptotic formulas for the amplitudes \( F \).

\[ F_j = \gamma_j(t, \kappa^2) K_j(t) \frac{\alpha_j(t)}{\alpha_j(t)} + \frac{1}{\sin \pi \alpha_j(t)} \frac{1 \pm \exp \left( -i\pi \alpha_j(t) \right)}{s} (s_{\delta_0}^2)^{\alpha_j(t) - 1}, \]

(for \( j = 1, 2, 3, 4, 6 \))

(3·11)
\[ F_5 = \gamma(t, \kappa^2) K_4(t) \frac{1 \pm \exp(-i\pi\alpha_s(t))}{\sin\pi\alpha_s(t)} (s)_{a_1(t)}, \]

where \( \alpha_j(t) \) and \( \gamma_j(t, \kappa^2) \) are a Regge trajectory and a reduced residue, respectively. In Table I, we show trajectories which contribute to each amplitude.

**Table I. Contributing trajectories to each \( F_j \).**

<table>
<thead>
<tr>
<th>( F )</th>
<th>( P )</th>
<th>( G (\pi^\pm \text{ production}) )</th>
<th>( C (\pi^0 \text{ production}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_1 )</td>
<td>((-1)^J)</td>
<td>((-1)^{J+1} \rho, A_1, \pi_\gamma)</td>
<td>((-1)^J \rho, \omega, \varphi)</td>
</tr>
<tr>
<td>( F_2 )</td>
<td>((-1)^{J+1})</td>
<td>((-1)^{J+1} \pi, B)</td>
<td>((-1)^J \rho, B)</td>
</tr>
<tr>
<td>( F_3 )</td>
<td>((-1)^J)</td>
<td>((-1)^{J+1} \rho, A_2, \pi_e)</td>
<td>((-1)^J \rho, \omega, \varphi)</td>
</tr>
<tr>
<td>( F_4 )</td>
<td>((-1)^{J+1})</td>
<td>((-1)^{J+1} A_1)</td>
<td>((-1)^{J+1})</td>
</tr>
<tr>
<td>( F_5 )</td>
<td>((-1)^J)</td>
<td>((-1)^{J+1} \pi, B)</td>
<td>((-1)^J \rho, B)</td>
</tr>
<tr>
<td>( F_6 )</td>
<td>((-1)^{J+1})</td>
<td>((-1)^J A_1)</td>
<td>((-1)^{J+1})</td>
</tr>
</tbody>
</table>

The forward differential cross section (2.19) is written in terms of the \( F \)'s as
\[
\begin{align*}
A &= |F_3|^2 + |F_1|^2 - 2 \text{Re}(F_3^* F_1), \\
B &= 2|F_5|^2.
\end{align*}
\]

From (3.11) and (3.12) we see the Regge pole contributions to the differential cross section at high energy and near forward direction.

**§ 4. Pion form factor**

Hereafter we discuss only the charged pion production. Let us first review the well-known conspiracy at \( t = 0 \). From the high energy asymptotic expression (3.11), the constraint relations (3.9) and (3.10) can be written as
\[
\begin{align*}
2m_\sigma \gamma_\sigma(0, \kappa^2) + (\mu^2 + \kappa^2) \gamma_\sigma(0, \kappa^2) &= 0, \\
\alpha_\sigma(0) &= \alpha_\sigma(0),
\end{align*}
\]
and
\[
\begin{align*}
\gamma_\sigma(0, \kappa^2) + 2m(\mu^2 + \kappa^2) \frac{\alpha_{A_1}(0)}{\alpha_{A_1}(0) + 1} \gamma_{A_1}(0, \kappa^2) &= 0, \\
\alpha_\sigma(0) &= \alpha_{A_1}(0) - 1,
\end{align*}
\]
respectively, where \( \pi_\sigma(A_1) \) stands for a trajectory, if any, which contributes to the amplitude \( F_3(F_5) \) (see Table I). If the pion belongs to Class III of \( O(3, 1) \), as is assumed by many authors\(^7\) to explain the forward peak of the photo-pion production cross section, (4.1) is realized and \( F_3 = F_5 = 0 \) at \( t = 0 \). On the other hand, if the pion belongs to Class II, as was assumed by Sawyer,\(^8\) then (4.2) is realized and \( F_5 = F_5 = 0 \) at \( t = 0 \). Finally if the pion chooses the evasive solution \( F_5 = F_5 = F_5 = F_5 = 0 \) at \( t = 0 \).
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In the case of Class III, the cross section for the electro-pion production in the forward direction, where $t=0$, becomes, from (3·12),

$$
A = \left| \gamma_{\pi e}(0, \kappa^2) \frac{\alpha_{\pi e}(0)}{\alpha_{\pi e}(0) + 1} \frac{1 + \exp(-i\pi \alpha_{\pi e}(0))}{\sin \pi \alpha_{\pi e}(0)} \left( \frac{s}{s_0} \right)^{\alpha_{\pi e}(0)} \frac{s_0}{m} \right|^2,
$$

$$
B = 0,
$$

where use has been made of the asymptotic form of $t$,

$$
-t \rightarrow \frac{1}{s^2} m^2 (\mu^2 + \kappa^2) + \frac{1}{s^2} m^2 (\mu^2 + \kappa^2)^2 (2m^2 + \mu^2 - \kappa^2) + \cdots. \quad (4·4)
$$

In the case of Class-II pion, the forward cross section is

$$
A = 0,
$$

$$
B = 2 \left| \gamma_{\pi e}(0, \kappa^2) \frac{1 + \exp(-i\pi \alpha_{\pi e}(0))}{\sin \pi \alpha_{\pi e}(0)} \left( \frac{s}{s_0} \right)^{\alpha_{\pi e}(0)+1} \frac{s_0}{m (\mu^2 + \kappa^2)^2} \right|^2. \quad (4·5)
$$

Our results (4·3) and (4·5) agree with Sawyer’s general statement\(^{9}\) that when a Class-III (Class II) trajectory is exchanged, spin-flip (non-flip) amplitudes are dominant in the forward direction.

The cross section (4·5) is incomplete by itself, since it cannot explain the forward peak of the photo-pion production. That is, if the pion belongs to Class II, there must be some contribution to $A$ from Regge cuts, fixed poles or something like that. An example of this is given by Amati et al.\(^{10}\) For the term $B$, we assume that it comes solely from the exchange of a Regge pole. This assumption can in principle be checked experimentally by seeing if the cross section $B$ reveals the $t$ dependence peculiar to the Regge model.

From (2·16) we see that the forward cross section is proportional to $A + \epsilon B$, so that if we plot the experimental values of $d\sigma^e/d\Omega^e|_{\epsilon=0}$ against $\epsilon$ with a fixed $\kappa^2$, we obtain a straight line with the tangent proportional to $B$. Comparing (4·3) and (4·5), we see a method of discrimination of Class-II and Class-III pion from experiment. That is, if the tangent is zero, the pion belongs to Class III. On the other hand, if the pion belongs to Class II, the tangent will have an appreciable value.

Finally we consider how to extract the pion electromagnetic form factor from the forward cross section for the electro-pion production. The pion pole terms of the invariant amplitudes are

$$
B_1^* = -\frac{\sqrt{2} g}{t - \mu^2} F_*(\kappa^2),
$$

$$
B_2^* = \frac{1}{\sqrt{2}} \frac{g}{t - \mu^2} F_*(\kappa^2),
$$

where $g$ is the $\pi N$ coupling constant ($g^2/4\pi \sim 15$) and $F_*(\kappa^2)$ is the electromagnetic form factor of the pion normalized as $F_*(0) = 1$. Combining (4·6)
with \((3·7)\), we see that among the amplitudes \(F_j\) only \(F_5\) has the pion pole at \(t = \mu^2\).

\[
F_5 = \frac{i}{\sqrt{2m}} \mu^2 \kappa (4\mu^2 + \kappa^2) \frac{gF_\pi(\kappa^2)}{t - \mu^2}.
\]

(4·7)

The fact that \(\bar{F}_2\) does not have a pion pole at \(t = \mu^2\) is in accord with the Regge asymptotic formulas \((3·11)\). We see, therefore, that the forward cross section for the electro-pion production is connected with the pion form factor \(F_\pi(\kappa^2)\) only when the spin-non-flip amplitude \(F_6\) makes appreciable contributions, i.e. only when the pion trajectory belongs to Class II. In this case, comparing \((4·7)\) with \((3·11)\), we get

\[
\gamma_{\pi, s}(\mu^2, \kappa^2) = \frac{i\pi g}{2\sqrt{2m}} \mu^2 \kappa (4\mu^2 + \kappa^2) F_\pi(\kappa^2) \alpha_{\pi}'(\mu^2).
\]

(4·8)

Assuming that the variation of the residue function \(\gamma_{\pi, s}(t, \kappa^2)\) is small in going from \(t = \mu^2\) to \(t = 0\),

\[
\gamma_{\pi, s}(0, \kappa^2) \simeq \gamma_{\pi, s}(\mu^2, \kappa^2),
\]

(4·9)

we have the following expression for the cross section \(B\) in terms of the pion form factor:

\[
B = \frac{g^2 \kappa^4 (4\mu^2 + \kappa^2)^3}{\left(\kappa^2 + \mu^2\right)^4} \left[F_\pi(\kappa^2)\right]^2 \left(1 + \cos \pi \alpha_{\pi}(0)\right) s_0 \left(-\frac{s}{s_0}\right)^{2(1 + \alpha_{\pi}(0))}.
\]

(4·10)

That is, from the part of the cross section which is proportional to \(\epsilon\), we can estimate the pion form factor \(F_\pi(\kappa^2)\).

§ 5. Discussion

It has been shown above that if and only if the pion belongs to Class II of \(O(3, 1)\), the electromagnetic form factor of the pion can be determined from the high energy forward cross section for electro-pion production.

The discrimination between Class-II and Class-III pion is seen from the \(\epsilon\) dependence of the forward cross section. For Class-II pion the order of magnitude of \(B\) is estimated from \((4·10)\) with \(F_\pi(\kappa^2) \simeq 1\). When we compare this with the value of \(A\) at \(\kappa^2 = 0\) obtained from the experiment of the forward photo-pion production cross section, we see that \(A\) and \(B\) are of the same order. For Class-III pion, \(B\) is zero.

Although the classification of the pion into \(O(3, 1)\) has not been determined at present, the Class-II pion is less probable, at least phenomenologically, since it must have a trajectory with \(\alpha_{\pi, 0} = 0.98\). Furthermore, we must assume Regge pole dominance for \(B\). Especially for the Class-II pion, we must have some non-Regge-pole contributions for \(A\) so that there is no guarantee in assuming pure Regge pole dominance for \(B\). To check this, we must inquire the \(t\) dependence of the cross section from the experiment and see whether it shows Regge-like behaviors.
Possibility of Obtaining Pion Electromagnetic Form Factor

Acknowledgements

The author would like to express his sincere thanks to Prof. M. Kato for valuable instructions and discussions.

Appendix I

Differential cross section in terms of t-channel helicity amplitudes

The matrix element of the electroproduction (2.1) is expressed as

\[ T = \frac{e^2}{k^2} j_\mu J^\mu_\nu, \]  

(A·1)

where

\[ J^\mu_\nu = \langle NN; \text{out} \mid \pi^\nu, \mu \rangle \]  

(A·2)

The polarization vectors \( e^{(h)} \) defined by (2.11) satisfy the following orthonormal and completeness conditions:

\[ \sum_h e^{(h)}_\mu e^{(h')}_\nu = \delta_{\mu\nu}, \quad \sum_h \delta_{h h'} e^{(h)}_\mu e^{(h')}^{*}_\nu = \delta_{\mu\nu}, \]  

(A·3)

where

\[ \varepsilon_h = \begin{cases} +1 & \text{for } h = +, -, s, \\ -1 & \text{for } h = 0. \end{cases} \]  

(A·4)

Using (A·3) and (A·4), \( |J_\mu J^\mu|^2 \) can be calculated as

\[ |J_\mu J^\mu|^2 = |J_\mu \partial_\mu j^\nu|^2 = \sum_h \varepsilon_h J^h \cdot e^{(h)} \cdot e^{(h')}, \]  

(A·5)

\[ \sum_{\text{spin}} |J_\mu j^\nu|^2 = \sum_{h, h'} \varepsilon_h \varepsilon_{h'} \sum_{\text{spin}} (J^t \cdot e^{(h')})(J^t \cdot e^{(h')}) \sum_{e^{\text{spin}}} (j^s \cdot e^{(h')})(j^s \cdot e^{(h')}) \]  

\[ = \frac{1}{2} \sum_{h, h'} \varepsilon_h \varepsilon_{h'} \sum_{\text{spin}} f^t_{\mu \nu} f^t_{\mu' \nu'} \]  

\[ \times \left[ e^{(h)} \cdot K^t e^{(h')}, K^t - e^{(h)} \cdot k^t e^{(h')} \cdot k^t - 2\varepsilon_h \varepsilon_{h'} k^t \right] \]  

\[ = f_1 \left[ \frac{1}{2} (K_a^t + K_y^t)^2 + k^2 \right] + f_2 \left[ \frac{1}{k^2} (k_a^t K_a^t - k_K^t K_t)^2 - k^2 \right] \]  

\[ + f_3 \frac{\sqrt{2}}{k} (k_a^t K_a^t - k_K^t K_t) K_a^t - f_4 [K_a^t K_y^t - K_y^t K_a^t], \]  

(A·6)

where

\[ f_1 = |f_0^t(0, 0; 0, 1)|^2 + |f_0^t(0, -1; 0, 1)|^2 + |f_0^t(0, 0; 0, 1)|^2 + |f_0^t(0, 0; 0, 1)|^2, \]  

\[ f_2 = |f_0^t(0, 0; 0, 1)|^2 + |f_0^t(0, 0; 0, 1)|^2, \]
\[ f_\varepsilon = \text{Re} \left[ f^{t\ast}_{(\varepsilon)}(a) f^{t\ast}_2(a) + f^{t\ast}_{(\varepsilon)}(a) f^{t\ast}_2(a) \right] \]
and where \( K = k_1 + k_2 \). (A.6) is rewritten as

\[ \sum \left| J_{\varepsilon} \right|^2 = \frac{f_\varepsilon}{2} \left[ 1 + \frac{1}{4k_t^2p_0^2} (K \cdot (p_1 - p_2))^2 \right] \]

\[ - f_\varepsilon \left[ 1 - \frac{1}{4k_t^2p_0^2} (K \cdot (p_1 - p_2))^2 \right] \]

\[ - f_\varepsilon \frac{\sqrt{2k^2}}{4k_t^2p_0^2 \sin \theta^\varepsilon} \left[ \frac{1}{p_1^2} ((K \cdot p_1)^2 - (K \cdot p_2)^2) - \frac{k_t^2 \cos \theta^\varepsilon}{k_t^2p_0^2} (K \cdot (p_1 - p_2))^2 \right] \]

\[ - f_\varepsilon \left[ \frac{1 - 1}{4k_t^2p_0^2} (K \cdot (p_1 - p_2))^2 \right] + \frac{1}{2 \sin \theta^\varepsilon} \left( \frac{1}{p_1^2} (K \cdot (p_1 + p_2))^2 - \frac{2k_t^2 \cos \theta^\varepsilon}{k_t^2p_0^2} (K \cdot (p_1 - p_2))^2 \right) \]

where \( k^t \) and \( p^t \) are \( |k^t| \) and \( |p^t| \), respectively. From (A.8) we get the expression for \( A, B, C \) and \( D \) in terms of \( f_{\varepsilon k p} \).

\[ A = f_\varepsilon \left[ \frac{1}{2} (1 + \alpha^2) + f_\varepsilon \frac{1}{2} \left(-1 + \alpha^2\right) - f_\varepsilon \frac{\sqrt{2k^2}}{\sin \theta^\varepsilon} \alpha \gamma + f_\varepsilon \left(-1 + \alpha^2 - \frac{2k^2}{\sin^2 \theta^\varepsilon}\right) \right] \]

\[ B = f_\varepsilon \left[ \left(-1 + \alpha^2 + \frac{k^2}{4k_t^2p_0^2} |q^t|^2 \sin \theta^\varepsilon\right) + f_\varepsilon \left(1 + \alpha^2 + \frac{k^2}{4k_t^2p_0^2} |q^t|^2 \sin^2 \theta^\varepsilon\right) \right] \]

\[ + f_\varepsilon \left(1 + \alpha^2 + \frac{k^2}{4k_t^2p_0^2} |q^t|^2 \sin \theta^\varepsilon - \frac{2r^2}{\sin^2 \theta^\varepsilon} - \frac{\beta^2}{2 \sin^2 \theta^\varepsilon} |q^t|^2 \sin \theta^\varepsilon \right) \]

\[ |q^t|^2 C = f_\varepsilon \left[ \frac{k^2}{8k_t^2p_0^2} + f_\varepsilon \frac{k^2}{4k_t^2p_0^2} + f_\varepsilon \frac{1}{\sin \theta^\varepsilon} \frac{\sqrt{2k^2 \beta}}{4k_t^2p_0^2} + f_\varepsilon \left(\frac{k^2}{4k_t^2p_0^2} - \frac{\beta^2}{2 \sin^2 \theta^\varepsilon}\right) \right] \]

\[ (\sqrt{2} |q^t|)^{-1} D = f_\varepsilon \frac{\alpha \kappa}{2k_t^2p_0^2} - f_\varepsilon \frac{\alpha \kappa}{k_t^2p_0^2} + f_\varepsilon \frac{\sqrt{2}}{k_t^2p_0^2} \frac{\alpha \beta}{2 \sin \theta^\varepsilon} \left(\frac{k^t}{\sqrt{s} - k_1p_0^t} \alpha \beta \right) - f_\varepsilon \frac{2\beta \gamma}{\sin \theta^\varepsilon} \]

where

\[ \alpha = (|k^t| q_0^t - k_0^t |q^t| \cos \theta^\varepsilon) \frac{1}{2k_t^2p_0^2} \]

\[ \beta = \frac{1}{p^t} + \frac{k_t^2 \cos \theta^\varepsilon}{k_t^2p_0^2} \]

\[ \gamma = \frac{1}{k_t^2p_0^2} \frac{\alpha \beta}{\sqrt{s} - k_1p_0^t} \]
At the forward direction \((\theta^\prime = 0)\), \(\alpha = 1\) and \(\gamma = 0\), so that we get

\[ A = f_1, \quad B = 2f_2. \tag{A.11} \]

When we consider that the kinematical factors \(\sin \theta^\prime\) and \(\cos \theta^\prime\) in \(f_{ab;e}^{(0)}\) (see Appendix II) become 0 and \(-1\) at \(\theta^\prime = 0\), respectively, we obtain \((2.19)\) at the forward direction.

\[ A = |f_{(0/2),(0/2);1}|^2, \quad B = 2|f_{(0/2),(0/2);4}|^2. \tag{A.12} \]

**Appendix II**

(a) *s*-channel helicity amplitudes in terms of the invariant amplitudes

\[
\begin{align*}
    f_{i/2; (0/2); 1}^{(0)} &= -\sqrt{2i}(\Imaginary{(\mathcal{F}_1 + \mathcal{F}_2)}\sin(\theta^\prime/2) + \frac{1}{2}(\mathcal{F}_3 + \mathcal{F}_4)\sin\theta^\prime\cos(\theta^\prime/2)), \\
    f_{i/2; (0/2); -1}^{(0)} &= -\frac{i}{\sqrt{2}}\sin\theta^\prime\cos(\theta^\prime/2)(\mathcal{F}_3 + \mathcal{F}_4), \\
    f_{i/2; (0/2); 1}^{(0)} &= \frac{i}{\sqrt{2}}\sin\theta^\prime\sin(\theta^\prime/2)(\mathcal{F}_3 - \mathcal{F}_4), \\
    f_{i/2; (0/2); -1}^{(0)} &= -\sqrt{2i}(\Imaginary{(\mathcal{F}_1 - \mathcal{F}_2)}\cos(\theta^\prime/2) - \frac{1}{2}(\mathcal{F}_3 - \mathcal{F}_4)\sin\theta^\prime\sin(\theta^\prime/2)), \\
    f_{i/2; (0/2); 0}^{(0)} &= i\frac{k^1_\pi}{\kappa}\cos(\theta^\prime/2)\{(\mathcal{F}_1 + \mathcal{F}_4) + \mathcal{F}_3 + \cos\theta^\prime(\mathcal{F}_3 + \mathcal{F}_4)\}, \\
    f_{i/2; (0/2); 0}^{(0)} &= -i\frac{k^1_\pi}{\kappa}\sin(\theta^\prime/2)\{(\mathcal{F}_1 - \mathcal{F}_4) - \mathcal{F}_3 + \cos\theta^\prime(\mathcal{F}_3 - \mathcal{F}_4)\},
\end{align*}
\]

where

\[ \mathcal{F}_j = K_j F_j, \quad (j = i \sim 6) \]

\[
\begin{align*}
    K_1 &= \frac{W - m}{2m}\sqrt{(p_{10}^1 + m)(p_{20}^1 + m)}, \\
    K_2 &= \frac{W + m}{2m}\sqrt{(p_{10}^2 - m)(p_{20}^2 - m)}, \\
    K_3 &= \frac{|q_1|^2}{2m}\sqrt{p_{10}^1 - m}, \\
    K_5 &= \frac{|q_2|^2}{2m}\sqrt{p_{20}^2 - m}, \\
    W &= p_{10}^1 + q_1^1 = p_{20}^2 + k^1_\pi, \quad \text{the total energy in the } s\text{-channel center-of-mass system,}
\end{align*}
\]

and where

\[
\begin{align*}
    F_1 &= B_1 - \frac{1}{W - m}B_3, \\
    F_2 &= -B_1 - \frac{1}{W + m}B_3, \\
    F_5 &= -B_1 + 2B_3 + \frac{1}{2}(W + m)B_6 - (W + m)B_8, \tag{A.15}
\end{align*}
\]
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\[ F_1 = B_4 - 2B_3 + \frac{1}{2}(W - m)B_5 - (W - m)B_6, \]
\[ F_5 = - (W + m)B_1 - (W + p_{00}^2)B_1 - 2(W - p_{00}^2)B_3 - B_5 \]
\[ + \frac{1}{2}(W + p_{00}^2)(W + m)B_6 + (W - p_{00}^2)(W + m)B_6, \]
\[ F_6 = - \frac{\kappa_0^2}{p_{00}^2 + m}B_1 + (W + p_{00}^2)B_1 + 2(W - p_{00}^2)B_3 - \frac{W + m}{p_{00}^2 + m}B_5 \]
\[ + \frac{1}{2}(W + p_{00}^2)(W - m)B_5 + (W - m)(W - p_{00}^2)B_6. \]

(b) \textit{t-channel helicity amplitudes in terms of the invariant amplitudes}

\[ f'_{(0;1)/(1;0)} = - \frac{i}{\sqrt{2}} \left\{ i\left( p_0^i \frac{p_0^j k_0^i}{m k^i} \right) B_1 - \frac{2p_0^i p_0^j}{m} B_2 + p_0^j k_0^i B_0 \right\} \sin \theta', \]
\[ f'_{(1;0)/(0;1)} = - \frac{i}{\sqrt{2}} \left\{ k^i B_1 - \frac{p_0^i}{m} B_3 + \frac{k^i p_0^j}{m} (1 - \cos \theta') B_2 \right\} (1 + \cos \theta'), \]
\[ f'_{-(0;1)/(1;0)} = - \frac{i}{\sqrt{2}} \left\{ k^i B_1 + \frac{p_0^i}{m} B_3 + \frac{k^i p_0^j}{m} (1 + \cos \theta') B_2 \right\} (1 - \cos \theta'), \]
\[ f'_{-(1/2)-(1/2)} = \frac{i}{\sqrt{2}} \left\{ k^i \left( B_1 + \frac{p_0^i}{m} B_3 + \frac{k^i p_0^j}{m} B_2 - p_0^j k_0^i B_0 \right) \sin \theta', \right. \]
\[ \left. f'_{(0;1)/(0;1)} = - \frac{i}{m \kappa} \left\{ - p_0^i k_0^i \cos \theta' B_1 + p_0^i k_0^i (2p_0^j B_2 - m k_0^j B_3) \cos \theta' \right. \right. \]
\[ + 2p_0^i k_0^j \left( 2p_0^j B_3 - \frac{m}{2p_0^j} B_2 - m k_0^j B_3 \right) \left. \right\}, \]
\[ f'_{(0;1)/(0;1)} = - \frac{i}{m \kappa} \left\{ k_0^i p_0^i \left( B_3 + k_0^i p_0^j \cos \theta' B_4 + \frac{2p_0^j k_0^i}{k_0^i} B_5 \right) \sin \theta'. \right. \right. \]

From (A·13) and (A·16), we get a relation between \( f' \) and \( f' \) at \( \theta' = 0 \), using \( \sin \theta' = \sin \theta = 0 \) and \( \cos \theta' = - \cos \theta = 1 \),

\[ |f'_{(0;1)/(0;1)}| = |f'_{(0;1)/(0;1)}|, \]
\[ |f'_{(0;1)/(0;1)}| = |f'_{(0;1)/(0;1)}| \quad \text{at} \quad \theta' = 0. \quad (A·17) \]

\textbf{References}

5) J. S. Ball and M. Jacob, Nuovo Cim. 54A (1968), 620.