The $\pi N$ Differential Cross Section in Crossed Form\textsuperscript{3)}

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We derive two differential cross-section formulas for positive and negative $u$ regions of the $s$ channel in terms of the $u$-channel $f_1, f_2$ amplitudes which are suitable for the application of Regge formalism. In particular the limits of these expressions are formed for $u=0$ and the direct channel formulas are recovered.

\textbf{§ 1. Introduction}

The $\pi^+p$ differential cross-section formula at $u=0$ for the physical $s$ channel in terms of the $u$-channel amplitudes was derived previously as\textsuperscript{3)}

\begin{equation}
\frac{d\sigma}{d\Omega} = \frac{4u^2}{(M^2-1)^2} \left( \frac{s + 2(M^2-1)}{s} (x + y - z) + \frac{4u\sqrt{u}}{(M^2-1)^2} \frac{4M(M^2-1)}{s} (x - y) + \frac{4u}{(M^2-1)^2} \frac{(M^2-1)^2}{s} (x + y + z) \right).\end{equation}

Here $M$ is the proton mass and $m_+=1$ units are used. $x$, $y$ and $z$ are the following isospin combinations of the $u$-channel amplitudes:

\begin{align*}
x &= \frac{1}{\sqrt{2}} f_1^{u=1/2} + \frac{1}{\sqrt{2}} f_1^{u=-1/2} = \frac{1}{\sqrt{2}} |F_2|^2, \\
y &= \frac{1}{\sqrt{2}} f_1^{u=1/2} + \frac{1}{\sqrt{2}} f_2^{u=1/2} = \frac{1}{\sqrt{2}} |F_3|^2, \\
z &= \frac{1}{\sqrt{2}} (f_1^{u=1/2} + \frac{1}{2} f_2^{u=1/2} + \frac{1}{2} f_2^{u=-1/2}) (f_2^{u=1/2} + \frac{1}{2} f_1^{u=1/2} + \frac{1}{2} f_1^{u=-1/2}) \times (f_2^{u=1/2} + \frac{1}{2} f_1^{u=1/2} + \frac{1}{2} f_1^{u=-1/2}) = \frac{1}{6} [F_3 F_3^* + F_2 F_2^*].
\end{align*}

$f_2$ is the spin-flip amplitude. $f_1$ is the usual combination of the spin-flip and non-flip amplitudes.

\[ f_1 + \cos \theta f_2 = f_{a0}, \]
\[ f_2 = f_{a1}. \]

The upper indices $\frac{1}{2}$ and $\frac{3}{2}$ refer to the isotopic spin states. $f_1$ and $f_3$ have poles at $u=0$ which come from the kinematical factors connecting them to the invariant amplitudes $A$ and $B$.

\begin{equation}
f_1 = \frac{(W_u + M)^2 - 1}{2u} [A + (W_u - M) B],\end{equation}

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\[ f_s = \frac{(W_u - M)^2 - 1}{2u} \left[ -A + (W_u + M)B \right]. \]  

Here \( W_u = u \).

We had previously pointed out to the difficulties in the crossing for \( u=0 \). It was observed that the low energy region and especially the resonances were not obtained if the amplitudes \( f_1^u \) and \( f_2^u \) are unambiguous functions of \( u \) and \( z_u = \cos \theta_u \) and not their ratios. Since \( f_1 \) and \( f_2 \) have poles at \( u=0 \) \( x, y \) and \( z \) also must have poles there. As \( u=0 \) and \( z_u = +1 \) on the backward cone one can argue using the above assumption that \( x, y \) and \( z \) are constants there after the poles are cancelled with the corresponding \( u \)-dependent factors in the differential cross-section formula. As a result one obtains the \( 1/s \) behaviour for the cross section which is in agreement with experiment at large energies. But the expression so obtained which is not a high energy approximation does not reproduce the direct channel resonances. Apparently a more careful investigation is needed to find out what is happening in the neighborhood of \( u=0 \). We would like to remember that the analyticity and crossing form the basis of this result.

The intriguing thing about Eq. (1) is how the complicated \( s \)-dependence could be contained in it if \( x, y \) and \( z \) are assumed to be constants. We have considered taking into account the fact that the differential cross sections are angle averaged quantities. We will discuss this point later.

Here we would like to make the following remarks. The differential cross section could have been written in terms of the Mandelstam amplitudes \( A \) and \( B \) instead of \( f_1 \) and \( f_2 \). But \( A \) and \( B \) are functions of \( u \) and \( s \) and at \( u=0 \) their \( s \)-dependence is not known. Moreover in a crossing procedure like the one using Regge form one needs a complete set of basis functions \( (P_1(z)) \) which are in the variable \( \cos \theta \) and not in one of \( s \) or \( u \). \( f_1 \) and \( f_2 \) are functions of the energy \( W \) and the angle variable \( z \). Therefore the Regge crossing is performed in terms of these amplitudes. After the crossing has been performed and the \( s \)-channel differential cross section has been expressed in terms of the \( u \)-channel \( f_1 \), \( f_2 \) amplitudes one can try to make use of the fact that \( u \) and \( \cos \theta_u \) are constants on the backward cone. However there are two difficulties which must be overcome if one can do this at all.

One is the pole the \( f_1 \) and \( f_2 \) amplitudes have at \( u=0 \). We shall see that the cross-section formulas we will develop in the positive and negative neighborhoods of \( u=0 \) are such that these poles are cancelled exactly by the kinematical factors multiplying these amplitudes.

The other difficulty is the singularity of the transformation between the two sets of variables \( u \), \( s \) and \( u \), \( z_u \). The point \( u=0 \), \( z_u = +1 \) is an indeterminacy point for \( s \), since

\[ \frac{1 - z_u}{u} = 2s - 2(M^2 + 1) + u \]  

\[ (M^2 - 1)^3. \]  

(5)
However we should realize that this indeterminacy of \( s \) on the backward cone is purely a kinematical property and should be separated from the dynamical dependence of the \( f_1^u, f_2^u \) amplitudes on their variables \( u \) and \( z_u \). Thus even though the relation (5) holds between \( u \) and \( z_u \) one can of course give examples of functions of \( u \) and \( z_u \) which are not indeterminate at \( u=0, z_u=+1 \). If the functions themselves were indeterminate at this point they would be functions of \( s \) alone and there would not be much sense in considering the amplitudes \( f_1 \) and \( f_2 \) functions of \( W_u \) and \( z_u \) as far as the backward cone is concerned. The results indicate that this is actually the case.

The other alternative is the possibility that the \( f_1, f_2 \) amplitudes are unambiguous functions of \( u \) and \( z_u \). In that case after the poles have been cancelled out we can try to expand the remaining functions of the two variables \( W_u \) and \( z_u \) in the neighborhood of \( u=0, z_u=+1 \). Now the cross section can be thought of as an angle averaged quantity. But for this alternative to be capable of producing the complicated \( s \)-dependence of the cross sections there must be a sharp maximum or minimum at or near \( u=0 \) so that the averaged quantity can be appreciably different from the local constant value. Otherwise averaging a smooth function will not help much. Moreover the derivatives, as we move off the backward cone, are \( s \)-dependent so that they cannot be used as parameters. These considerations indicate that the amplitudes \( f_1 \) and \( f_2 \) must depend on the ratio \( (1-z_u)/u \). Such dependence can be seen for instance in the relativistic Rutherford formula.

In § 2 the differential cross-section formulas for \( u>0 \) and \( u<0 \) are derived. The general formula is given by Eq. (6). It takes the form (10) for \( u>0 \) and the form (22) for \( u<0 \). These formulas are suitable for obtaining the \( s \)-channel differential cross sections in terms of \( u \)-channel \( f_1, f_2 \) amplitudes using Regge representation for them.

The limits of these two formulas are formed for \( u \) approaching zero from positive and negative directions and Eqs. (14) and (23) are obtained. Finally by expressing the limits of the \( f_1, f_2 \) amplitude combinations the differential cross section is recovered (Eq. (16)) in terms of the direct channel invariant amplitudes.

### § 2. Differential cross sections

Equation (1) indicates that it is not valid for \( u<0 \) since it contains a term with \( \sqrt{u} \). This is so because the \( u \)-channel kinematical factors must be analytically continued into the \( s \) channel. Thus \( W_u \) becomes pure imaginary outside of the backward cone of the \( s \) channel making the \( \sqrt{u} \) term imaginary. We start with the expression

\[
\frac{d\sigma}{d\Omega} = |f_1^{s,5/2}|^2 + |f_2^{s,5/2}|^2 + \cos \theta_s (f_1^{s,5/2}f_2^{s,5/2} + f_2^{s,5/2}f_1^{s,5/2})
\]

(6a)

and use the relations (3) and (4) to express \( f_1^s \) and \( f_2^s \) in terms of \( A^s \) and \( B^s \).
After that crossing is used to replace $A^*$ and $B^*$ by $A^u$ and $B^u$. Finally $A^u$ and $B^u$ are expressed in terms of $f_1^*$ and $f_2^*$ by the inverse relations of (3) and (4). A lengthy calculation gives us the general formula for arbitrary $u$:

$$
\frac{d\sigma}{d\Omega} = [Q^i |a|^2 + P^i |c|^2 - \cos \theta \cdot Q^P (ac^* + ca^*) ]X \\
+ [Q^i |b|^2 + P^i |d|^2 - \cos \theta \cdot Q^P (bd^* + db^*) ]Y \\
- [Q^i ab^* + P^i cd^* - \cos \theta \cdot Q^P (ad^* + b^* c) ]Z \\
- [Q^i ba^* + P^i dc^* - \cos \theta \cdot Q^P (da^* + c^* b) ]Z^*.
$$

(6b)

Here

$$X = 9x, \\
Y = 9y, \\
Z + Z^* = 9z, \\
Z = F_1 F_2^*. 
$$

(7)

The other quantities are defined as follows:

$$Q = \frac{E_u + M}{3W_u}, \\
P = \frac{E_u - M}{3W_u}, \\
E_u = \frac{W_u^2 + M^2 - 1}{2W_u}, \\
a = \frac{W_u - r}{E_u + M'}, \\
b = \frac{W_u + r}{E_u - M'}, \\
c = \frac{W_u + p}{E_u + M'}, \\
d = \frac{W_u - p}{E_u - M'}, \\
E_u = \frac{W_u^2 + M^2 - 1}{2W_u}, \\
r = W_u - 2M, \\
p = W_u + 2M.
$$

(8)

Equation (6) goes over to Eq. (1) for positive values of $u$. But a mismatch of this form with the one we will derive for $u < 0$ forces us to keep all $u$-depend-
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ence in Eq. (6) until the very end when we form the limit for \( u \to +0 \). In particular we keep the \( u \)-dependence in \( \cos \theta_s \).

\[
\cos \theta_s = \frac{-s^2 + 2s(M^2 + 1) + (M^2 - 1)^2 - 2us}{s^2 - 2s(M^2 + 1) + (M^2 - 1)^2}.
\]  

Equation (6) first takes the following form:

\[
\frac{d\sigma}{d\Omega} = \frac{1}{(E_u + M)^3} \left[ W_u^2 K - 2 W_u J + L \right] X
\]

\[
+ \frac{1}{(E_u - M)^3} \left[ W_u^2 K + 2 W_u J + L \right] Y - \frac{1}{E_u^2 - M^2} \left[ W_u^2 K - L \right] (Z + Z^*). 
\]

Here

\[
J = Q^2 - pP^2 + \cos \theta_s (r-t)QP, \\
K = Q^2 + pP^2 - 2 \cos \theta_s QP, \\
L = r^2 Q^2 + p^2 P^2 + 2 \cos \theta_s rpQP. 
\]

It is interesting to note that if one defines vectors \( Q, P, rQ \) and \( pP \) with the angle between \( Q \) and \( P \) defined as \( \theta_s \) one can write

\[
K = (Q - P)^2, \\
J = (rQ + pP) \cdot (Q - P), \\
L = (rQ + pP)^2. 
\]

Next we make the denominators of the terms in Eq. (6) equal and express \( J, K \) and \( L \) in terms of \( s \) and \( u \) using the definitions (8).

\[
J = -\frac{2M}{9s} (M^2 - 1 + u), \\
K = \frac{1}{9s} [s + 2(M^2 - 1) + u], \\
L = \frac{1}{9s} [(M^2 - 1)^2 - u(s - 4M^2)].
\]

Finally we form the limit by keeping the lowest order terms in \( u \):

\[
\frac{d\sigma}{d\Omega} = -\frac{4u^2}{9s} \frac{4(M^2 + 1)}{(M^2 - 1)^2} (X + Y) \\
- \frac{4u^2}{9s} \frac{2s}{(M^2 - 1)^2} (Z + Z^*) + \frac{4u}{9s} (X + Y + Z + Z^*), \quad u \to +0. 
\]

This form is suitable for comparison of \( d\sigma/d\Omega \) with the formula obtained in terms of \( A \) and \( B \), if one expresses the combinations \( X + Y, Z + Z^* \) and \( X + Y + Z + Z^* \) in terms of \( A \) and \( B \). Using Eq. (3) we find: (\( A \) and \( B \) stand for the proper isospin combinations. Thus \( A \) means \( A^{u,1/2} + \frac{1}{2} A^{u,3/2} \). See Eq. (2).)
\[ X + Y \rightarrow \frac{2(M^2 - 1)^2}{4u^2} (A - MB) (A^* - MB^*), \quad u \rightarrow \pm 0, \quad (a) \]

\[ Z + Z^* \rightarrow -\frac{2(M^2 - 1)^2}{4u^2} (A - MB) (A^* - MB^*), \quad u \rightarrow \pm 0, \quad (b) \]

\[ X + Y + Z + Z^* \rightarrow \frac{1}{|u|} \left[ 2MA^* - (1 + M^2)B \right] \left[ 2MA - (1 + M^2)B^* \right], \quad u \rightarrow \pm 0. \quad (c) \]

One thus finds

\[
\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left[ \frac{2(M^2 - 1) + s}{s} |A|^3 M(AB^* + BA^*) + \frac{1 + sM^2 - M^4}{s} |B|^3 \right] \quad (16)
\]

which differs from the expression in terms of s-channel amplitudes by a (−) sign for the second term which comes from the change of sign of the B amplitudes in crossing.

Next we derive a similar expression for \( u < 0 \). This distinction is necessary because in the general form, Eq. (6), the coefficients of \( Z \) and \( Z^* \) are complex conjugates of each other and for \( u > 0 \) they become real and hence equal since \( a, b, c \) and \( d \) are real in this case (see Eq. (8)). \( Q, P \) and \( \cos \theta \) are always real. But the coefficients of \( X \) and \( Y \) are different because \( |a|^2 = |b|^2, |c|^2 = |d|^2 \) and \( ac^* + ca^* = bd^* + db^* \). Thus we see that:

\[
|a|^2 = \frac{W_u^2 - 2W_u r + r^2}{(E_u + M)^2} = \frac{2W_u (W_u - 2M) + (W_u - 2M)^2}{4W_u^2}, \quad (17)
\]

\[
|b|^2 = \frac{W_u^2 - 2W_u b + b^2}{(E_u - M)^2} = \frac{2W_u (W_u + 2M) + (W_u + 2M)^2}{4W_u^2}.
\]

Similar expressions can be written for other pairs to see that they are not equal.

On the other hand for \( u < 0 \) the coefficients of \( Z \) and \( Z^* \) become different, because \( W_u \) is pure imaginary now, which makes the coefficient complex. But now the coefficients of \( X \) and \( Y \) become equal. This can be seen by noticing that

\[
|a|^2 = |b|^2, \quad |c|^2 = |d|^2 \quad \text{and} \quad ac^* + ca^* = bd^* + db^*.
\]

Let us show this again only for the first pair.

\[
|a|^2 = \frac{W_u - r}{E_u + M} \frac{W_u^* - r}{E_u + M} = \frac{|W_u|^2 - r(W_u + W_u^*) + r^2}{|E_u|^2 + M(E_u + E_u^*) + M^2} = \frac{|W_u|^2 + r^2}{|E_u|^2 + M^2}, \quad (18)
\]

\[
|b|^2 = \frac{W_u + r}{E_u - M} \frac{W_u^* + r}{E_u - M} = \frac{|W_u|^2 + r(W_u + W_u^*) + r^2}{|E_u|^2 - M(E_u + E_u^*) + M^2} = \frac{|W_u|^2 + r^2}{|E_u|^2 + M^2}.
\]

Here we used the definitions of \( a \) and \( b \) as given by Eqs. (8) and the relations...
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\[ W_u + W_u^* = 0, \]
\[ E_u + E_u^* = 0 \]  \hspace{1cm} (19)

for $u<0$. The second relation follows again from Eq. (8) connecting $E_u$ to $W_u$. We thus find for the coefficients of $X(Y)$ and $Z, Z^*$:

\[ Q^2|a|^2 + P^2|c|^2 - \cos \theta_s QP(ac^* + ca^*) = \frac{1}{|E_u|^2 + M^2} [K|W_u|^2 + L], \]  \hspace{1cm} (20)

\[ Q^2ab^* + P^2cd^* - PQ \cos \theta_s (ad^* + b^*c) = \frac{1}{|E_u|^2 - M(E_u - E_u^*) - M^2} \times [K|W_u|^2 + J(W_u - W_u^*) - L]. \]  \hspace{1cm} (21)

Replacing now all quantities by their definitions we find the differential cross section.

\[ \frac{d\sigma}{d\Omega} = \frac{1}{|E_u|^2 + M^2} \{K|W_u|^2 + L\} (X + Y) - \frac{1}{|E_u|^2 - M(E_u - E_u^*) - M^2} \times \{K|W_u|^2 + J(W_u - W_u^*) - L\} Z - \frac{1}{|E_u|^2 - M(E_u - E_u^*) - M^2} \times \{K|W_u|^2 + J(W_u - W_u^*) - L\} Z^*. \]  \hspace{1cm} (22)

Keeping again only the lowest order terms in $u$ we obtain

\[ \frac{d\sigma}{d\Omega} \rightarrow \frac{4u^2}{9s} \frac{8M^2}{(M^2 - 1)^2} (X + Y) - \frac{4u^2}{9s} \frac{2[s - 2(3M^2 + 1)]}{(M^2 - 1)^2} (Z + Z^*) \]
\[ + \frac{4u}{9s} (X + Y + Z + Z^*), \quad u \rightarrow 0. \]  \hspace{1cm} (23)

This equation is different from Eq. (14) which we had previously derived for $u>0$. But as before when the limit $u \rightarrow 0$ is formed and relations (15) are used it goes into the proper form for $u=0$, which is Eq. (16).

We notice that both in Eqs. (14) and (22) $(X+Y)$ and $(Z+Z^*)$ have poles in $u^2$ and $(X+Y+Z+Z^*)$ has a pole in $u$ which exactly cancel the powers of $u$ multiplying them. The cross-section formulas (6a) and (6b) can be used within the framework of duality. One can put the direct channel resonances into the s-channel amplitudes and express the differential cross section in terms of these. Since no analytic continuation is necessary in the direct channel, the resonances can be expressed either as Breit-Wigner forms or Regge forms. As the inelasticity increases and the resonances become broader, in addition to parity reflected states more daughters should be included to express the resonances realistically. In this way the amplitudes are saturated with the direct channel resonances. Of course it is an assumption of the duality that infinite number of poles exist in order to obtain the poles in the crossed channel. The differential cross section used here is Eq. (6a) and typical amplitudes are of the form:
Here $e$ and $o$ are even and odd continuations. The upper signs refer to $f_1$ and the lower ones to $f_2$. Positive and negative parity states are of course related by parity reflection. Similar forms are written also for the daughters.

On the other hand the amplitudes, after they are expressed in terms of the $u$-channel amplitudes, can be saturated with the $u$-channel Regge poles, analytically continued into the $s$ channel. In this case formula (10) is used within the backward cone and Eq. (22) outside of this. The quantities $X$, $Y$, $Z$ and $Z^*$ are formed using Eqs. (7) and (2). The $u$-channel amplitudes in Eqs. (2) are of the same form as given by Eq. (24). The continuation of the full resonance form will of course be more complicated but in principle this can be done. Such an application of duality is more realistic in the sense that the widths of resonances are taken into account and they are not treated as delta functions as in the Veneziano model for instance. But the simple form of such a model gets of course lost.

**Acknowledgments**

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**References**

1) For the general formalism of the pion-nucleon problem the reader is referred to V. Singh, Phys. Rev. 129 (1963), 1889.

For the applications of the high energy crossing to the backward scattering we refer to the following papers:


All these papers make use of the justification of the high energy Regge form at $u=0$ given by D. Z. Freedman and Jiunn-Ming Wang, Phys. Rev. Letters 17 (1966), 569 which is by no means a trivial thing for the unequal mass scattering. In this paper we are not concerned with the high energy behavior but rather with the problems arising when we move across the line $u=0$.