Phase Transitions in the Plane Ising Lattices
Decorated with Higher Ising Spins

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Multiple phase transitions in the decorated plane Ising-lattices obtained by modifying the models which have been discussed recently by several authors are investigated. The decorating spin in this paper is, so to speak, a higher Ising spin: we mean by this that the spin variable takes three or more spin values while the usual Ising spin variable takes two spin values; viz. plus and minus the unit. By this modification various types of decorated lattices of different plane Ising-lattices such as those of square, honeycomb and dice which were previously found to possess only a single transition temperature in the case of decoration with the usual Ising spins can possess three transition temperatures, if the exchange-coupling parameters are chosen appropriately. It is also found in the case of decoration with the higher Ising spin that there can exist even five and seven transition temperatures in certain suitable cases.

§ 1. Introduction

Nakano\(^1\) has recently suggested that the spin ordering does not decrease monotonously with rising of temperature in a certain kind of decorated lattice of various lattices and has shown that those of simple-cubic and body-centred-cubic lattices can have three transition temperatures if the values of transition temperatures of these cubic lattices estimated by various approximatic methods are not far from true. Syozi\(^2\) has found that even the rectangular lattice can have three transition temperatures if the whole bonds are not decorated but a half of those is decorated, and Nakano has shown that also the half-decorated triangular lattice can do the same.\(^3\)

Before the existence of multiple transition temperatures in such decorated lattices has been discussed, Vaks, Larkin and Obchennikov\(^4\) showed a complicated plane lattice which has three transition temperatures if the exchange-coupling parameters satisfy a certain condition. This condition, however, is rather severe and the region of the exchange-coupling parameters which satisfy this condition is quite small. The condition of three transition temperatures in the case of the above-mentioned decorated lattice is more easily realized.

It has been shown by Syozi and Miyazima\(^5\) that the existence of three transition temperatures is realized more easily by increasing the number of decorating spins per bond. By means of this increasing, the plane lattices decorated on every bond (wholly-decorated lattices) which Nakano shows not to have three transition temperatures can have three transition temperatures.
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The author suggests a more realistic decoration which facilitates the realization of the condition of existence of three transition temperatures in the decorated lattice. We assume the decorating spin to be a higher Ising spin; viz. the spin whose spin variable is not confined to plus and minus unities but is allowed to take more than two values. By means of this modification the wholly-decorated lattices can be made to possess three transition temperatures. Some lattices which are partly decorated can be modified to possess even five transition temperatures by appropriately choosing the relevant exchange-coupling parameters. The honeycomb and dice lattices which are decorated on every bond can be made to possess even seven transition temperatures if the exchange-coupling parameters on each bond which are different by the direction of the bond are taken appropriately.

We explain the model in § 2 and give the general formulation of the theory. Equations for determining the transition temperatures of the system are also obtained. The decorated lattices of various types, such as those of square, honeycomb and dice lattices, in which the decorating spins are the higher Ising spins and the exchange-coupling parameters are taken the same on all bonds, are investigated in § 3. It is found that phase transitions occur at three different temperatures in these decorated lattices in so far as the relevant coupling parameters are chosen appropriately.

In § 4 we investigate the rectangular lattice decorated with the higher Ising spins and find that this system can have even five transition points in a certain limited region of coupling-parameter values. The decorated triangular, honeycomb and dice lattices of the same sort are classified into two classes in § 5, both of which are investigated by dividing into three cases.

The investigation is developed in §§ 6 and 7 for the decorated triangular, honeycomb and dice systems of the above-mentioned two classes. It is found that these systems can have three, five and even seven transition temperatures for certain appropriate values of the relevant exchange-coupling parameters.

The last section § 8 is arranged for a brief summary and some concluding remarks.

§ 2. Illustration of the model and the general formulation

A lattice is decorated with higher Ising spins, which can take \( I + 1 \) different values. We take the triangular lattice as the matrix lattice. The decorated triangular lattice is shown in Fig. 1, where the parameters of the exchange coupling which act on each bond and the way of labelling \( (i, j)^{(r)} \) \( (r = 0, 1, 2 \text{ or } 3) \) which classifies the lattice sites are also shown. On the basis of the result obtained in this case one can investigate the rectangular, honeycomb and dice systems because the partition function of the rectangular system can be regarded as that of a special triangular system and the partition functions of the honeycomb...
Fig. 1. Decorated triangular lattice and exchange couplings between every pair of spins on the respective bonds. The lattice sites of the matrix lattice and the decorating sites are shown by white and black circles respectively.

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and dice systems can be transformed into that of the triangular system by means of the star-triangle transformation.

The Hamiltonian of the Ising spin system on the decorated triangular lattice is expressed as

$$H = -\sum_{i,j} [J^{(0)}_{\text{m}(i,j)}\mu^{(0)}_{\text{m}(i,j)} + J^{(0)}_{\text{d}(i,j)}\mu^{(0)}_{\text{d}(i,j)}]$$

$$- \sum_{i,j} \left\{J^{(1)}_{\text{m}(i,j)}\nu^{(1)}_{\text{m}(i,j)} + J^{(1)}_{\text{d}(i,j)}\nu^{(1)}_{\text{d}(i,j)} + J^{(2)}_{\text{m}(i,j)}\nu^{(2)}_{\text{m}(i,j)} + J^{(2)}_{\text{d}(i,j)}\nu^{(2)}_{\text{d}(i,j)} + J^{(3)}_{\text{m}(i,j)}\nu^{(3)}_{\text{m}(i,j)} + J^{(3)}_{\text{d}(i,j)}\nu^{(3)}_{\text{d}(i,j)} \right\},$$

(1)

where $\mu^{(r)}_{\text{m}(i,j)}$ and $\nu^{(r)}_{\text{d}(i,j)} (r = 1, 2, 3)$ denote the spin variable on the $(i,j)^{(r)}$-th site of the matrix lattice and on the $(i,j)^{(r)}$-th site of the decorating lattice respectively. $II_r$ and $J_r (r = 1, 2, 3)$ are the exchange-coupling parameters on various bonds as shown in Fig. 1. The summation $\sum_{i,j}$ is taken over all sites of the matrix lattice. The partition function $Z(l; \{I_r\}, \{J_r\})$ of our system is expressed as

$$Z(l; \{I_r\}, \{J_r\}) = \sum_{\{x_{(r)}\}} e^{-H/lT},$$

(2)

where $I_r' = I_r/kT$ and $J_r' = J_r/kT (r = 1, 2, 3)$, and $k$ and $T$ denote the Boltzmann constant and the absolute temperature respectively. The summation $\sum_{\{x_{(r)}\}}$ is taken over all spin values; that is, $x_1 = \pm 1$ and $x_2 = \pm 1, \pm (l - 2), \pm (l - 4), \ldots$. By making summation first on the decorated spin variables $\nu^{(r)}_{\text{d}(i,j)}$ we obtain

$$Z(l; \{I_r\}, \{J_r\}) = Z_0(\{X_r\}) \prod_{r=1}^3 A^r(l; J_r'),$$

(3)

where

$$A^r(l; J_r') = \frac{(l+1) \text{sh} 2(l+1)J_r'}{\text{sh} 2J_r'}, \quad (r = 1, 2 \text{ or } 3)$$

(4)

$$X_r'(l; I_r', J_r') = II_r' + \frac{1}{2} \log \frac{\text{sh} 2(l+1)J_r'}{(l+1) \text{sh} 2J_r'} , \quad (r = 1, 2 \text{ or } 3)$$

(5)

$N$ represents the total number of matrix lattice points, and $Z_0$ denotes the partition function of the triangular lattice with the exchange couplings $kTX_r'$ acting on the respective bonds. According to a calculation by Houtappel,$^9$ we have

$$\log Z_0(\{X_r\})/2 = \lim_{N \to \infty} \frac{1}{N} \log Z_0(\{X_r\})/2^N$$

$$= \frac{1}{2(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} dp dq \log \{c_1c_2c_3 + s_1s_2s_3 - s_1 \cos p - s_2 \cos q - s_3 \cos (p + q)\},$$

(6)
where $c_r$ and $s_r$ represent $ch \, 2X_{r'}$ and $sh \, 2X_{r'}$ respectively ($r=1, 2$ or $3$). The partition function of the decorated triangular lattice per site of the triangular lattice is given by

$$
\log Z_t(l; \{I_{r'}\}, \{J_{r'}\})/2 = \lim_{N \to \infty} \log Z(l; \{I_{r'}\}, \{J_{r'}\})/2^N
$$

$$
= \sum_{r=1}^{3} \log A(l; J_{r'}) + \log Z_{ta}(\{X_{r'}\})/2.
$$ (7)

The transition temperatures are determined from the condition that the expression under the logarithm sign in (6) vanishes for certain values of $p$ and $q$. After a short calculation, we can obtain equations for determining the transition temperatures:

$$
\prod_{r=1}^{3} (1 + \varepsilon^i \, th \, X_{r'}) = 2 \prod_{r=1}^{3} th \, X_{r'}, \quad (i=0, 1, 2 \text{ or } 3)
$$ (8)

where

$$
\varepsilon_0^i = \varepsilon_0^i = 1, \quad \varepsilon_1^i = -\varepsilon_1^i = 1, \quad -\varepsilon_2^i = \varepsilon_2^i = 1, \quad \varepsilon_3^i = \varepsilon_3^i = -1
$$

and $\varepsilon^i$ denotes the product $\varepsilon_0^i \varepsilon_1^i \varepsilon_2^i \varepsilon_3^i (i=0, 1, 2, 3)$. The transition temperatures of the decorated triangular lattice are determined from (8) into which (5) is substituted. A similar equation for the decorated rectangular lattice which is shown in Fig. 2(a) is derived by putting $X_{r'}$ equal to zero in (8); that is, we have

$$
(1 + th[X_{r'}) (1 + th[X_{r'}) = 2,
$$ (9)

where $X_{r'}$ and $X_{r'}$ is given by (5).

By applying the star-triangle transformation to the honeycomb and dice lattices one can relate the partition functions $Z_{ha}$ and $Z_{do}$ of these lattices with $Z_{ta}$ of the triangular lattice per site; that is, one gets

$$
\log Z_{ha}(\{X_{r'}\})/2 = \frac{1}{2} \log B(\{X_{r'}\}) + \frac{1}{2} \log Z_{ta}(\{Y_{r'}\})/4
$$ (10)

and

$$
\log Z_{do}(\{X_{r'}\})/2 = \frac{3}{8} \log B(\{X_{r'}\}) + \frac{1}{8} \log Z_{ta}(\{Y_{r'}\})/8,
$$ (11)

Fig. 2. Decorated lattices and the parameters of exchange couplings
(a): square case.  (b): honeycomb case.  (c): dice case.
where \( kTX_r' \) and \( kTY_r'(r=1, 2 \) or 3) denote the effective exchange couplings on the respective bonds as shown in Figs. 3(a) and (b) and

\[
B'({\{X_r'}\}) = 2^4 \prod_{r=1}^{8} \text{ch}(\varepsilon_1'X_r' + \varepsilon_2'X_r' + \varepsilon_3'X_r').
\]

(12)

The parameters \( X_r' \) and \( Y_r' \) are related with one another in the honeycomb case as

\[
\text{exp}(4Y_r') = \frac{\text{ch}(X_r' + X_r' + X_5') \text{ch}^2(\varepsilon_1'X_r' + \varepsilon_2'X_r' + \varepsilon_3'X_r')}{\prod_{r=1}^{8} \text{ch}(\varepsilon_1'X_r' + \varepsilon_2'X_r' + \varepsilon_3'X_r')}, \quad (r=1, 2 \text{ or } 3)
\]

(13)

and in the dice case as

\[
\text{exp}(2Y_r') = \frac{\text{ch}(X_r' + X_r' + X_5') \text{ch}^3(\varepsilon_1'X_r' + \varepsilon_2'X_r' + \varepsilon_3'X_r')}{\prod_{r=1}^{8} \text{ch}(\varepsilon_1'X_r' + \varepsilon_2'X_r' + \varepsilon_3'X_r')}, \quad (r=1, 2 \text{ or } 3)
\]

(14)

One can see from (10) and (11) that the singular points of \( Z_{10}({\{X_r'}\}) \) and \( Z_{30}({\{X_r'}\}) \) are determined from the equations which are obtained by substituting (13) and (14) into Eqs. (8) which determine those of \( Z_{10}({\{X_r'}\}) \). Thus we can obtain equations for determining the transition temperatures in the honeycomb case

\[
\text{th}|X_r'|\text{th}|X_r'| + \text{th}|X_r'|\text{th}|X_2'| + \text{th}|X_r'|\text{th}|X_3'| = 1,
\]

(15)

and in the dice case

\[
(1 - \text{th}|X_r'|\text{th}|X_2'| - \text{th}|X_2'|\text{th}|X_3'| - \text{th}|X_r'|\text{th}|X_1'|)^s = 4 \text{th}|X_r'|\text{th}|X_r'|\text{th}|X_5'| \text{th}|X_2'| \text{th}|X_3'| + \text{th}|X_r'| + \text{th}|X_r'|,
\]

(16)

where \( X_r'(r=1, 2 \text{ or } 3) \) is given by (5).

The effective exchange-coupling parameters \( X_r'(l; I', J') \) between a pair of spins on matrix lattice sites are dependent on the temperature, and these temperature dependences predominate the thermodynamical properties of the system. Therefore we investigate the temperature dependence of the function

\[
X'(l; \alpha, J') = -l\alpha J' + \frac{1}{2} \log \frac{\text{sh} 2(l+1)J'}{(l+1)\text{sh} 2J'},
\]

(17)

where we have abbreviated the suffix \( r \) in (5) and have utilized the notation \( \alpha = -I/|J| \). The temperature dependence of (17) is represented by the curves

Fig. 3. Star-triangular transformations. (a): honeycomb case. (b): dice case.
shown in Fig. 4 for the cases of several values of the parameters \( l \) and \( a (=\alpha/(1+|\alpha|)) \). Curves in classical limit as \( l \) tends to infinity are obtained by replacing \( lJ' \) with \( J' \).

We denote the minimum value of \( X'(l; \alpha, J') \) by \( X_m \), which corresponds to the value \( J_m' \) of the parameter \( J' \). The relations of \( X_m'(l, a) \) and \( J_m'(l, a) \) \((=|W_m'|/(1+|W_m'|))\) with the parameter \( a \) are shown in Fig. 5, where one can see that \( X_m'(l, a) \) is never smaller than \(-\frac{1}{2} \log(l+1)\) and tends to this value as \( a \) approaches one half.

Fig. 4. Relations of \( X' \) with \( J'(=|lJ'|/(1+|lJ'|)) \) given by (17). (a): \( a=0.45 \). (b): \( a=0.48 \).

Fig. 5. (a): Relations of \( X_m'(l, a) \) with \( a \).
(b): Relations of \( j_m'(l, a) (=|W_m'|(1+|W_m'|)) \) with \( a \).
§ 3. The decorated plane lattices with isotropic exchange couplings

We study in this section respectively the square, honeycomb and dice lattices which are decorated with the higher Ising spins and have the isotropic exchange couplings. These lattices and the exchange couplings are obtained by putting \( I_r = I \) and \( J_r = J (r = 1, 2 \) or \( 3 \) in Fig. 2. The decorated triangular lattice of this sort cannot have three transition temperatures because the triangular lattice cannot be decomposed into two equivalent sublattices as in the square lattice. In this section, therefore, we investigate the existence or non-existence of three phase transitions of the above-mentioned lattices.

The partition function of the decorated lattice can be reduced to that of the spin system on the matrix lattice in which the effective exchange coupling \( kT X' \) with \( X' \) given by (17) acts between every pair of nearest neighbouring spins. The relation of \( X' \) with temperature is shown in Fig. 4. Such a system makes a phase transition at a certain temperature where the absolute value \( X' \) equals a certain critical value \( X_0' \) characteristic for the type of the matrix lattice; e.g. \( X_0' = 0.4407, 0.6585 \) and \( 0.4157 \) respectively for the square, honeycomb and dice lattices. Therefore the transition temperatures are determined by the equation

\[
-l \alpha J' + \frac{1}{2} \log \frac{\text{sh} (l + 1) J'}{(l + 1) \text{sh} 2 J'} = \pm X_0',
\]

(18)

Fig. 6. Magnetic phase diagram on the \( a-J' \) plane for several values of \( l \). F, AF and P represent the domains of ferro-, anti-ferro- and para-magnetic states respectively. (a): square case, (b): honeycomb case, (c): dice case.
where the plus and minus signs correspond respectively to the ferro- or antiferromagnetic transitions.

By solving (18) numerically, we can see that there exist three phase transitions if \( l \) and \( \alpha \) satisfy certain conditions; that is, for example, \( \alpha \) is larger than 0.468 and smaller than 0.5 and \( l=3 \) in the square case. These conditions are shown in Table I for several values of \( l \) in the square, honeycomb and dice cases. It is seen that these lattices which are decorated with the usual Ising spins (i.e. \( l=1 \)) cannot have three transition temperatures.

The ordered states in the system are ferro-, para- and antiferro-magnetic ones in the regions of temperature which satisfy the inequalities \( X'>X_c' \), \( |X'|<X_c' \) and \( X'<-X_c' \) respectively. Therefore it can be seen from the temperature dependence of \( X' \) shown in Fig. 4 that the system passes through four different phases successively, namely ferro-, para-, antiferro- and para-magnetic states, as the temperature rises infinitely from zero. We show in Figs. 6(a), (b) and (c) the magnetic phase diagrams respectively of the decorated square, honeycomb and dice lattices on the \( j'-\alpha \) plane for several values of \( l \), where \( j'=rac{|J'|}{1+|J'|} \) and \( \alpha=\alpha/(1+|\alpha|) \).

§ 4. The decorated rectangular lattice with anisotropic exchange couplings

The decorated rectangular lattice is classified into two types as illustrated in Figs. 7(a) and (b); that is, the one is decorated only on a part of bonds (i.e. \( J_i=0 \)) and the other is decorated on every bond, in which the exchange couplings on each bond are taken different by the direction of the bond.

The transition temperatures of the decorated rectangular lattice are determined by the equation

\[
\exp(-|2X_i'|)=\text{th}|X_i'|
\]

which is derived from (9). The parameters \( X_i' \) and \( X_i'' \) in (19) are given by

\[
X_i'=p_i \quad \text{and} \quad X_i''=H_i'
\]
for type (a) as illustrated in Fig. 7(a), and
\[ X_r' = \rho_r; \quad \rho_r = I_r' + \frac{1}{2} \log \frac{\text{sh} 2(l+1)J_r'}{(l+1)\text{sh} 2J_r'} \quad (r = 1 \text{ or } 2) \]  
(21)

for type (b) as illustrated in Fig. 7(b). In this section we shall use the notations for the temperature variable and the exchange-coupling ratios;

\[ j' = \frac{I_r}{kT + |J_J|}, \quad j_c' = \frac{I_J}{kT + |J_J|} (T_c = \text{transition temperature}), \]
\[ a_r = -\frac{I_r}{|J_J| + |I_J|} (r = 1 \text{ or } 2), \quad b_r = \frac{J_J}{|J_J| + |J_J|}, \quad c_r = \frac{I_J}{|J_J| + |I_J|}. \]

In order to solve (19) numerically, we picture in each type a pair of curves which show the relations of the left-hand and the right-hand sides in Eq. (19).
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with the temperature variable \( j' \).

In type (a) we can get three intersection points between the pair of curves, as shown in Fig. 7(a) in so far as the parameters \( l, a_1 \) and \( c_1 \) are taken appropriately. By assuming several different values of the exchange-coupling ratio \( b_2 \) when \( l, a_1 \) and \( a_2 \) are fixed, we show in Fig. 7(b) similar curves in the case of type (b) where it is seen that we can get three or five intersection points between the pair of curves for an appropriate parameter \( b_2 \). That is to say, we can have three transition temperatures in type (a) and three or five ones in type (b).

We show in Fig. 8 the dependence of the transition temperatures \( (j_0') \) on the exchange-coupling ratio \( c_1 \) by fixing the ratio \( a_1 \) for type (a) and on the parameter \( b_2 \) by fixing the ratios \( a_1 \) and \( a_2 \) for type (b). It is seen that the system of type (a) possesses three transition temperatures for example in the regions \( 0.5 > a_1 > 0.47 \) and \( 1 > c_1 > 0.22 \) when \( l = 2 \). The system of type (b) also possesses three transition temperatures in the case \( 1 > b_2 > 0.795, \ 0.745 > b_2 > 0.255 \) or \( 0.205 > b_2 > 0 \) and even five transition temperatures in the case \( 0.795 > b_2 > 0.745 \) or \( 0.255 > b_2 > 0.205 \) when \( l = 3 \) and \( a_1 = a_2 = 0.48 \). By similar investigation, we can see that in type (a) three phase transitions occur in the case \( l = 1 \) in so far as the relevant coupling parameters are chosen appropriately and that in type (b) five phase transitions do not occur in the case \( l = 1 \), whatever values we may assume for the exchange-coupling ratios.

§ 5. Investigation of various decorated lattices of the triangular, honeycomb and dice lattices

There are three different directions of the bond lines in all of the triangular, honeycomb and dice lattices. The decoration of the bond lines in these directions and the definitions of the exchange couplings on the bond lines are made in such way as shown in Fig. 9, 11, 13, 15, 17 and 19, where we have made two types of classification. The first is related to the decoration or non-decoration of the bond line of each direction. One, two and three of three bond lines each with different directions are decorated respectively in the types which are distinguished by the Roman letters (a), (b) and (c). The second classification is related to the exchange couplings which are assumed to act on each bond. The classes are distinguished by the Greek letters \( \alpha \) and \( \beta \). In the case of (\( \alpha \)) the exchange couplings on two of the three kinds of bond lines are assumed the same and in the case of (\( \beta \)) those on three kinds of bond lines are assumed all different from one another.

The transition temperatures of the decorated triangular, honeycomb and dice lattices are found by solving the equations which are obtained by substituting (5) into (8), (15) and (16) respectively. The transition temperatures of the respective decorated lattices in the class (\( \alpha \)) can be determined by the equations
\[ X_t' = g(X_t'), \quad \text{(triangular)} \]
\[ \exp(-2 X_t') = -g(X_t'), \quad \text{(honeycomb)} \]
\[ \exp(-2 X_t') = -g_a(X_t'), \quad \text{(dice)} \]

where

\[ g(X_t') = \frac{1 - 2 \, \text{th} \, |X_t'| - \text{th}^3 X_t'}{1 + 2 \, \text{th} \, |X_t'| - \text{th}^3 X_t'} \]
\[ g_a(X_t') = \frac{(1 - \text{th} \, |X_t'|)^4 - 8 \, \text{th}^3 X_t'}{(1 + \text{th} \, |X_t'|)^4 - 8 \, \text{th}^3 X_t'} \]

which are derived respectively from Eqs. (8), (15) and (16) where \( X_t' \) is replaced by \( X_t \). The parameters \( X_t' \) and \( X_s' \) in (22) are expressed in type (a) as

\[ X_t' = U_s', \quad X_s' = p_t, \quad \text{(23)} \]

in type (b) as

\[ X_t' = P_1, \quad X_s' = U_s', \quad \text{(24)} \]

and in type (c) as

\[ X_r' = P_r. \quad (r=1 \text{ or } 2) \quad \text{(25)} \]

Equations for determining the transition temperatures in the class (\( \beta \)) are obtained, by rewriting Eqs. (8), (15) and (16) respectively, as

\[ \text{th} \, X_t' = f(X_t', X_s') \quad \text{or} \quad -f(X_t', -X_s'), \quad \text{(triangular)} \]
\[ \exp(-2 X_t') = -f(|X_t'|, |X_s'|), \quad \text{(honeycomb)} \]
\[ \exp(-2 X_t') = f_a(X_t', X_s'), \quad \text{(dice)} \]

where

\[ f(X_t', X_s') = \frac{1 - |\text{th} \, X_t' + \text{th} \, X_s'| - \text{th} \, X_t' \, \text{th} \, X_s'}{1 + |\text{th} \, X_t' + \text{th} \, X_s'| - \text{th} \, X_t' \, \text{th} \, X_s'} \]

and

\[ f_a(X_t', X_s') = \frac{(1 - \text{th}^2 X_t') (1 - \text{th}^2 X_s') - 4 \sqrt{\text{th} \, |X_t'| \text{th} \, |X_s'| ((1 + \text{th}^2 X_t') (1 + \text{th}^2 X_s'))}}{(1 + \text{th} \, |X_t'|)^2 (1 + \text{th} \, |X_s'|)^2 - 8 \, \text{th} \, |X_t'| \text{th} \, |X_s'|} \]

The parameters \( X_t', X_s' \) and \( X_r' \) in (26) are expressed in type (a) as

\[ X_t' = P_1, \quad X_s' = U_{r'}, \quad (r=2 \text{ or } 3) \quad \text{(27)} \]

in type (b) as

\[ X_r' = P_r, \quad (r=1 \text{ or } 2), \quad X_s' = U_{r'} \quad \text{(28)} \]

and in type (c) as

\[ X_r' = P_r, \quad (r=1, 2 \text{ or } 3) \quad \text{(29)} \]
We shall find in §§ 6 and 7 the transition temperatures of various decorated lattices by solving numerically Eqs. (22) in the class (α) and Eqs. (26) in class (β). In these sections we shall use the notations for the exchange-coupling ratios such as

\[ a_r = \frac{-I_r}{|J_r| + |I_r|}, \quad (r = 1, 2 \text{ or } 3) \]

\[ b_r = \frac{J_r}{|J_1| + |J_r|}, \quad (r = 2 \text{ or } 3) \]

\[ c_r = \frac{I_r}{|J_1| + |I_r|}, \quad (r = 2 \text{ or } 3) \]  

§ 6. The decorated triangular, honeycomb and dice systems in the class (α)

The transition temperatures of the decorated triangular, honeycomb and dice systems of the types (a), (b) and (c) in the class (α) are determined by Eqs. (22) into which the effective exchange-coupling parameters (23), (24) and (25) of each type are substituted respectively. In order to solve these equations numerically, we picture in each case a pair of curves which shows the relations of the left-hand and right-hand sides in the respective equations (22) with the

Fig. 9. Curves of (22) for type (α) in the class (α).
(t): triangular case. \( l = 2, a_1 = 0.45 \).
(h): honeycomb case. \( l = 3, a_1 = 0.48 \).
(d): dice case. \( l = 3, a_1 = 0.48 \).
temperature variable $j'$. Thus the transition temperatures are determined from the intersection points of the pair of curves. The curves indicated by (t), (h) and (d) in Fig. 9 corresponds to the triangular, honeycomb, dice cases of type (a), and Figs. 11 and 13 represent the similar curves in the cases of the types (b) and (c).

In the triangular case of type (a), we do not have more than a single intersection point between the pair of curves, as seen in Fig. 9 (t), whatever we may assume the parameter $l$ and the exchange-coupling ratios $a_1$ and $c_2$ in (30).

![Fig. 10](https://example.com/fig10.png)

**Fig. 10.** Dependence of $j'_c$ on $c_2$ and the ordered states in various cases of type (a) in the class (a). (F: ferromagnetic, AF: antiferromagnetic and P: paramagnetic.) (h): honeycomb case. (d): dice case. $l=3$, $a_1=0.48$.

![Fig. 11](https://example.com/fig11.png)

**Fig. 11.** Curves of (22) for the type (b) in the class (a). (t): triangular case. (h): honeycomb case. (d): dice case. $l=3$, $a_1=0.48$. 
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Fig. 12. Dependence of \( J' \) and \( c_2 \) and the ordered states in various cases of type (b) in the class (a).
(F: ferromagnetic, AF: antiferromagnetic and P: paramagnetic.)
(t): triangular case.
(h): honeycomb case.
(d): dice case. \( l=3, a_1=0.48 \).

Fig. 13. Curves of (22) for type (c) in the class (a).
(t): triangular case.
(h): honeycomb case.
(d): dice case. \( l=3, a_1=a_2=0.48 \).
K. Yamada

Fig. 14. Dependence of $j_{2}$ on $b_{2}$ and the ordered states in various cases of type (c) in the class (a).
(F: ferromagnetic, AF: antiferromagnetic and P: paramagnetic.)
(t): triangular case.
(h): honeycomb case.
(d): dice case. $l=3$, $a_1=a_2=0.48$.

That is, the triangular lattice of type (a) in the class (a) does not possess three transition temperatures but only a single transition temperature.

In the honeycomb and dice cases of type (a), it is seen as shown in Figs. 9 (h) and (d) that the pair of curves have three intersection points if $l$ and the exchange-coupling ratios $a_1$ and $c_2$ are taken appropriately. That is to say, we can have three transition temperatures in these cases. Nakano and Yamada have already discussed the honeycomb and dice lattices of this sort which are decorated with the usual Ising spin (i.e. $l=1$) and have shown that they can possess three transition temperatures for the parameters appropriately chosen. We can see that only if $l$ is larger than the unity three phase transitions occur in the honeycomb and dice cases of type (a) in so far as the exchange-coupling parameters are taken appropriately.

We show in Figs. 10 (h) and (d) the dependence of the transition temperatures ($j_{2}^{'})$ in the honeycomb and dice cases of type (a) on the exchange-coupling ratio $c_2$ by fixing the values of $l$ and $a_1$. It is seen there that these systems possess three transition temperatures; for example, in the case $l=3$, if $0.5>a_1>0.48$ and $1>c_2>0.17$ for the honeycomb case and if $0.5>a_1>0.48$ and $1>c_2>0.08$ for the dice case.

By similar investigation in each lattice case of the types (b) and (c), it is seen that we can find three phase transitions in all lattice cases of the types (b) and (c), two ones in the triangular case of type (b) and five ones in the dice
Table II. Some regions of the exchange-coupling ratios for the existence of multiple transition temperatures of each type in the class (a).

<table>
<thead>
<tr>
<th>Type of the decoration</th>
<th>Lattice</th>
<th>( l ) and fixed ratios</th>
<th>Regions of variable ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>triangular</td>
<td></td>
<td>none</td>
<td>none</td>
</tr>
<tr>
<td>(a)</td>
<td>honeycomb</td>
<td>( l=3 ) and ( a_1=0.48 )</td>
<td>none</td>
</tr>
<tr>
<td></td>
<td>dice</td>
<td>( l=3 ) and ( a_1=0.48 )</td>
<td>none</td>
</tr>
<tr>
<td>(b)</td>
<td>triangular</td>
<td>( l=3 ) and ( a_1=0.48 )</td>
<td>(-0.03)&lt;(c_2&lt;-0.06 )</td>
</tr>
<tr>
<td></td>
<td>honeycomb</td>
<td>( l=3 ) and ( a_1=0.48 )</td>
<td>none</td>
</tr>
<tr>
<td></td>
<td>dice</td>
<td>( l=3 ) and ( a_1=0.48 )</td>
<td>none</td>
</tr>
<tr>
<td>(c)</td>
<td>triangular</td>
<td>( l=3 ) and ( a_1=a_2=0.48 )</td>
<td>none</td>
</tr>
<tr>
<td></td>
<td>honeycomb</td>
<td>( l=3 ) and ( a_1=a_2=0.48 )</td>
<td>none</td>
</tr>
<tr>
<td></td>
<td>dice</td>
<td>( l=3 ) and ( a_1=a_2=0.48 )</td>
<td>none</td>
</tr>
</tbody>
</table>

case of type (c), if \( l \) and the exchange-coupling ratios are taken appropriately. On the basis of Figs. 11 and 13 which correspond to the types (b) and (c), the magnetic phase diagrams on the \( j_c'-b_1 \) (or \( c_2 \)) plane are pictured respectively in Figs. 12 and 14, by fixing the other parameters. We show in Table II several regions of the relevant parameters in various lattice systems of the types (a), (b) and (c), in which these system possess multiple transition temperatures.

The triangular lattice reduce to the rectangular lattice in the limit that any one of exchange couplings vanishes. Therefore one can make use of the first equation in (22) for the triangular lattice of type (b) to determine the transition temperatures of the decorated square lattice by replacing the left-hand side in that equation with zero. As shown in Fig. 11(t), the triangular system of the type (b) can possess two transition temperatures in the case that the curve which represents the relation of the right-hand side in the first equation in (22) with the temperature variable \( j' \) intersects with the coordinate axis at three points. In this case the decorated square lattice can possess three transition temperatures. It is realized so far as \( l\geq2 \), as we have already discussed in § 3. That is, under this condition the triangular lattice of type (b) has two transition temperatures as seen in Fig. 11(t).

We remark further on the honeycomb and dice lattice systems that the rectangular lattice system can be regarded as a certain special case of the honeycomb or dice lattice system in which any one of the exchange couplings is infinitely
large. On the basis of this fact we can find easily the condition for \( l \) that the honeycomb and dice lattice systems of various types can have three or five transition temperatures so far as the exchange-coupling parameters are taken appropriately. In § 8 we shall show in Table IV these conditions for \( l \).

§ 7. The decorated triangular, honeycomb and dice systems in the class (\( \beta \))

We can find the transition temperatures of the decorated triangular, honeycomb and dice systems of the types (a), (b) and (c) in the class (\( \beta \)) by solving Eqs. (26) into which Eqs. (27), (28) and (29) are substituted respectively. The relations on the left- and right-hand sides of (26) with \( j' \) are pictured in Figs. 15, 17 and 19 in the cases of types (a), (b) and (c) respectively.

We show the magnetic phase diagrams corresponding to the types (a), (b) and (c) on the plane of the temperature and one exchange-coupling parameter in Figs. 16, 18 and 20 in the cases on the types (a), (b) and (c) respectively, by fixing the remaining exchange-coupling parameters. We can see in these figures that all of the triangular, honeycomb and dice systems of the various type

<table>
<thead>
<tr>
<th>Type of the decoration</th>
<th>Lattice</th>
<th>( l ) and fixed ratios</th>
<th>Regions of variable ratio</th>
<th>Number of transition temperatures</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>triangular</td>
<td></td>
<td>( l=3, a_1=0.48 ) and ( c_1=-0.20 )</td>
<td>none</td>
<td>0.12&gt;( c_2 &gt;0.07 ) or (-0.61&gt;( c_2 &gt;1.0 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>honeycomb</td>
<td></td>
<td>( l=3, a_1=0.48 ) and (</td>
<td>c_2</td>
<td>=0.60 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>dice</td>
<td></td>
<td>( l=3, a_1=0.48 ) and (</td>
<td>c_2</td>
<td>=0.50 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>triangular</td>
<td></td>
<td>( l=3, a_1=a_2=0.48 ) and ( b_2=0.75 )</td>
<td>none</td>
<td>1.0&gt;</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>honeycomb</td>
<td></td>
<td>( l=3, a_1=a_2=0.48 ) and ( b_2=0.25 )</td>
<td>none</td>
<td>0.31&gt;</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>dice</td>
<td></td>
<td>( l=3, a_1=a_2=0.48 ) and ( b_2=0.70 )</td>
<td>none</td>
<td>0.17&gt;</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>triangular</td>
<td></td>
<td>( l=3, a_1=a_2=a_3=0.48 ) and ( b_2=0.20 )</td>
<td>none</td>
<td>1.0&gt;</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>honeycomb</td>
<td></td>
<td>( l=3, a_1=a_2=a_3=0.48 ) and ( b_2=0.05 )</td>
<td>none</td>
<td>0.90&gt;</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>dice</td>
<td></td>
<td>( l=3, a_1=a_2=a_3=0.48 ) and ( b_2=0.05 )</td>
<td>none</td>
<td>0.80&gt;</td>
</tr>
</tbody>
</table>
Phase Transitions in the Plane Ising Lattices

Fig. 15. Curves of \(26\) for type (a) in the class (\(\beta\)).

(t): triangular case. \(c_2 = -0.20\).

(h): honeycomb case. \(c_2 = 0.60\).

(d): dice case. \(c_2 = 0.50\).

\(l=3, \alpha_1=0.48\).

Fig. 16. Dependence of \(j_0'\) on \(c_2\) and the order states in various cases of type (a) in the class (\(\beta\)).

(F: ferromagnetic, AF: antiferromagnetic and P: paramagnetic.)

(t): triangular case. \(c_5 = -0.20\).

(h): honeycomb case. \(c_5 = 0.60\).

(d): dice case. \(c_5 = 0.50\).

\(l=3, \alpha_1=0.48\).
Fig. 17. Curves of (26) for type (b) in the class (β).
(t): triangular case. \( b_3 = 0.75 \).
(h): honeycomb case. \( b_3 = 0.25 \).
(d): dice case. \( b_2 = 0.70 \).
\( l = 3, a_1 = a_2 = 0.48 \).

Fig. 18. Dependence of \( j_{c'} \) on \( c_3 \) and the properties of the ordered states in the various cases of type (b) in the class (β).
(F: ferromagnetic, AF: antiferromagnetic and P: paramagnetic.)
(t): triangular case. \( b_3 = 0.75 \).
(h): honeycomb case. \( b_3 = 0.25 \).
(d): dice case. \( b_2 = 0.70 \).
\( l = 3, a_1 = a_2 = 0.48 \).
Phase Transitions in the Plane Ising Lattices

---

**Fig. 19.** Curves of (26) for type (c) in the class \((\beta)\).

- (t): triangular case, \(b_2=0.20\).
- (h): honeycomb case, \(b_2=0.25\).
- (d): dice case, \(b_2=0.65\).

\(l=3, a_1=a_2=a_3=0.48\).

**Fig. 20.** Dependence of \(j_{c'}\) on \(b_3\) and the ordered states in the various cases of type (c) in the class \((\beta)\).

(F: ferromagnetic, AF: antiferromagnetic and P: paramagnetic.)

- (t): triangular case, \(b_2=0.20\).
- (h): honeycomb case, \(b_2=0.25\).
- (d): dice case, \(b_2=0.65\).

\(l=3, a_1=a_2=a_3=0.48\).
in this class can possess three transition temperatures and the systems of the
types (b) and (c) in this class can possess even five ones if we take \( l \) large
enough and adopt certain appropriate exchange-coupling parameters.

The dice system of the type (c) can have even seven transition temperatures;
it does, for example, when \( l = 3, 0.87 > b_4 > 0.84, a_1 = a_2 = a_3 = 0.48 \) and \( b_2 = 0.65 \).
The honeycomb system of type (c) has never seven transition temperatures in
so far as \( 6 \geq l \geq 1 \), but we can show that this system has seven transition tem­
peratures in the classical limit that \( l \) tends to infinity. We give in Table III
some regions of the exchange-coupling parameters in various lattice systems of
the types (a), (b) and (c), in which these systems have multiple transition

\section*{§ 8. Summary and concluding remarks}

We have shown that various systems of the plane Ising lattices which are
decorated with the higher Ising spins possess three or five transition temperatures
if \( l \) and the exchange-coupling parameters are taken appropriately. It has also
been seen that the dice and honeycomb lattices which are decorated on every bond
can possess even seven transition temperatures in so far as \( l \) and the different
exchange-couplings on each bond are chosen appropriately. That is, if the mag­
nitude \( l \) of the higher Ising spin which decorates the lattice is sufficiently large
the plane decorated lattice of Ising spins can possess multiple transition tem­
peratures. We show in Table IV the possibility of occurrence of multiple phase
transitions in the cases of various types of decorated lattices. The possibility
depends on the value \( l \) of the decorating spin and we show in Table IV the
minimum \( l \) value which makes the occurrence of multiple phase transitions possible.

We assume in this paper that the magnitude \( l \) is the same for all decorating
spins. If \( l \) is assumed to be different from one direction of the bond to another
direction, the multiple phase transitions can occur in a larger variety of the
decorated-lattice types. We cannot find, however, the system which possesses
more than seven transition temperatures.

It is shown that the systems which possess multiple transition temperatures
exhibit the same logarithmic singularity in the heat capacity as the systems of
the matrix lattices.

We have been interested in the ordered state in the triangular lattice with
the antiferromagnetic exchange couplings which are taken nearly equal but a
little different by the direction of the bond. The interaction of the smallest
magnitude scarcely contributes to determine the spin arrangement in the matrix
triangular lattice in the ordered state. That is, the lattice looks in the ordered
state like the square net which is obtained by eliminating every bond of the
smallest exchange coupling, and the simple antiferromagnetic spin arrangement
characteristic of this square net appears in the ordered state.

We can show the existence of the multiple phase transitions also in the
Phase Transitions in the Plane Ising Lattices

hempleaf and kagomé lattices. The super-decorated Ising lattices, which have been investigated by Syozi and Nakano\(^7\) and which have been shown to have three transition temperatures in so far as the number of the decorating spins per

Table IV. Possibility of the existence of multiple transition temperatures and the minimum \(l\) values for this existence in each lattice case of the types (a), (b) and (c) of the two classes (a) and (b).

<table>
<thead>
<tr>
<th>Type</th>
<th>Lattice</th>
<th>Existence and the minimum (l) values</th>
<th>Number of transition temperatures</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Isotropic</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>square</td>
<td>no</td>
<td>yes (l \geq 1)</td>
<td>no</td>
</tr>
<tr>
<td>triangular</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>honeycomb</td>
<td>no</td>
<td>yes (l \geq 3)</td>
<td>no</td>
</tr>
<tr>
<td>dice</td>
<td>no</td>
<td>yes (l \geq 2)</td>
<td>no</td>
</tr>
<tr>
<td>(a) rectangular</td>
<td>no</td>
<td>yes (l \geq 1)</td>
<td>no</td>
</tr>
<tr>
<td>(b) rectangular</td>
<td>no</td>
<td>yes (l \geq 2)</td>
<td>yes</td>
</tr>
<tr>
<td>(a-a) triangular</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>honeycomb</td>
<td>no</td>
<td>yes (l \geq 1)</td>
<td>no</td>
</tr>
<tr>
<td>dice</td>
<td>no</td>
<td>yes (l \geq 1)</td>
<td>no</td>
</tr>
<tr>
<td>(a-b) triangular</td>
<td>yes</td>
<td>(l \geq 1)</td>
<td>yes</td>
</tr>
<tr>
<td>honeycomb</td>
<td>no</td>
<td>yes (l \geq 2)</td>
<td>no</td>
</tr>
<tr>
<td>dice</td>
<td>no</td>
<td>yes (l \geq 2)</td>
<td>no</td>
</tr>
<tr>
<td>(a-c) triangular</td>
<td>no</td>
<td>yes (l \geq 1)</td>
<td>no</td>
</tr>
<tr>
<td>honeycomb</td>
<td>no</td>
<td>yes (l \geq 2)</td>
<td>yes</td>
</tr>
<tr>
<td>dice</td>
<td>no</td>
<td>yes (l \geq 2)</td>
<td>yes</td>
</tr>
<tr>
<td>(b-a) triangular</td>
<td>no</td>
<td>yes (l \geq 1)</td>
<td>no</td>
</tr>
<tr>
<td>honeycomb</td>
<td>no</td>
<td>yes (l \geq 1)</td>
<td>no</td>
</tr>
<tr>
<td>dice</td>
<td>no</td>
<td>yes (l \geq 1)</td>
<td>no</td>
</tr>
<tr>
<td>(b-b) triangular</td>
<td>no</td>
<td>yes (l \geq 1)</td>
<td>yes</td>
</tr>
<tr>
<td>honeycomb</td>
<td>no</td>
<td>yes (l \geq 2)</td>
<td>yes</td>
</tr>
<tr>
<td>dice</td>
<td>no</td>
<td>yes (l \geq 2)</td>
<td>yes</td>
</tr>
<tr>
<td>(b-c) triangular</td>
<td>no</td>
<td>yes (l \geq 1)</td>
<td>yes</td>
</tr>
<tr>
<td>honeycomb</td>
<td>no</td>
<td>yes (l \geq 2)</td>
<td>yes</td>
</tr>
<tr>
<td>dice</td>
<td>no</td>
<td>yes (l \geq 2)</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
</tbody>
</table>
bond exceeds a certain value, can possess three transition temperatures even in
the case of a single decorating spin per bond if the decorating spin is replaced
by the higher Ising spin.

Hattori and Nakano have investigated recently the detailed magnetic and
thermal properties of certain models of decorated square lattice which are de-
signed to allow exact calculation even in the case of the presence of finite
magnetic field according to Fisher. The first model is a decorated square lattice
which has only a single transition temperature. The second model is a semi-
decorated model of the type investigated by Syozi which has three transition
temperatures and shows more interesting magnetic and thermal behaviours. If
we replace the decorating spin in Hattori and Nakano's first model by the higher
Ising spin, even this model can have three transition temperatures and will have
involved magnetic and thermal properties in the presence of finite magnetic field.
We shall investigate this problem in the near future.

Acknowledgements

The author would like to express his thanks to Professor H. Nakano, who
has suggested this problem, for his guidance and encouragement during the course
of this work.

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3) V. G. Vaks, A. I. Larkin and Yu. N. Obchinickov, JETP 49 (1965), 1180 [Soviet Phys.—
   JETP 22 (1966), 820].