Generalized $O(4)$ Harmonics and
Bethe-Salpeter Equation for Spinor-Spinor Particle System

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The generalized $O(4)$ harmonics which are derived from the representations of the $O(4)$ group are applied to the analysis of the Bethe-Salpeter equation for the spinor-spinor particle system exchanging vector particles at the vanishing total four-momentum.

The separation of the angular variables is made by introducing the helicity amplitudes which have definite transformation properties under the $O(4)$ group. After carrying out angular integrations, we get a system of one-dimensional equations which contains Goldstein's equation for the representation of $M=0$ and Goldstein's equation and Kummer's one for the representation of $M=\pm 1$, where $M$ denotes the four-dimensional helicity quantum number. All the equations belonging to $M=\pm 1$ are analytically solved in the case of massless particles exchange.

On discussing the normalization properties of the solutions of Kummer's equation, it is found that the sign of norm is $\pm (-)^{n-l}$, where the double sign $\pm$ depends on the type of combination of $M=+1$ and $-1$ amplitudes, and $n$ and $l$ stand for the four-dimensional and three-dimensional angular momentum quantum numbers, respectively. The properties of the amplitudes under space inversion are also discussed.

§ 1. Introduction

In recent years, there has been a great development in the group-theoretical approach to the scattering problem in connection with the Regge theory. There are two main currents: The one is the $O(4)$ group-theoretical approach by Freedman and Wang and the other is the $O(3,1)$ group-theoretical one by Toller. Toller has completed the formulation based directly on the homogeneous Lorentz group, i.e. the non-compact group $O(3,1)$, while Freedman and Wang have established a formulation based on the compact group $O(4)$ of the complex Lorentz group by using the Hall-Wightman theorem. Although the investigators have adopted different group structures, they have equally succeeded in expanding the scattering amplitude in the crossed channel as well as in classifying the Regge poles in terms of the irreducible representations of each group.

In this paper, we are to introduce the representations of the $O(4)$ group in the investigation of the Bethe-Salpeter (B-S) equation for the spinor-spinor particle system by using the Wick-rotation instead of the Hall-Wightman theorem.
As is known, the B-S equation in the ladder approximation plays an important role as the only dynamical model in which the structures of the above formulations are established. In particular, the scalar-scalar model, including the Wick-Cutkosky model,\(^6\)\(^7\) has been well investigated and has provided useful informations for the Regge pole theory. On the other hand, it is a matter of regret that the solutions of the B-S equation for the spinor-spinor particle system is little known in spite of the various possible applicabilities of the equation. This is partly because the B-S amplitude consists of many components and partly because the kernel of the equation is not of the Fredholm type.

In the early stage of its research, the B-S amplitude for fermion-antifermion system was investigated in the form

\[
\phi(p) = \phi^s(p) + i\gamma_\mu \phi^A_\mu(p) + \gamma_5 \left[ \phi^p(p) + \gamma_\mu \phi^s_\mu(p) + \sigma_{\mu\nu} \phi^s_\nu(p) \right].
\]  

(1.1)

In the equal-mass case at the vanishing total four-momentum, the B-S equation separates into three decoupled systems \(\phi^s\), \((\phi^A_\mu, \phi^T_\mu)\) and \((\phi^p, \phi^r)\). Goldstein\(^5\) found an explicit solution to the pseudo-scalar part in the case of exchange particles which are massless and proved that the solution exists for any positive value of coupling constant. The situation of this continuous spectrum was later proved to be inevitable by Green\(^7\). On the other hand, Kummer\(^8\) discovered a discrete set of solutions to the axialvector-tensor part for vector (or axialvector) coupling by imposing the Lorentz condition on the axialvector part.

In this connection, we are to investigate the above by our method based on \(O(4)\) group. For this purpose, we introduce the off-mass-shell helicity amplitude which has definite transformation properties under the \(O(4)\) group without using the decomposition (1.1). Accordingly, our method is very analogous to the one adopted by Mueller,\(^9\) who has achieved the classification of Lorentz poles in the investigation of \(N\bar{N}\) scattering, but our object in this paper is to the analysis of the B-S equation itself.

In the next section, we summarize the theory of the representation of \(O(4)\) in the same prescriptions as Freedman and Wang adopted in their work.\(^9\) Next we introduce a generalized \(O(4)\) harmonics and modified generalized \(O(4)\) harmonics. In § 3, we are to deal the B-S equation for fermion-antifermion system with vector coupling by introducing the helicity amplitudes. Here we come to Goldstein's and Kummer's equation through the process different from theirs. The normalization properties of Kummer's solutions are investigated in § 4. The properties of the helicity amplitudes under the space inversion are discussed in the final section. Appendix (A) is devoted to a discussion of the B-S equation for fermion-antifermion system with scalar coupling, and Appendix (B) is to a discussion of the B-S equation for scalar-spinor particle system with scalar coupling.


\section*{2. \textit{O}(4) group and \textit{O}(4) harmonics}

The rotation group \textit{O}(4) in a four-dimensional Euclidean space has six generators, i.e., \( L_1, L_2, L_3 \) which generate ordinary rotations in the three-dimensional space, and \( K_1, K_2, K_3 \) which generate rotations in the planes involving the fourth axis; \( K \) is called boost by Freedman and Wang in analogy with the standard Lorentz-group terminology.

The most convenient parametrization of finite transformations of \( \textit{O}(4) \) is

\begin{equation}
\begin{aligned}
g = e^{-i\varphi L_1} e^{-i\alpha L_2} e^{-i\gamma L_3} e^{-i\delta K_1} e^{-i\beta K_2} e^{-i\lambda K_3}
\end{aligned}
\end{equation}

\[ 0 \leq \varphi, \alpha, \gamma < 2\pi, \quad 0 \leq \delta, \beta \leq \pi. \tag{2.1} \]

In this parametrization, the Haar measure on the group manifold becomes

\begin{equation}
\begin{aligned}
dg = d\varphi d(\cos \theta) \sin^2 \delta d\delta d\alpha d(\cos \beta) d\gamma,
\end{aligned}
\end{equation}

and hence the volume of the group manifold is equal to \( 16\pi^4 \); its finiteness implies the compactness of \( \textit{O}(4) \).

It is well known that the generators \( A = \frac{1}{2} (L + K) \) and \( B = \frac{1}{2} (L - K) \) make independently two sets of the generators of three-dimensional rotation. The \( \textit{O}(4) \) group is, therefore, isomorphic to a direct product of two three-dimensional rotation groups.

The irreducible representation of \( \textit{O}(4) \) is denoted by a pair of eigenvalues \((a, b)\) of the Casimir operators \( A^2 = a(a+1) \) and \( B^2 = b(b+1) \). The most direct choice of a basis set for the irreducible representation \((a, b)\) is to adopt \( \{|ab\rangle\} \), the set of eigenvectors of the operators \( A_1 \) and \( B_2 \). We adopt, however, another basis set \( \{|ab lm\rangle\} \), which consists of eigenvectors of the total angular momentum \( L' \) and its third component \( L_3 \). Transformation between these two bases is easily found by considering \( L = A + B \). For application it is convenient to introduce a pair of numbers \( n = a + b \) and \( M = a - b \) and to label the irreducible representation and its basis sets by \( (nM) \) instead of \( (ab) \).

In the basis set \( \{|nM lm\rangle | M \leq l \leq n, -l \leq m \leq l\} \), the representation matrix of the transformation (2.1) can be written as

\begin{equation}
\begin{aligned}
D^{\text{re}r'}_{\text{m}m'}(g) = \langle nM lm | g | n'M l'm' \rangle \\
= \sum_{m'} D^r_{mm'}(\varphi, \theta, \delta) D^r_{m'm'}(\alpha, \beta, \gamma), \tag{2.3}
\end{aligned}
\end{equation}

where the \( D^r \) are the ordinary representation matrices of three-dimensional rotations, and the boost matrix is given by

\begin{equation}
\begin{aligned}
d^\text{re}_{\text{m}m'}(\delta) = \langle nM lm | e^{-i\delta K_3} | nM l'm' \rangle \\
= \sum_n C \left( \frac{n+M}{2}, \frac{n-M}{2}, l; \mu, m-\mu \right) C \left( \frac{n+M}{2}, \frac{n-M}{2}, l'; \mu, m-\mu \right) \times e^{-i(3\mu-m)^3}. \tag{2.4}
\end{aligned}
\end{equation}
The last equality is easily verified by transforming $|nM:lm\rangle$ to $|nM:ab\rangle$, in which $K_3$ is diagonal, with the aid of Clebsch-Gordan coefficient. The orthogonality of the boost matrices is

$$\sum_{\mu} \int_0^\pi d\delta \sin^3\delta \left[d_{l'M}^{nM}(\delta)\right]^* d_{l'm'}^{nM}(\delta) = \delta_{\mu_1\mu_2} \delta_{M'M'} \frac{\pi (2l+1) (2l'+1)}{2(n+M+1) (n-M+1)}. \tag{2.5}$$

The orthogonality of the representation matrices of $O(4)$ is readily derived from those of $D^k$ and of boost matrices:

$$\int dg \left[D^{nM}_{l'm'}(g)\right]^* D^{n'M'}_{l'm'}(g) = \delta_{n,n'} \delta_{M,M'} \times \delta_{l,l'} \delta_{m,m'} \frac{16\pi^4}{(n+M+1) (n-M+1)}. \tag{2.6}$$

A product of two bases is expanded as

$$\langle n_1M_1: l_1m_1| n_2M_2: l_2m_2 \rangle = \sum_{nMl} F(n_1M_1; n_2M_2; nM|l_1l_2; m_1m_2) \times \langle nM: l, m_1+m_2 |,$$

where the coefficient $F$ is given by

$$F(n_1M_1; n_2M_2; nM|l_1l_2; m_1m_2) = \sum_{n,Ml} C \left[\frac{l_1 + M_1}{2}; \frac{n - M_1}{2}; \mu_1, m_1 - \mu_1\right] C \left[\frac{l_2 + M_2}{2}; \frac{n - M_2}{2}; \mu_2, m_2 - \mu_2\right] \times C \left[\frac{n_1 + M_1}{2}; \frac{n_2 + M_2}{2}; \mu_3, m_3 - \mu_3\right] \times C \left[\frac{n_1 - M_1}{2}; \frac{n_2 - M_2}{2}; \mu_4, m_4 - \mu_4\right]. \tag{2.7}$$

The ranges of the summations $\sum_{nMl}$ of which explicit forms are omitted are determined by the non-vanishing value of the coefficient $F$. When multiplied by $g|n_1M_1: l_1'm_1'\rangle|n_2M_2: l_2'm_2'\rangle$, where $g$ is defined by (2.1), Eq. (2.7) yields

$$D^{n_1M_1}_{l_1m_1l_1'm_1'}(g) D^{n_2M_2}_{l_2m_2l_2'm_2'}(g) = \sum_{n,Ml} \sum_{n',M'} F(n_1M_1; n_2M_2; nM|l_1l_2l_3; m_1m_2m_3) \times F(n_1M_1; n_2M_2; nM|l_1'l_2'l_3'; m_1'm_2'm_3') D^{n'M}_{l_1'm_1l_1'm_1'}(g), \tag{2.8}$$

that is, we obtain the coupling rule for the representation matrices of $O(4)$. This rule plays an important role in the application to the B–S equation through the generalized $O(4)$ harmonics.

Now, we introduce a definition of the generalized $O(4)$ harmonics:
\[ H_{\text{lm};}\text{st},(\delta, \theta, \varphi) = \left[ \frac{(n+M+1)(n-M+1)}{2\pi^2(2s+1)} \right]^{1/2} \left[ D_{\text{lm};}\text{st},(g_\circ) \right], \quad (2.10) \]

where
\[ g_\circ = e^{-i\varphi L_\varphi} e^{-i\theta L_\theta} e^{-i\delta K_\delta}. \quad (2.11) \]

Definition (2.10) is rewritten by using the reduction formula (2.3):
\[ H_{\text{lm};}\text{st},(\delta, \theta, \varphi) = \left[ \frac{(n+M+1)(n-M+1)}{2\pi^2(2s+1)} \right]^{1/2} \left[ d_{\text{lm};}\text{st},(\delta) D_{\text{n};}\text{m},(\varphi, 0, 0) \right]. \quad (2.12) \]

It is to be noticed that the generalized \( O(4) \) harmonics for the special case \( s = 0 \), in which \( M = \lambda = 0 \), reduce to the scalar \( O(4) \) harmonics \( H_{\text{m};}\text{m},(\delta, \theta, \varphi) \). In fact, we can prove the equality
\[ H_{\text{m};}\text{m},(\delta, \theta, \varphi) = H_{\text{m};}\text{m},(\delta, \theta, \varphi) \quad (2.13) \]

by using formulas
\[ d_{\text{m};}\text{m},(\delta) = \left[ \frac{(2l+1)(n-l)!}{(n+1)(n+l+1)!} \right]^{1/2} \sin^2 C_{n+1}^{\delta} \cos \delta, \quad (2.14) \]

where \( C_n^\delta \) denotes the Gegenbauer polynomial, and
\[ D_{\text{m};}\text{m},(\varphi, 0, 0) = \left[ \frac{4\pi}{2l+1} \right]^{1/2} Y_{\text{m};}\text{m},(\theta, \varphi). \quad (2.15) \]

The orthogonality of the generalized \( O(4) \) harmonics is derived from those of boost matrices (2.5), and of \( D_{\text{m};}\text{m},(\varphi, 0, 0) \):
\[ \sum \int d\Omega_{\varphi} \left[ H_{\text{m};}\text{m},(\delta, \theta, \varphi) \right]^* H_{\text{m};}\text{m},(\delta, \theta, \varphi) = \delta_{\text{m};}\text{m},(\delta, \theta, \varphi) \quad (2.16) \]

where \( d\Omega_{\varphi} = \sin^2 \delta d\phi d\theta d(\cos \theta) d\varphi \). It should be remarked that the two suffixes \( s \) appearing in (2.16) is not summed.

The coupling rule for the generalized \( O(4) \) harmonics is easily obtained from (2.9) through the definition (2.10). We confine ourselves to the investigation of the coupling of \( H_{\text{m};}\text{m},(\delta, \theta, \varphi) \) and \( H_{\text{m};}\text{m},(\delta, \theta, \varphi) \), which becomes important in dealing with the spin \( s_1 \) and \( s_2 \) particles system. In the above case, one of the coefficients \( F \) appearing in (2.9) is converted into a more compact form by using Wigner's \( 9-j \) symbol: \(^{12,49)}\)

\(^{49)} \text{Wigner's } 9-j \text{ symbol is defined by}
\[ \sum \frac{C(j_1 j_2 j_3 ; m_1 m_2 m_3)}{m_1 + m_2 + m_3 + m_4} \times \frac{C(j_1 j_2 j_3 ; m_1 m_3 m_4)}{m_1 + m_3 + m_4} \]
\[ = \delta_{j_1 j_2 j_3} \delta_{j_3 j_1 j_2} \delta_{j_2 j_3 j_1} \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ j & j & j \\ j_1 & j_2 & j_3 \\ j_3 & j_2 & j_1 \end{array} \right). \]
The 9-j symbol appearing here can be reduced to the Racah coefficient\(^{(11,12)}\) as follows:

\[
\begin{pmatrix}
\frac{n+M}{2} & \frac{k}{2} \\
\frac{n-M}{2} & \frac{k}{2} \\
\frac{k}{2} & 0
\end{pmatrix} = \frac{\sqrt{(2s+1)(n+M+1)(n-M+1)}}{(2s+1)} W \left( \frac{n+M}{2} \frac{n-M}{2} \frac{k}{2} ; \frac{k}{2} \right).
\]

The Racah coefficients are convenient for the numerical evaluation, but we prefer to use the 9-j symbol because the additivity properties between elements can be easily seen. The 9-j symbol is invariant under the transposition and yields the sign \((-)^{s}\) under the interchange of two columns or rows, where \(s\) denotes the sum of all nine elements.

Now the coupling rule in the case mentioned above becomes

\[
H_{k_1l_{1}m_{1}}(\delta, \theta, \varphi) H_{l_{1}^{*}m_{1}^{*}s_{1}^{*}}(\delta, \theta, \varphi)
\]

\[
= \left[ \frac{2s+1}{2\pi^2(2s+1)} \right]^{1/2} \sum_{nM} F(k0; s_1+s_2 s_1-s_2; nM,l,l'; l',m_2) \times K_{k_1s_1l_{1}m_{1}l_{1}^{*}m_{1}^{*}s_{1}^{*}}^{nM}(\delta, \theta, \varphi),
\]

where the functions \(K(\delta, \theta, \varphi)\), which we call modified generalized \(O(4)\) harmonics, are defined by

\[
K_{k_1s_1l_{1}m_{1}s_{1}^{*}}^{(nM)}(\delta, \theta, \varphi) = (k+1)(2s+1) \begin{pmatrix}
\frac{n+M}{2} & \frac{k}{2} \\
\frac{n-M}{2} & \frac{k}{2} \\
\frac{k}{2} & 0
\end{pmatrix} H_{l_{1}m_{1}s_{1}^{*}}^{nM}(\delta, \theta, \varphi).
\]

The orthonormality of the 9-j symbol

\[
\sum_{s} (k+1)^{s}(2s+1)^{s} \begin{pmatrix}
\frac{n+M}{2} & \frac{k}{2} \\
\frac{n-M}{2} & \frac{k}{2} \\
\frac{k}{2} & 0
\end{pmatrix} \begin{pmatrix}
\frac{n+M}{2} & \frac{k'}{2} \\
\frac{n-M}{2} & \frac{k'}{2} \\
\frac{k'}{2} & 0
\end{pmatrix} = \delta_{kk'}
\]
and Eq. (2.16) yield the orthogonality of the modified generalized $O(4)$ harmonics:

$$\sum_{\alpha}d\Omega_{\alpha}[K^{(s_{1}s_{2})nM}_{k_{1}l_{1}m_{1}l_{2}}(\delta, \theta, \varphi)]^* K^{(s_{1}s_{2})nM'}_{k_{1}l_{1}m_{1}l_{2}}(\delta, \theta, \varphi) = \delta_{nn'}\delta_{MM'}\delta_{kl}\delta_{ll'}\delta_{mm'}.$$  \hspace{1cm} (2.22)

The superscript $(s_{1}s_{2})$ will be omitted throughout the next section, where we deal with a spinor-spinor particle system, in which $s_{1}=s_{2}=\frac{1}{2}$.

Next we introduce the symmetrized harmonics with respect to the suffix $M$, which are necessary for constructing eigenstates of the B–S amplitudes for the relative time or space inversion, as follows:

$$H_{\text{sym}}^{nM}(\delta, \theta, \varphi) \equiv \begin{cases} \frac{1}{\sqrt{2}} \left[H_{\text{unsym}}^{nM}(\delta, \theta, \varphi) + \frac{M}{|M|} H_{\text{unsym}}^{n-M}(\delta, \theta, \varphi) \right] & \text{for } M \neq 0, \\
H_{\text{unsym}}^{nM}(\delta, \theta, \varphi) & \text{for } M = 0 \end{cases}$$

(2.23)

for the generalized harmonics, and with this definition

$$K^{(s_{1}s_{2})nM}_{k_{1}l_{1}m_{1}l_{2}}(\delta, \theta, \varphi) = (k+1)(2s+1) \left( \begin{array}{c} n + M \\ 2 \\
\frac{k}{2} \\
\frac{2}{2} \\
\frac{0}{2} \\
\frac{s}{s}
\end{array} \right) H_{\text{sym}}^{nM}(\delta, \theta, \varphi)$$

(2.24)

for the modified generalized harmonics. It should be noticed that the definitions (2.23) and (2.24) leave the orthogonality relations

$$\sum_{\alpha}d\Omega_{\alpha}[H_{\text{sym}}^{nM}(\delta, \theta, \varphi)]^* H_{\text{sym}}^{n'M'}(\delta, \theta, \varphi) = \delta_{nn'}\delta_{MM'}\delta_{kl}\delta_{ll'}\delta_{mm'}.$$  \hspace{1cm} (2.25)

and

$$\sum_{\alpha}d\Omega_{\alpha}[K^{(s_{1}s_{2})nM}_{k_{1}l_{1}m_{1}l_{2}}(\delta, \theta, \varphi)]^* K^{(s_{1}s_{2})n'M'}_{k_{1}l_{1}m_{1}l_{2}}(\delta, \theta, \varphi) = \delta_{nn'}\delta_{MM'}\delta_{kl}\delta_{ll'}\delta_{mm'}.$$  \hspace{1cm} (2.26)

(2.25) is derived from (2.16) together with (2.23), and (2.26) from (2.21) and (2.25) together with (2.24).

The symmetrized harmonics and unsymmetrized ones are connected by the sum rule,

$$\sum_{M} f(M) H_{\text{sym}}^{nM}(\delta, \theta, \varphi) [H_{\text{sym}}^{n'M'}(\delta', \theta', \varphi')]^* = \sum_{M} f(M) H_{\text{unsym}}^{nM}(\delta, \theta, \varphi) [H_{\text{unsym}}^{n'M'}(\delta', \theta', \varphi')]^*$$

(2.27)

for an even function $f(M)$, which is readily derived from the definition (2.23). The sum rule similar to (2.27) for the modified generalized harmonics holds for only $s_{1}=s_{2}$, that is,
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\[ \sum_M f(M) K_{c, l, m, l}^{(s, \ell)}(\delta, 0, \varphi) [K_{c, l, m, l}^{(s, \ell)}(\delta', 0', \varphi')]^* \]

\[ = \sum_M f(M) K_{c, l, m, l}^{(s, \ell)}(\delta, 0, \varphi) [K_{c, l, m, l}^{(s, \ell)}(\delta', 0', \varphi')]^* \]

for an even function $f(M)$. \hfill (2.28)

It seems difficult to define harmonics, in a compact way, which are convenient for constructing the eigenstates of parity in a system of different spin particles, for example, the scalar-spinor particle system discussed in the Appendix (B).

In concluding this section, we enumerate several properties of the boost matrices and the ordinary three-dimensional rotation matrices which will be used in the following sections: symmetry properties\textsuperscript{1,11}

\[ [d_{d1}^{aM}(\delta)]^* = d_{d1}^{aM}(-\delta) = (-)^{-d}d_{d1}^{aM}(\delta), \]

\[ d_{d1}^{aM}(\delta) = d_{d1}^{aM}(\delta), \]

\[ d_{d1}^{aM}(\pi - \delta) = (-)^{a + M - \ell + \ell}d_{d1}^{aM}(\delta), \]

\[ [D^{\ell}_{m1}(\varphi, 0, 0)]^* = (-)^{m - \lambda}D^{\ell}_{m1}(\varphi, 0, 0), \]

\[ D^{\ell}_{m1}(\varphi, \pm \pi, \pi - 0, 0) = e^{\pm i\varphi}D^{\ell}_{m1}(\varphi, 0, 0), \]

and the coupling rule for $D^{\ell}$

\[ D^{\ell}_{m1}D^{\ell}_{n2} = \sum \epsilon C(l_1l_2; m_1m_2)C(l_1l_2; l_1l_2)D^{\ell}_{m_1 + m_2, l_1 + l_2}, \] \hfill (2.34)

§ 3. Spinor-spinor model with vector coupling

We try to solve the fermion-antifermion B–S equation at the vanishing total four-momentum by introducing the helicity amplitude, where we consider a model of the fermion-antifermermion system exchanging vector particles in the ladder approximation. The B–S equation of our model reads in the Wick-rotated form

\[ [1 + i\gamma \rho] \phi(p) [1 + i\gamma \rho] = -\frac{\lambda}{\rho^2} \int d^4p' \frac{\bar{\tau}_\mu \phi(p') \tau_\mu}{(p - p')^2 + \mu^2} , \] \hfill (3.1)

where masses of constituent particles are put equal to unity, \( \lambda \) is the product of two coupling constants, and \( \mu \) in the denominator is the mass of exchange particle. The B–S amplitude \( \phi(p) \) is expressed as a $4 \times 4$ matrix. We note that the charge conjugated B–S amplitude for the fermion-fermion system satisfies (3.1) with the sign of \( \lambda \) changed.

In order to define the helicity amplitude of the B–S amplitude, we introduce the off-mass-shell Wick-rotated Dirac equation by

\[ [\overline{\gamma} \rho] u(p, \lambda) = 0, \quad [\gamma \rho + \gamma] v(p, \lambda) = 0 , \] \hfill (3.2)

where $|p| = (p^2 + \mu^2)^{1/2}$. The helicity \( \lambda \) takes the values $\pm \frac{1}{2}$ according to

\[ \frac{S \cdot p}{|p|} \begin{pmatrix} u(p, \pm\frac{1}{2}) \\ v(p, \pm\frac{1}{2}) \end{pmatrix} \begin{pmatrix} u(p, \pm\frac{1}{2}) \\ v(p, \pm\frac{1}{2}) \end{pmatrix} , \] \hfill (3.3)
where $S$ is the spin operator. It is convenient to use the linear combinations of $u(p, \lambda)$ and $v(p, \lambda)$ defined by

$$u^{(\pm)}(p, \lambda) = \frac{1}{\sqrt{2}} [u(p, \lambda) \pm v(p, \lambda)] ,$$

(3.4)

which have the orthonormality

$$u^{(+)*}(p, \lambda) u^{(+)}(p, \lambda') = \delta_{\lambda \lambda'},$$

$$u^{(-)*}(p, \lambda) u^{(-)}(p, \lambda') = \delta_{\lambda \lambda'},$$

(3.5)

and the completeness

$$\sum_{\lambda} [u^{(+)}(p, \lambda) u^{(+)*}(p, \lambda) + u^{(-)}(p, \lambda) u^{(-)*}(p, \lambda)] = \delta_{ij}.$$  

(3.6)

If the Kramer set of $\gamma$ matrices

$$\gamma = -\sigma_3 \otimes \sigma, \quad \gamma_4 = \sigma \otimes 1$$

(3.7)

is employed, $u^{(\pm)}(p, \lambda)$ are represented in adequate phase convention by

$$u^{(+)}(p, \lambda) = \begin{pmatrix} e^{i\lambda_3} & 0 \\ 0 & D_{1/2}^2(\phi, 0, 0) \end{pmatrix},$$

$$u^{(-)}(p, \lambda) = \begin{pmatrix} e^{-i\lambda_3} & 0 \\ 0 & D_{1/2}^2(\phi, 0, 0) \end{pmatrix},$$

(3.8)

where $\delta$, $\theta$, and $\phi$ denote the polar angles of $p$.

Now, the helicity amplitudes $\phi^{(i)}(p; s \lambda)$ of the B–S amplitude are defined, with the helicity spinors $u^{(\pm)}(p, \lambda)$, by

$$\begin{pmatrix} \phi^{(1)}(p; s \lambda) \\ \phi^{(2)}(p; s \lambda) \\ \phi^{(3)}(p; s \lambda) \\ \phi^{(4)}(p; s \lambda) \end{pmatrix} = \sum_{\lambda_1 - \lambda_2} (-)^{1/2 - \lambda_2} C(\frac{1}{2} \frac{1}{2} s; \lambda_1 - \lambda_2) \begin{pmatrix} u^{(+)*}(p, \lambda_1) \phi(p) u^{(+)}(p, \lambda_2) \\ u^{(+)*}(p, \lambda_1) \phi(p) u^{(-)}(p, \lambda_2) \\ u^{(-)*}(p, \lambda_1) \phi(p) u^{(+)}(p, \lambda_2) \\ u^{(-)*}(p, \lambda_1) \phi(p) u^{(-)}(p, \lambda_2) \end{pmatrix}.$$  

(3.9)

The phase factor $(-)^{1/2 - \lambda_2}$ is very important in the final result. The quantum numbers $s$ and $\lambda$ mean the total spin and the helicity of the fermion-antifermion system, respectively, and take the values $s=0, 1$ and $-s \leq \lambda \leq s$.

We can derive a system of integral equations for the helicity amplitudes $\phi^{(i)}(p; s \lambda)$ from (3.1) by using the orthogonality and completeness of $u^{(\pm)}(p, \lambda)$, those of the Clebsch-Gordan coefficients and the relation

$$\gamma_p u^{(\pm)}(p, \lambda) = |p| u^{(\mp)}(p, \lambda).$$  

(3.10)

Before writing down the equations for the helicity amplitudes, we introduce the notation
Generalized $O(4)$ Harmonics and Bethe-Salpeter Equation

\[-\gamma (pp'; s \lambda; s' \lambda') = \sum_{k=1}^{2 \lambda} \sum_{k''=1}^{2 \lambda} (-\gamma)^{k-\lambda} C\left(\frac{1}{2} \delta s; \lambda - \lambda\right) \times C\left(\frac{1}{2} \delta s'; \lambda - \lambda\right) \sum_{\mu} \left[ u^{(+)\mu}(p, \lambda) \gamma_\mu u^{(-)}(p', \lambda') \right] \left[ u^{(+)\mu}(p', \lambda') \gamma_\mu u^{(-)}(p, \lambda) \right], \tag{3.11} \]

which will appear in the kernel when the original amplitude is expressed in terms of the helicity ones. The notation $w(pp'; s \lambda; s' \lambda')$ can be rewritten in a more compact form by the following procedures: substitution of (3.8) in (3.11), summation over the suffix $\mu$, and uses of (2.32) and (2.34) to make $D^{(\kappa)}(\varphi, \theta, 0)$ and $D^{(\kappa)}(\varphi', \theta', 0)$, of (2.4) and (2.29) to make $[D^{(\kappa)}]^{\ast}$ and $D^{(\kappa)}$, and finally of (2.12). The result is

\[ w(pp'; s \lambda; s' \lambda') = \pi^2 \sqrt{(2s+1)(2s'+1)} \sum_{\lambda m} H^{(\lambda s \lambda)}_{im}(\vartheta, \theta, 0) [H^{(\lambda s' \lambda')}_{im'}(\vartheta', \theta', \varphi')]^{\ast}. \tag{3.12} \]

The quantity which is obtained by interchanging $u^{(+)}$ and $u^{(-)}$ on the r.h.s. of (3.11) also appears in the kernel and it can be proved to be equal to $(-\gamma) w(pp'; s \lambda; s' \lambda')$ in the same steps as those mentioned above.

Now, the equations for the helicity amplitudes become

\[ [\phi^{(1)} - \frac{\lambda}{\pi^2} \frac{1}{(p-p')^2 + \mu^2}] (p; s \lambda) = -\frac{4\lambda}{\pi^2} \int \frac{d^4 p'}{2\pi^4} \phi^{(0)}(p'+00), \tag{3.13} \]

\[ [\phi^{(2)} - \frac{\lambda}{\pi^2} \frac{1}{(p-p')^2 + \mu^2}] (p; s \lambda) = -\frac{\lambda}{\pi^2} \int \frac{d^4 p'}{2\pi^4} \sum_{\nu^\prime} (-\gamma) w(pp'; s \lambda; s' \lambda') \phi^{(3)}(p'; s' \lambda'), \tag{3.14} \]

\[ [\phi^{(3)} - \frac{\lambda}{\pi^2} \frac{1}{(p-p')^2 + \mu^2}] (p; s \lambda) = -\frac{\lambda}{\pi^2} \int \frac{d^4 p'}{2\pi^4} \sum_{\nu^\prime} (-\gamma) w(pp'; s \lambda; s' \lambda') \phi^{(3)}(p'; s' \lambda'), \tag{3.15} \]

and

\[ [\phi^{(4)} - \frac{\lambda}{\pi^2} \frac{1}{(p-p')^2 + \mu^2}] (p; s \lambda) = -\frac{\lambda}{\pi^2} \int \frac{d^4 p'}{2\pi^4} \phi^{(0)}(p'+00), \tag{3.16} \]

where $[f+g](x) = f(x) + g(x)$ and $f^{(\ast)}(x) = f(x) + f(x)$. The appearance of $\delta_{\alpha \eta}$ in (3.13) and (3.16) is the special feature of the vector-coupling model and it allows us to solve the equations easily (cf. Appendix (A) where we discuss the scalar coupling model).

Next, we investigate the angular dependence of the kernels of Eqs. (3.13) ~ (3.16). The propagator $[ (p-p')^2 + \mu^2]^{-1}$ is expanded in terms of the scalar $O(4)$ harmonics in the well known formula

\[ \frac{1}{(p-p')^2 + \mu^2} = \sum_{k=0}^{\infty} 2\pi^2 R_k([p], [p']^\ast; \mu^2) \frac{k+1}{k} \sum_{\lambda m} H^{(\lambda)}_{km}(\vartheta, \theta, \varphi) H^{(\lambda)}_{km'}(\vartheta', \theta', \varphi'), \tag{3.17} \]
where
\[ R_k(|p|, |p'|; \mu^2) = \left\{ p^2 + p'^2 + \mu^2 - \sqrt{\frac{p^2 + (|p| - |p'|)^2}{2}} \right\}^{k+1} \]
and in the special case of \( \mu = 0 \)
\[ R_k(|p|, |p'|; 0) = \frac{|p'^k \theta(|p| - |p'|)|}{|p'|^{k+2}} + \frac{|p|^k \theta(|p'| - |p|)}{|p|^{k+2}}. \]

Equations (3.17) and (3.12) are combined by the coupling rule (2.19) for the case \( s_1 = s_2 = \frac{1}{2} \) in the form
\[ \frac{w(pp'; ss'; s'k')}{(p-p')^2 + \mu^2} = -\sum_{k=0}^{\infty} 4\pi^2 R_k(|p|, |p'|; \mu^2) \sum_{nMlm} \frac{K^{nM}_{k; lm, n; \theta, \varphi} K^{n'M'}_{k'; l'm', n'; \theta', \varphi', \varphi}}{k+1}, \]
where we have used the relation
\[ \sum_{l_1l_2m_1m_2} F(k0; 10; nM|l_1l_2l; m_1m_2) F(k0; 10; n'M'|l_1l_2l'; m_1m_2) = \delta_{nm} \delta_{MM'} \delta_{ll'}, \]
which is directly derived from the definition (2.8) of the coefficient \( F \). Thus we have been able to separate the angular variables of the kernels in terms of the modified generalized harmonics. However, the sign factors \((-)^n\) and \((-)^{n'}\) appearing in (3.14) and (3.15) are troublesome in performing the angular integration. The sum rule on the 9-\( j \) symbol
\[ (-)^n \begin{pmatrix} \frac{1}{2} & n+M & \frac{k}{2} \\ \frac{1}{2} & 2 & 2 \\ s & s & 0 \end{pmatrix} = \sum_{k'} (-)^{M+1+(k+k')/2} (k'+1)^3 \]
\[ \times \begin{pmatrix} \frac{1}{2} & n+M & \frac{k}{2} \\ n-M & \frac{1}{2} & k \\ k' & k' & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & n+M & k' \\ \frac{1}{2} & 2 & 2 \\ s & s & 0 \end{pmatrix}, \]
enables us to cure the situation. In fact, from (3.22) and (2.20), Eq. (3.20) multiplied by the sign factor \((-)^n\) or \((-)^{n'}\) is re-expanded in the form which has no \( s \) or \( s' \) dependency except for the modified generalized harmonics as follows:
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\[ \frac{(-)^{s}w(pp'; s\lambda; s'\lambda')}{(p-p')^{2} + \mu^{2}} = -4\pi^{2} \sum_{k'} (-)^{M+1+(k+k')/2} (k' + 1) R_{k'}(\lambda | p', | p |; \mu) \]

\[ \times \sum_{n, M, l, m} \left\{ \begin{array}{ccc} 1 & n+M & k \\ 2 & 2 & 2 \\ n-M & 1 & k' \\ 2 & 2 & 2 \\ k' & k' & 0 \\ 2 & 2 & 2 \end{array} \right\} K_{n; l; \lambda}^{[M]}(\delta, \theta, \phi) K_{n; l; \lambda'}^{[M]}(\delta', \theta', \phi') \] (3.23)

and

\[ \frac{(-)^{s}w(pp'; s\lambda; s'\lambda')}{(p-p')^{2} + \mu^{2}} = -4\pi^{2} \sum_{k'} (-)^{M+1+(k+k')/2} (k' + 1) R_{k'}(\lambda | p', | p |; \mu) \]

\[ \times \sum_{n, M, l, m} \left\{ \begin{array}{ccc} 1 & n+M & k' \\ 2 & 2 & 2 \\ n-M & 1 & k \\ 2 & 2 & 2 \\ k & k & 0 \\ 2 & 2 & 2 \end{array} \right\} K_{n; l; \lambda}^{[M]}(\delta, \theta, \phi) K_{n; l; \lambda'}^{[M]}(\delta', \theta', \phi') \] (3.24)

where we have symmetrized the quantum number $M$ by (2.28).

We expand the helicity amplitude with the quantum numbers $n, [M], l$ and $m$ in terms of the modified generalized harmonics as

\[ \phi_{\lambda_{i}n[M]}^{(i)}(p; s\lambda) = \sum_{k} \chi_{k}^{(i)n[M]}(|p|) K_{n; l; \lambda}^{[M]}(\delta, \theta, \phi), \] (3.25)

where we call $\chi_{k}^{(i)n[M]}(|p|)$ the partial wave amplitudes in analogy with a three-dimensional one. We can readily derive integral equations for the partial wave amplitude by substituting (3.25) in (3.13)~(3.16) and by using the orthogonality (2.25) and (2.26) together with (3.23) and (3.24). Their explicit forms are as follows: from (3.13) and (3.16) in the case $s=1$

\[ \sum_{k} (k+1) \left\{ \begin{array}{ccc} 1 & n+M & k \\ 2 & 2 & 2 \\ 1 & n-M & k \\ 2 & 2 & 2 \\ 1 & 1 & 0 \end{array} \right\} [\chi^{(i)} - p^{2}\chi^{(i)} + i |p| \chi^{(i)B \alpha}]_{n[M]}^{[M]}(|p|) = 0 \] (3.26)

and

\[ \sum_{k} (k+1) \left\{ \begin{array}{ccc} 1 & n+M & k \\ 2 & 2 & 2 \\ 1 & n-M & k \\ 2 & 2 & 2 \\ 1 & 1 & 0 \end{array} \right\} [\chi^{(i)} - p^{2}\chi^{(i)} + i |p| \chi^{(i)B \alpha}]_{n[M]}^{[M]}(|p|) = 0, \] (3.27)
from (3·13) and (3·16) in the case \( s=0 \) (hence \( M=0 \))

\[
\sum_k (k+1) \begin{pmatrix} \frac{1}{2} & n & k \\ \frac{1}{2} & n & k \\ \frac{1}{2} & n & k \\ 0 & 0 & 0 \end{pmatrix} \left[ \chi^{(3)} - p^2 \chi^{(1)} + \frac{1}{\sqrt{2}} p \chi^{(1B)} \right]_{k}^{(n)} (|\rho|) \\
= -8\lambda \int d|\rho'||p'|^3 R_{\alpha}(|\rho|, |\rho'|;\mu^2) \sum_k (k+1) \begin{pmatrix} \frac{1}{2} & n & k \\ \frac{1}{2} & n & k \\ \frac{1}{2} & n & k \\ 0 & 0 & 0 \end{pmatrix} \chi^{(1\alpha\nu)}_{k}^{(n)} (|\rho'|) \\
\tag{3·28}
\]

and

\[
\sum_k (k+1) \begin{pmatrix} \frac{1}{2} & n & k \\ \frac{1}{2} & n & k \\ \frac{1}{2} & n & k \\ 0 & 0 & 0 \end{pmatrix} \left[ \chi^{(1)} - p^2 \chi^{(1)} + \frac{1}{\sqrt{2}} p \chi^{(1B)} \right]_{k}^{(n)} (|\rho|) \\
= -8\lambda \int d|\rho'||p'|^3 R_{\alpha}(|\rho|, |\rho'|;\mu^2) \sum_k (k+1) \begin{pmatrix} \frac{1}{2} & n & k \\ \frac{1}{2} & n & k \\ \frac{1}{2} & n & k \\ 0 & 0 & 0 \end{pmatrix} \chi^{(1\alpha\nu)}_{k}^{(n)} (|\rho'|) \\
\tag{3·29}
\]

and from (3·14) and (3·15)

\[
\left[ \chi^{(3)} - p^2 \chi^{(3)} + \frac{1}{\sqrt{2}} p \chi^{(3B)} \right]_{k}^{(n)} (|\rho|) \\
= 4\lambda \sum_{k'} (-)^{M+1+(k+k')/2} (k+1) \begin{pmatrix} \frac{1}{2} & n+M & k' \\ \frac{1}{2} & n+M & k' \\ \frac{1}{2} & n+M & k' \\ \frac{1}{2} & n+M & k' \end{pmatrix} \\
\times \int d|\rho'||p'|^3 R_{\alpha}(|\rho|, |\rho'|;\mu^2) \chi^{(3\alpha\nu)}_{k'}^{(n)} (|\rho'|) \\
\tag{3·30}
\]

and

\[
\left[ \chi^{(3)} - p^2 \chi^{(3)} + \frac{1}{\sqrt{2}} p \chi^{(3B)} \right]_{k}^{(n)} (|\rho|) 
\]
It should be remarked, in the above equations, that the quantum number $M$ takes only the value 0 or $\pm 1$ because $|M| \leq s = 0$ or 1. In the following, we discuss the solutions of the equations according to $[M] = [0]$ or $[\pm 1]$.

(i) The case $[M] = [0]$

In this case, $k$ or $k'$ is restricted to $n+1$ or $n-1$ for $n \geq 1$ and is to 1 for $n = 0$, lest the 9-$j$ symbols should vanish.

We get from (3.26) minus (3.27)

$$\sum_k (k+1) \begin{pmatrix} 1 & n & k \cr 2 & 2 & 2 \end{pmatrix} \chi_k^{J=4,n=0}(p) = 0 \quad (3.32)$$

and from (3.28) minus (3.29)

$$(1+\lambda)[\sum_k (k+1) \begin{pmatrix} 1 & n & k \cr 2 & 2 & 2 \end{pmatrix} \chi_k^{J=0,n=0}(p)]$$

$$= 8\lambda \int dp' |p'|^2 R_k(|p|, |p'|; \mu) \sum_k (k+1) \begin{pmatrix} 1 & n & k \cr 2 & 2 & 2 \end{pmatrix} \chi_k^{J=0,n=0}(p'). \quad (3.33)$$

It becomes clear from (3.25) and (2.24) that Eq. (3.32) implies the vanishing of the helicity amplitude $\phi_{J=4,n=0}(p; 1\lambda)$. Equation (3.33) is equivalent to Goldstein's one$^6$ (integrated over the angular variables), and its solution for $\mu = 0$ is known to be
\[ \sum_{k} (k+1) \left( \begin{array}{ccc} 1 & n & k \\ 2 & 2 & 2 \\ \\ 0 & 0 & 0 \end{array} \right) \chi_{k+1}^{(\Sigma) n(k)}(|\rho|) = 2B_{n+2}^{(n)} |\rho|^{n} F(\rho + n + 2, 1 - \rho; n + 2; -\rho^{2}), \]  

(3.34)  

where

\[ \rho = -\frac{1}{2}(n+1) + \left[\frac{1}{4}(n+1)^{2} - 4\right]^{1/2}. \]  

(3.35)  

It is well known that the solution (3.34) exists for any positive value of \( \lambda \). Thus, from (3.25), (2.24) and (3.34) with (3.35), we can get the helicity amplitude \( \phi_{n(\Sigma) n(\Sigma)}(\rho; 00) \) which corresponds to the pseudo-scalar part of the original amplitude with \( 4 \times 4 \) components.

It is difficult to find the other solutions because of the complicated features of the equations. When \( n=0 \) exceptionally, it is easy to find the solutions. The solution found by Bastai et al.\(^{10}\) should be contained in this case. We do not, however, discuss this problem farther.

(ii) \( \text{The case } [M] = [\pm 1] \)

In this case \( k \) and \( k' \) are restricted to \( n(\geq 1) \). Hence the system of equations is much simplified.

Equations \( \text{(3.26)} \) and \( \text{(3.27)} \) become

\[ [\chi^{(1)} - p^{2} \chi^{(1)} + i p |\chi^{(1 \Sigma)}|]_{n(\pm 1)}^{(n \Sigma \pm 1)}(|\rho|) = 0 \]  

(3.36)  

and

\[ [\chi^{(1)} - p^{2} \chi^{(1)} + i p |\chi^{(1 \Sigma)}|]_{n(\pm 1)}^{(n \Sigma \pm 1)}(|\rho|) = 0. \]  

(3.37)  

Equations \( \text{(3.30)} \) and \( \text{(3.31)} \) become

\[ [\chi^{(3)} - p^{2} \chi^{(3)} + i p |\chi^{(1 \Sigma)}|]_{n(\pm 1)}^{(n \Sigma \pm 1)}(|\rho|) = -4\lambda \int d|p'|| |p| R_{n}(|\rho|, |p'|; \mu^{2})_{n(\pm 1)}^{(n \Sigma \pm 1)}(|\rho'|) \]  

(3.38)  

and

\[ [\chi^{(5)} - p^{2} \chi^{(5)} + i p |\chi^{(1 \Sigma)}|]_{n(\pm 1)}^{(n \Sigma \pm 1)}(|\rho|) = -4\lambda \int d|p'|| |p| R_{n}(|\rho|, |p'|; \mu^{2})_{n(\pm 1)}^{(n \Sigma \pm 1)}(|\rho'|), \]  

(3.39)  

where we have used the relation

\[ \left( \begin{array}{ccc} 1 & n \pm 1 & n \\ 2 & 2 & 2 \end{array} \right) = (-1)^{n+1} \left( \begin{array}{ccc} n \pm 1 & 1 & n \\ 2 & 2 & 2 \end{array} \right) \]  

(3.39)  

for \( n \geq 1 \).
It should be noticed that all the equations for the partial wave amplitudes are degenerate with respect to \([M] = [+1]\) and \([-1]\) values.

We get another equation of the Goldstein type from (3·38) minus (3·39):

\[
(1 + p^2)\chi^{(\text{GS})}_n^{[\pm 1]}(|\rho|) = 4\lambda \int d|\rho'||\rho'||^3 \frac{R_n(|\rho|, |\rho'||; \mu^2)}{n + 1} \chi^{(\text{GS})}_n^{[\pm 1]}(|\rho'|). \tag{3.41}
\]

The amplitude appearing here corresponds to the longitudinal vector part and is solved for \(\mu = 0\) in a similar fashion to (3·34).

By eliminating \(\chi^{(\text{GS})}\) from (3·36) plus (3·37) and (3·38) plus (3·39), we get the equation

\[
(1 + p^2)\chi^{(\text{GS})}_n^{[\pm 1]}(|\rho|) = -4\lambda (1 - p^2) \int d|\rho'||\rho'||^3 \frac{R_n(|\rho|, |\rho'||; \mu^2)}{n + 1} \chi^{(\text{GS})}_n^{[\pm 1]}(|\rho'|), \tag{3.42}
\]

which is equivalent to Kummer’s equation obtained by imposing the Lorentz condition on the axial-vector part.\(^9\) Here, we can obtain this equation without imposing any condition because the \(O(4)\) decomposition extracts automatically the effective components for Kummer’s equation. The solution for \(\mu = 0\) has the discrete spectrum specified by the quantum number \(N(n + 1)\) and is given by

\[
\chi^{(\text{GS})}_n^{[\pm 1]}(|\rho|) = 2B_n^{[\pm 1]} P_n^{(1 - p^2)} (1 + p^2)^{\frac{N + n + 1}{2}} \left(\frac{1 - p^2}{1 + p^2}\right), \tag{3.43}
\]

where \(P\) denotes Jacobi’s polynomial\(^1\) and

\[
\alpha = 2N - n + 2\left[2N(N-n) + (n + \frac{1}{2})^2\right]. \tag{3.44}
\]

The discrete eigenvalue is given by

\[
-2\lambda \chi_n^{[\pm 1]} = 3N(N-n) + n(n + \frac{1}{2}) + (2N-n)\left[2N(N-n) + (n + \frac{1}{2})^2\right]. \tag{3.45}
\]

The other amplitudes \(\chi^{(1\pm 1)}(|\rho|)\) and \(\chi^{(2\pm 1)}(|\rho|)\) are readily obtained from the algebraic equations (3·36) and (3·37), especially the amplitude \(\chi^{(2\pm 1)}(|\rho|)\) vanishes.

Finally we notice that the spin \(s\) is restricted to unity for the amplitude belonging to \([M] = [+1]\).

\section*{§ 4. Normalization of Kummer’s solution}

In the previous section, we have investigated the solutions of the fermion-antifermion B-S equation by introducing the helicity amplitudes which have definite transformation properties under the \(O(4)\) group. In particular we have seen that all the helicity amplitudes belonging to the representation \([M] = [+1]\) are analytically found in the case \(\mu = 0\).

The purpose of this section is to investigate the normalization properties of Kummer’s solution for \(\mu = 0\) which belongs to the representation \([M] = [+1]\).
The complete form of Kummer's solution is given from (3.25) with (3.43) by

\[ \phi_{\text{Kum}}^{(2\otimes 3)}(p; s\lambda) = 2\mathcal{D}^\text{Kum}_{\pm 1} \left| \frac{p}{1 - p^2} \right|^\nu (1 - p^2) P_{n+\nu-1}^2 \frac{1}{1 + p^2} K_{\nu+1}^{\text{Kum}}(\delta, \theta, \varphi). \]  

(4.1)

It is impossible at least for \( \mu = 0 \) that the equation of Goldstein type (3.41) and Kummer's equation (3.42) have the non-vanishing solutions simultaneously because of the different signs of \( \lambda \). If we take Kummer's solution as non-vanishing, the solution of (3.41) must vanish. We have therefore

\[ \phi_{\text{Kum}}^{(2\otimes 3)}(p; s\lambda) = 0. \]  

(4.2)

The other solutions belonging to the representation \([M] = [\pm 1]\) are readily obtained from (3.25) with (3.36) and (3.37):

\[ \phi_{\text{Kum}}^{(1\otimes 3)}(p; s\lambda) = 0 \]  

(4.3)

and

\[ (1 - p^2) \phi_{\text{Kum}}^{(1\otimes 3)}(p; s\lambda) = -2i p \phi_{\text{Kum}}^{(2\otimes 3)}(p; s\lambda). \]  

(4.4)

Now, the Wick-rotated normalization condition \((\vartheta, \varphi)\rightarrow(\vartheta, -\varphi)\) for the original amplitude with the quantum numbers \(n, [M], N, l, m \) reads

\[ \int d^4p \text{Tr} \left[ \phi_{\text{Kum}}^{(2\otimes 3)}(p) (1 + i\gamma^5 p) \phi_{\text{Kum}}^{(2\otimes 3)}(p) (1 + i\gamma^5 p) \right] = \lambda_{\text{Norm}}^{(2\otimes 3)}(s) \]  

(4.5)

where we neglect the possible \( l \) dependence of the derivative of \( \lambda \) for simplicity. The conjugate amplitude \( \tilde{\phi} \) is related to the original one by

\[ \tilde{\phi}_{\text{Kum}}^{(2\otimes 3)}(p) = i\lambda_{\text{Norm}}^{(2\otimes 3)}(p, -p_\lambda). \]  

(4.6)

We can rewrite the normalization condition (4.5) in terms of the helicity amplitudes (3.9) by using the completeness of the helicity spinors \( u^{(\pm)}(p, \lambda) \) and the relation

\[ \gamma_u u^{(\pm)}(p, -p_\lambda) = e^{\pm i\lambda} u^{(\pm)}(p, \lambda), \]  

(4.7)

which comes from (3.8) by noticing that the polar coordinate of \((p, -p_\lambda)\) is given by \((|p|, \pi - \delta, \theta, \varphi)\). In our case, the situation is much simplified by virtue of the relations (4.2) \(\sim\) (4.4), and we get

\[ \frac{1}{2} \sum_{\pm} \int d^4p \left( \frac{1 + p^2}{1 - p^2} \right)^\nu (1 - p^2) P_{n+\nu-1}^2 \frac{1}{1 + p^2} \lambda_{\text{Norm}}^{(2\otimes 3)}(s) \]  

(4.8)

Next, we investigate the time-reversed helicity amplitude appearing in (4.8). By means of (2.31) we can derive the relation
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\[ K_{\pm \ell}^{\pm \lambda}(\pi-\theta, \theta, \phi) = \pm (-)^{n-\ell} K_{\pm \ell}^{\pm \lambda}(\theta, \theta, \phi), \quad (4.9) \]

where the symmetrization of the quantum number $M$ plays an important role. Hence the helicity amplitude has a definite $\rho_r$-parity as follows:

\[-\lambda^\pm \ell \varphi_{\pm \ell}^{\pm \lambda}(p_\ell - p_\ell; s \lambda) = \pm (-)^{n-\ell} \lambda^\pm \ell \varphi_{\pm \ell}^{\pm \lambda}(p_\ell; s \lambda). \quad (4.10)\]

Equation (4.8) is converted, by substituting (4.10) and moreover (4.1), into

\[ \pm (-)^{n-\ell+1} 2^{-\alpha-n-1} B_N^{\alpha \pm \ell} \left[ \frac{1}{2} \int_0^1 dx \frac{1}{(1-x)^{\alpha} - (1-x)^{\alpha-1}} \right] (1+ x)^{\alpha+1} \Gamma(N+n-1+1) \]

\[ = \lambda_{\alpha \pm \ell} \left| \frac{d\lambda_{\alpha \pm \ell}(s)}{ds} \right|_{s=0}, \quad (4.11) \]

where we put \( x = - (1-p^2)/(1+p^2) \). The integration in (4.11) is easily carried out by using the orthogonality of Jacobi's polynomial and its recurrence formula.\(^19\)

The result is

\[ \pm (-)^{n-\ell+1} 2^{-\alpha-n-1} B_N^{\alpha \pm \ell} \left[ \frac{(\alpha-2N+n) N! \Gamma(\alpha+N-n)}{\alpha(\alpha+2N+n)(N-n-1)! \Gamma(\alpha+N+1)} \right] \]

\[ = \lambda_{\alpha \pm \ell} \left| \frac{d\lambda_{\alpha \pm \ell}(s)}{ds} \right|_{s=0}. \quad (4.12) \]

It is quite natural to assume the r.h.s. of (4.12) to be negative.\(^4\) Therefore the sign of the norm of Kummer's solution becomes $\pm (-)^{n-1}$. It should be noticed that the sign $(\pm)$ appearing here is not the sign of $M$ itself but of $[M]$ which depends on the type of combinations of $M = +1$ and $M = -1$ amplitudes.

It is interesting to remember that the sign of the norm in the unequal-mass Wick-Cutkosky model at the vanishing total four-momentum becomes $(-)^{n-1}.\(^{10, 20}\)

\[ § 5. \text{Further discussion} \]

First we discuss the space inversion of relative coordinates, under which the B–S amplitude specified by certain quantum numbers $\tau$ is transformed as

\[ [\phi_\tau(p)]^\tau = - \gamma \phi_\tau(-p, -p_\ell \gamma_4). \quad (5.1) \]

Equation (5.1) is applicable not only to the bound state with vanishing total four-momentum, but also to that with a finite total energy if it is limited to the rest frame.

We can convert (5.1) into that for the helicity amplitudes by using the relations on the helicity spinors

\[ \gamma \phi^{ (+)}(-p, p_\ell, \lambda) = e^{i\pi/2} \phi^{ (-)}(p, -\lambda) \quad (5.2) \]

\(^4\) The validity of this assumption can be demonstrated in the Wick-Cutkosky model\(^10\) and some other models\(^20\) but unfortunately not in our model.
and
\[ \gamma_{d^4}(-p, p_0, \lambda) = e^{x_2/2}u^{(\lambda)}(-p, -\lambda), \quad (5.3) \]
which are derived from (3·8) and (2·33) by noticing that the polar coordinates of \((-p, p_0)\) are given by \((|p|, \theta, \pi - \theta, \varphi \pm \pi)\), where the upper and lower signs correspond to \(0 \leq \varphi < \pi\) and \(0 \leq \varphi - \pi < \pi\), respectively. The resultant equation is
\[ [\phi_r^{(i)}(p; s, \lambda)]^n = -(-)^i\phi_r^{(i-1)}(-p, p_0; s, \lambda). \quad (i = 1 \ldots 4) \quad (5.4) \]

From (2·30) and (2·33), we have further a relation on the modified generalized harmonics,
\[ K_{n/m}^{(M)}(\theta, \pi - \theta, \varphi \pm \pi) = \frac{M}{|M|} (-)^i K_{n/m}^{(M+1)}(\theta, \varphi \pm \pi), \quad (5.5) \]
where we put \(M/|M| = 1\) for \(M = 0\). From this, if the helicity amplitudes \(\phi_r^{(i)}(p; s, \lambda)\) have the form of (3·25), Eq. (5·4) reduces to
\[ [\phi_r^{(i)}(p; s, \lambda)]^n = -\frac{M}{|M|} (-)^{i+s} \phi_r^{(i+s)}(p; s, \lambda). \quad (i = 1 \ldots 4) \quad (5.6) \]

Hence we have the four \(p\)-parity-definite amplitudes
\[ [\phi_r^{(i)}(p; s, \lambda)]^n = -\frac{M}{|M|} (-)^{i+s} \phi_r^{(i+s)}(p; s, \lambda), \]
\[ [\phi_r^{(i)}(p; s, \lambda)]^n = \frac{M}{|M|} (-)^{i+s} \phi_r^{(i+s)}(p; s, \lambda), \]
\[ [\phi_r^{(i)}(p; s, \lambda)]^n = -\frac{M}{|M|} (-)^{i+s} \phi_r^{(i+s)}(p; s, \lambda), \]
\[ [\phi_r^{(i)}(p; s, \lambda)]^n = \frac{M}{|M|} (-)^{i+s} \phi_r^{(i+s)}(p; s, \lambda), \]
\[ [\phi_r^{(i)}(p; s, \lambda)]^n = -\frac{M}{|M|} (-)^{i+s} \phi_r^{(i+s)}(p; s, \lambda), \]
\[ [\phi_r^{(i)}(p; s, \lambda)]^n = \frac{M}{|M|} (-)^{i+s} \phi_r^{(i+s)}(p; s, \lambda), \]
\[ [\phi_r^{(i)}(p; s, \lambda)]^n = -\frac{M}{|M|} (-)^{i+s} \phi_r^{(i+s)}(p; s, \lambda), \]

where we put \(M/|M| = 1\) for \(M = 0\). In particular, the \(p\)-parity becomes \(\pm (-)^{i}\) for Kummer's solution \(\phi_r^{(i)}(p; s, \lambda)\), the sign of the norm of which is equal to \(\pm (-)^{i}\) as discussed in the previous section.

We notice that the symmetrization of the quantum number \(M\) plays an essential role to make both definite \(p\)-parity amplitudes and definite \(p_{i}\)-parity ones, and all the solutions which have been solved in § 3 have automatically the definite parity form.

The physical meanings of the quantum numbers \(n\) and \(M\) which have been used throughout this paper are the four-dimensional total angular momentum and the four-dimensional total spin helicity of a two-particle system, respectively. It is well known that if we treat the four-dimensional total angular momentum \(n\) as the eigenvalue instead of \(\lambda\) in Eqs. (3·26) \~ (3·31), the eigenvalue of \(n\) means the Lorentz (or Toller) pole and \(n-l\) gives the number of the daughter trajectory.
in the Regge pole theory. Kummer's solution which has the discrete spectrum will be Reggeized as in the Wick-Cutkosky model by using the prescriptions presented in this paper. Mueller\(^{19}\) investigated the N-N scattering in connection with the classification of Lorentz poles by an analogous method to ours. Similar investigations were made by Ito\(^{22}\) and Lipinski and Snider.\(^{23}\) However, there are many questions still unsolved in the Reggeization of our model. The analysis of the B-S equation in finite energy is hardly done, and it is quite unknown what happens in this energy range.

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**Appendices**

(A) *Spinor-spinor model with scalar coupling*

In this appendix, we deal with the B-S equation for the fermion-antifermion system exchanging scalar particles. We confine ourselves only to writing down the system of equations for the helicity amplitudes which corresponds to Eqs. (3.13) ~ (3.16) of vector coupling model.

The Wick-rotated B-S equation reads

\[
[1 + i\gamma \cdot p] \phi(p) [1 + i\gamma \cdot p'] = \frac{\lambda}{\pi^2} \int d^4 p' \frac{\phi(p')}{(p - p')^2 + \mu^2}.
\]  

(A.1)

Then the helicity amplitudes defined by (3.9) satisfy the equations

\[
[\phi^{(1)} - p^2 \phi^{(4)} + i|p|\phi^{(1\otimes 0)}]\cdot \sum_{\nu,\nu'} \frac{\varpi(p'p; \lambda\lambda') \phi^{(1)}(p'; \lambda')}{(p - p')^2 + \mu^2}.
\]

(A.2)

\[
[\phi^{(2)} - p^2 \phi^{(3)} + i|p|\phi^{(1\otimes 0)}] = \frac{\lambda}{2\pi^2} \int d^4 p' \sum_{\nu,\nu'} \frac{\varpi(p'p; \lambda\lambda') \phi^{(2)}(p'; \lambda')}{(p - p')^2 + \mu^2}.
\]

(A.3)

\[
[\phi^{(3)} - p^2 \phi^{(3)} + i|p|\phi^{(1\otimes 0)}] = \frac{\lambda}{2\pi^2} \int d^4 p' \sum_{\nu,\nu'} \frac{\varpi(p'p; \lambda\lambda') \phi^{(3)}(p'; \lambda')}{(p - p')^2 + \mu^2}.
\]

(A.4)

and

\[
[\phi^{(4)} - p^2 \phi^{(1)} + i|p|\phi^{(1\otimes 0)}] = \frac{\lambda}{2\pi^2} \int d^4 p' \sum_{\nu,\nu'} \frac{\varpi(p'p; \lambda\lambda') \phi^{(4)}(p'; \lambda')}{(p - p')^2 + \mu^2}.
\]

(A.5)
where \( \omega \) is given by \((3\cdot12)\), and \( \omega^{(1)} \) and \( \omega^{(2)} \) are defined by

\[
\omega^{(1)}(p^\prime; s\lambda\lambda') = \pi^2 \sum_n \frac{H^{\pm}_{m\nu}(\delta, \theta, \varphi) H^{\pm\nu}_{m\nu}(\delta', \theta', \varphi')}{(p - p')^2 + \mu^2}
\]  

\( (A\cdot6) \)

and

\[
\omega^{(2)}(p^\prime; s\lambda\lambda') = \pi^2 \sum_n \frac{H^{\pm}_{m\nu}(\delta, \theta, \varphi) H^{\pm\nu}_{m\nu}(\delta', \theta', \varphi')}{(p - p')^2 + \mu^2}
\]  

\( (A\cdot7) \)

(B) Scalar-spinor model with scalar coupling

The B–S equation for the scalar-spinor particle system exchanging scalar particles in the ladder approximation at the vanishing total momentum was already investigated by Sugano and Munakata.4) They found that this model has the discrete spectrum. In this appendix, we re-examine the model in our method.

The Wick-rotated basic equation of our model reads

\[
[1 + p^2][1 + i\not{p}] \phi(p) = \frac{\lambda}{\pi^2} \int d^4p' \frac{\phi(p')}{(p - p')^2 + \mu^2} \]  

\( (B\cdot1) \)

where we put the masses of constituent particles equal to unity. The helicity amplitudes defined by

\[
\phi^{(1)}(p, \lambda) = u^{(+\lambda)}(p, \lambda) \phi(p), \]

\( (B\cdot2) \)

\[
\phi^{(2)}(p, \lambda) = u^{(-\lambda)}(p, \lambda) \phi(p)
\]

satisfy the equations

\[
[1 + p^2][\phi^{(1)} + i\not{p}\phi^{(2)}] \phi(p, \lambda)
\]

\[
= \frac{2\lambda}{\pi^2} \int d^4p' \sum_{\lambda^\prime} \omega^{(1)}(p^\prime; \frac{1}{2}\lambda\lambda') \phi^{(1)}(p^\prime, \lambda') \]  

\( (B\cdot3) \)

and

\[
[1 + p^2][\phi^{(2)} + i\not{p}\phi^{(1)}] \phi(p, \lambda)
\]

\[
= \frac{2\lambda}{\pi^2} \int d^4p' \sum_{\lambda^\prime} \omega^{(2)}(p^\prime; \frac{1}{2}\lambda\lambda') \phi^{(2)}(p^\prime, \lambda') ,
\]  

\( (B\cdot4) \)

where \( \omega^{(1)} \) and \( \omega^{(2)} \) are given by \((A\cdot6)\) and \((A\cdot7)\).

The kernels of \((B\cdot3)\) and \((B\cdot4)\) are expanded in terms of the generalized \( O(4) \) harmonics as follows:

\[
\frac{\omega^{(1)}(p^\prime; \frac{1}{2}\lambda\lambda')}{(p - p')^2 + \mu^2} = \pi^2 \sum_{nM\ell m} \frac{R_{n+M}(|p|, |p'|; \mu^2)}{n + M + 1} H^{\pm}_{n+M/2}(\delta, \theta, \varphi) H^{\pm\nu}_{n+M/2}(\delta', \theta', \varphi')
\]  

\( (B\cdot5) \)

and

\[
\frac{\omega^{(2)}(p^\prime; \frac{1}{2}\lambda\lambda')}{(p - p')^2 + \mu^2} = \pi^2 \sum_{nM\ell m} \frac{R_{n-M}(|p|, |p'|; \mu^2)}{n - M + 1} H^{\pm}_{n-M/2}(\delta, \theta, \varphi) H^{\pm\nu}_{n-M/2}(\delta', \theta', \varphi')
\]  

\( (B\cdot6) \)
where we have used the relation
\[
\left( \begin{array}{ccc}
\frac{1}{2} & n+M & n-M \\
2 & 2 & 2 \\
0 & n-M & n-M \\
2 & 2 & 2 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
\end{array} \right) = \frac{1}{2(n-M+1)} \text{ for } n \geq |M| = \frac{1}{2},
\]
 otherwise.

If we expand the helicity amplitudes by
\[
\phi_{\pm}^{(i)nm}(\rho, \lambda) = \chi^{(i)nm}(\rho) R_{\pm 1/2}^{(0)}(\theta, \varphi), \quad (i = 1, 2)
\]
the partial wave amplitudes \( \chi^{(i)nm}(\rho) \) satisfy
\[
[1 + \rho^2] [\chi^{(i)} + i \rho |\chi^{(y)}|^{2M}(\rho)]
= 2 \lambda \int d|\rho'| |\rho'|^3 \frac{R_{n+M}(\rho, |\rho'|; \mu^2) \chi^{(i)nm}(\rho')}{n+M+1}
\tag{B.9}
\]
and
\[
[1 + \rho^2] [\chi^{(y)} + i \rho |\chi^{(i)}|^{2M}(\rho)]
= 2 \lambda \int d|\rho'| |\rho'|^3 \frac{R_{n-M}(\rho, |\rho'|; \mu^2) \chi^{(y)nm}(\rho')}{n-M+1}.
\tag{B.10}
\]

In the following, we investigate the solutions of Eqs. (B.9) and (B.10) in the case \( \mu = 0 \).

The B–S equation in the Wick-Cutkosky model integrated over the angular variables reads
\[
(1 + \rho^2) \varphi_{\pm L}(\rho)
= 2 \lambda \varphi_{\pm L}(\rho)
\tag{B.11}
\]
and its solution is given by
\[
\varphi_{\pm L}(\rho) = \frac{\rho^L}{(1 + \rho^2)^{L+3/2}} C_{\kappa} L^{L+3/2} \left( \frac{1 - \rho^2}{1 + \rho^2} \right), \quad (\kappa = 0, 1, 2 \ldots)
\tag{B.12}
\]
and
\[
\lambda_{\pm L} = (L + \kappa + 1)(L + \kappa + 2).
\tag{B.13}
\]

Here we expand the partial wave amplitudes \( \chi^{(i)nm}(\rho) \) in terms of \( \varphi_{\pm L}(\rho) \) as
\[
\chi^{(i)nm}(\rho) = \sum_{\pm L} \lambda_{\pm L} \varphi_{\pm L}(\rho).
\]
\[ Z_{c}^{(1)nM}(|p|) = \sum_{k=0}^{\infty} a_{k} \varphi_{k+n,M}(|p|) \]
\[ Z_{c}^{(2)n-M}(|p|) = \sum_{k=0}^{\infty} b_{k} \varphi_{k-n,M}(|p|) \]

where we assume that at least either \( a_{0} \) or \( b_{0} \) is not zero. The meaning of the additional quantum number \( \nu \) is not yet known in this step.

We have only to consider the \( M = \frac{1}{2} \) case, because Eqs. (B·9) and (B·10) for \( M = -\frac{1}{2} \) are converted to those for \( M = \frac{1}{2} \) by interchange of \( \chi^{(1)} \) and \( \chi^{(2)} \).

By substituting (B·14) in (B·9) and (B·10) and by using (B·11), we get the equations

\[ \sum_{k=0}^{\infty} a_{k} \frac{\lambda - \lambda_{k+y,n+1/2}}{\lambda_{k+y,n+1/2}} \left( \frac{1+x}{2} \right) C_{k+y}^{n+2}(x) = i \sum_{k=0}^{\infty} b_{k} \frac{\lambda}{\lambda_{k+y,n-1/2}} C_{k+y}^{n+2}(x) \]

and

\[ \sum_{k=0}^{\infty} b_{k} \frac{\lambda - \lambda_{k+y,n-1/2}}{\lambda_{k+y,n-1/2}} C_{k+y}^{n+2}(x) = i \sum_{k=0}^{\infty} a_{k} \frac{\lambda}{\lambda_{k+y,n+1/2}} \frac{1-x}{2} C_{k+y}^{n+2}(x) \]

where we put \( x = (1-p^{2})/(1+p^{2}) \).

The formulas of the Gegenbauer polynomial\(^{4}\)

\[ C_{k+y}^{n+1}(x) = \frac{n+1}{k+y+n+1} \left[ C_{k+y}^{n+2}(x) - C_{k+y}^{n+2}(x) \right] \]

and

\[ xC_{k+y}^{n+2}(x) = \frac{1}{2(k+y+n+2)} \left[ (\kappa+y+1)C_{k+y+1}^{n+2}(x) + (\kappa+y+2n+3)C_{k+y-1}^{n+2}(x) \right] \]

convert Eqs. (B·15) and (B·16) into algebraic equations for \( a_{k} \) and \( b_{k} \):

\[ a_{k-1} \frac{\lambda - \lambda_{k+y-1,n+1/2}}{4\lambda_{k+y-1,n+1/2}} (\kappa+y+2n+2) + a_{k} \frac{\lambda - \lambda_{k+y-2,n+1/2}}{2\lambda_{k+y-2,n+1/2}} \]
\[ + a_{k-1} \frac{\lambda - \lambda_{k+y-3,n+1/2}}{4\lambda_{k+y-3,n+1/2}} (\kappa+y+1) \]
\[ = -ib_{k} \frac{\lambda(n+1)}{\lambda_{k+y,n-1/2}(\kappa+y+n+1)} + ib_{k-1} \frac{\lambda(n+1)}{\lambda_{k+y-2,n-1/2}(\kappa+y+n-1)} \]
\[ (\kappa = 0, 1, 2 \cdots) \]

and

\[ -ia_{k-1} \frac{\lambda(\kappa+y+2n+2)}{4\lambda_{k+y-1,n+1/2}(\kappa+y+n+1)} + ia_{k-1} \frac{\lambda}{2\lambda_{k+y-2,n+1/2}} \]
\[ -ia_{k-1} \frac{\lambda(\kappa+y-2)}{4\lambda_{k+y-3,n+1/2}(\kappa+y+n-1)} \]
Generalized $O(4)$ Harmonics and Bethe-Salpeter Equation

\[
\begin{align*}
&= -b_e \frac{\left(\lambda - \lambda_{\nu+n+1/2} \right) (n + 1)}{\lambda_{\nu+n+1/2} (\nu + n + 1)} + b_{e-2} \frac{\left(\lambda - \lambda_{\nu+n-1/2} \right) (n + 1)}{\lambda_{\nu+n-1/2} (\nu + n - 1)}, \\
&\quad (\nu = 0, 1, 2 \cdots) \tag{B.20}
\end{align*}
\]

where $a_e$ and $b_e$ with negative $\kappa$ are put equal to zero.

The coefficients $a_e$ and $b_e$ can be obtained by successive iterations. We find easily that $b_0 = 0$, $a_0 \neq 0$ and that in order to obtain $b_i$ consistently $\lambda$ must be

\[
\lambda = \frac{1}{2} \lambda_{\nu+n+1/2} = \frac{1}{2} \left( \nu + n + \frac{3}{2} \right) \left( \nu + n + \frac{5}{2} \right), \quad \nu + n + \frac{3}{2} = 2, 3, 4 \cdots \tag{B.21}
\]

Thus, we get the discrete eigenvalue $\lambda$ specified by the quantum numbers $\nu$ and $n$.

References

7) H. S. Green, Phys. Rev. 97 (1955), 540.
20) N. Nakanishi, Phys. Rev. 147 (1966), 1153.