

where α , A , θ_0 , and n are oil "constants." Values of α were taken from Dyson, Naylor, and Wilson [2]; a Slotte relation was fitted to the values of viscosity at 30 (or 40), 60, and 100 deg C given in [2] or, for Vitrea 79, kindly supplied by A. Dyson of Shell Research Limited. No data on the thermal conductivity of the oils is available except for oil A at 40 deg C, for which $K = 0.117$ watt/m deg C; the same value was used for all the oils and seems unlikely to be seriously wrong.

Table 1 Oil Properties

	A Units vary	θ_0 (deg C)	n	$\alpha(m^2/N)$ at 30 deg C)
Oil A [2]	2.53×10^5	22	3.5	2.5×10^{-8}
Oil B [2]	1.11×10^4	15	3	2.70×10^{-8}
Oil F [2]	1.91×10^4	8	3	3.10×10^{-8}
Vitrea 79	5.70×10^6	20.3	4	2.5×10^{-8}

The data in Table 1 refer to viscosity in N -sec/m²: One N -sec/m² = 10 poise. In fact α falls as temperature rises, while the viscosity becomes more sensitive to temperature when the pressure increases. For the temperature rises considered the first effect is small, and the second will give rise to a greater reduction in film thickness than our theory predicts. Dyson and Wilson [11] obtain a rather different value of n for oil A ($n = 3.96$ at 40 deg C) but the important quantity, which is

$$-\frac{1}{\eta} \frac{d\eta}{d\theta} = \frac{n}{\theta_0 + \theta} \quad (42)$$

is unchanged.

APPENDIX B

Higher-Order Formulas

The first-order correction to the film thickness as found in the text can be supplemented by higher terms as follows. The modified Reynolds' equation (13) can be expanded in series to give

$$\frac{dp}{dx} = 12\eta_s U \frac{h - h_0}{h^3} \left[1 - \frac{n}{2} t + \frac{n(n+2)}{12} t^2 - \frac{n(n+2)}{24} t^3 + \dots \right] \quad (43)$$

Substituting for t from equation (17)

$$\frac{dp}{dx} = 12\eta_s U \frac{h - h_0}{h^3} [1 - a_1 z + a_2 z^2 - a_3 z^3 + \dots] \quad (44)$$

where $z = \frac{3}{4} \frac{\eta_s U^2}{K(\theta_0 + \theta_s)} \left(\frac{h - h_0}{h} \right)^2$ and $a_1 = n/2$; $a_2 = n(19n + 14)/84$; $a_3 = n(471n^2 + 959n + 490)/5880$. Thus, proceeding as before (see equation (30)),

$$\frac{1}{\alpha} = 12\eta_0 U \int_0^\infty \left[\frac{h - h_0}{h^3} - b_1 e^{\alpha p} \frac{(h - h_0)^3}{h^5} + b_2 e^{2\alpha p} \frac{(h - h_0)^5}{h^7} - b_3 e^{3\alpha p} \frac{(h - h_0)^7}{h^9} + \dots \right] dx \quad (45)$$

The use of the isothermal pressure distribution in the integral is now more dubious than before, and the consequent error in the higher terms uncertain. It is, nevertheless, of interest to obtain them. Substituting for $e^{\alpha p}$ and making Crook's approximation for the film shape, it is found that the effective value of α is

$$\bar{\alpha} = \alpha \left[1 - 0.4808a_1 \frac{\eta_0 U^2}{K(\theta_0 + \theta_s)} + 0.2463a_2 \left(\frac{\eta_0 U^2}{K(\theta_0 + \theta_s)} \right)^2 - 0.1325a_3 \left(\frac{\eta_0 U^2}{K(\theta_0 + \theta_s)} \right)^3 + \dots \right] \quad (46)$$

For oil A at 40 deg C this becomes

$$\bar{\alpha} = \alpha [1 - 1.566 \times 10^{-2} U^2 + 2.86 \times 10^{-4} U^4 - 4.88 \times 10^{-6} U^6 + \dots] \quad (47)$$

The higher terms shift the maximum value of $(\bar{\alpha}U)$, and so the maximum film thickness, only slightly, and it seems clear that the simple expression in the text, equation (36), can be used up to the maximum, while neither will be accurate much above it.

DISCUSSION

A. Dyson³

An analytical solution, even of limited applicability, often provides greater physical insight than a numerical solution obtained from a computer, and the authors are to be congratulated on solving a difficult problem. My aim in this contribution is to derive a solution in the form of a series expansion by an approach which differs from that of the authors, and to compare a suggested higher-order analytical approximation with published results obtained by numerical methods.

Equations (1) and (6) of the paper give

$$K \frac{d^2\theta}{dy^2} + \frac{1}{\eta_c} \left(\frac{dp}{dx} y \right)^2 \left(\frac{\theta_0 + \theta}{\theta_0 + \theta_c} \right)^n = 0$$

where the reference condition has been changed from the surface, suffix s , to the central plane of the lubricant film, suffix c . This equation may be put in the form

$$\frac{d^2\phi}{dz^2} + z^2\phi^n = 0 \quad (48)$$

where

$$z = y \left(\frac{dp}{dx} \right)^{1/2} [K\eta_c(\theta_c + \theta_0)]^{-1/4}$$

$$\phi = (\theta + \theta_0)/(\theta_c + \theta_0)$$

Equation (6) of the paper may be written in the form

$$K \frac{d^2\theta}{dy^2} + \frac{dp}{dx} y \frac{d\theta}{dy} = 0$$

Integration with respect to y between the limits 0 and $h/2$, with the second term being integrated by parts, gives

$$K \left(\frac{d\theta}{dy} \right)_s = -1/2 \frac{dp}{dx} U(h - h_0) \quad (49)$$

where the usual continuity argument has been applied to give

$$\int_0^{h/2} u dy = 1/2 U h_0$$

Equation (49) may be rewritten in terms of the dimensionless variables which were introduced previously:

$$\left(\frac{d\phi}{dz} \right)_s = 1/2 U(h - h_0) \eta_c^{1/4} \left(\frac{dp}{dx} \right)^{1/2} [K(\theta_c + \theta_0)]^{-3/4} \quad (50)$$

From the definition of z , it is known that

$$z_s = 1/2 h \left(\frac{dp}{dx} \right)^{1/2} [K(\theta_c + \theta_0)\eta_c]^{-1/4} \quad (51)$$

and from equation (1) of the paper

$$\eta_c = \eta_s \phi_s^n \quad (52)$$

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Equations (50), (51), and (52) give

$$\psi = \phi_s^{-(n+1)} z_s^{-2} \left(\frac{d\phi}{dz} \right)_s^2 \quad (53)$$

$$\chi \equiv \eta_o/\eta_s = -\frac{1}{3} z_s^3 \phi_s^n \left(\frac{d\phi}{dz} \right)_s^{-1} \quad (54)$$

where η_o is an effective viscosity defined by

$$\frac{dp}{dx} = 12\eta_o U (h - h_o) h^{-3}$$

and

$$\psi \equiv \frac{\eta_s U^2 (h - h_o)^2}{K(\theta_s + \theta_o) h^2}$$

The parameter ψ describes the external conditions defining the problem and the parameter χ represents the desired solution.

Elimination of z and of ϕ between equations (48), (53), and (54) enables χ to be expressed as a function of ψ .⁴ Differentiation of equations (53) and (54), with the use of equation (48) gives

$$\frac{d\psi}{dz} = -z^{-3} \phi^{-(n+2)} \frac{d\phi}{dz} \left[2\phi \frac{d\phi}{dz} + (n+1) z \left(\frac{d\phi}{dz} \right)^2 + 2z^3 \phi^{n+1} \right] \quad (55)$$

$$\frac{d\chi}{dz} = \left[-z^2 \phi^{n-1} / 3 \left(\frac{d\phi}{dz} \right)^2 \right] \left[3\phi \frac{d\phi}{dz} + nz \left(\frac{d\phi}{dz} \right)^2 + z^3 \phi^{n+1} \right] \quad (56)$$

Division of equation (55) by equation (56) with the use of (53) and (54) gives

$$[2 - 3(n+1)\chi\psi - 6\chi] \psi \frac{d\chi}{d\psi} + 3\chi(1 - n\chi\psi - \chi) = 0$$

⁴ I am indebted to my colleague, Mr. A. Jones, for a useful suggestion concerning the method of elimination.

This equation may be solved by expressing χ as a power series in ψ , substituting in the equation and equating to zero the coefficients of terms in all powers of ψ :

$$\begin{aligned} \chi = 1 - (3n\psi/7) + 9n\psi^2(12n+7)/539 \\ - 9n(202n^2 + 273n + 98)\psi^3/18,865 \\ + 27n(48,010n^3 + 105,882n^2 + 81,683n + 22,638) \\ \times \psi^4/27,599,495 + \dots \quad (57) \end{aligned}$$

Equation (44) of the paper, expressed in the present notation, is

$$\begin{aligned} \chi = 1 - (3n\psi/8) + 3n\psi^2(19n+14)/448 \\ - 9n\psi^3(471n^2 + 959n + 490)/12,544 + \dots \quad (58) \end{aligned}$$

In the present treatment, the coefficients a of Appendix B of the paper must be modified to give new values:

$$\begin{aligned} a_1' &= 4n/7 \\ a_2' &= 16n(12n+7)/539 \\ a_3' &= 64n(202n^2 + 273n + 98)/56,595 \end{aligned}$$

With these modified values, equation (46) of Appendix B of the paper gives

$$\begin{aligned} \bar{\alpha}/\alpha = 1 - 0.2747X + 0.08774[1 + (7/12n)]X^2 \\ - 0.0303(1 + 1.3515n^{-1} + 0.4841n^{-2})X^3 + \dots \quad (59) \end{aligned}$$

where

$$X \equiv \frac{n\eta_o U^2}{K(\theta_o + \theta_s)} \equiv \frac{U^2}{K} \left(-\frac{d\eta_o}{d\theta} \right)_s$$

Equation (59) is a perturbation that is valid only for small values of X . It is required of a full solution that $\bar{\alpha}/\alpha$ tend to zero as X tends to infinity, and the simple binomial expression

$$\bar{\alpha}/\alpha \approx (1 + aX)^{-b}$$

satisfies this condition and reproduces the first three terms of the expansion of equation (59) provided

$$\begin{aligned} b^{-1} &= 1.3255 + 1.3565n^{-1} \\ a &= 0.3641 + 0.3726n^{-1} \end{aligned}$$

For a numerical evaluation, it is convenient to take values of b

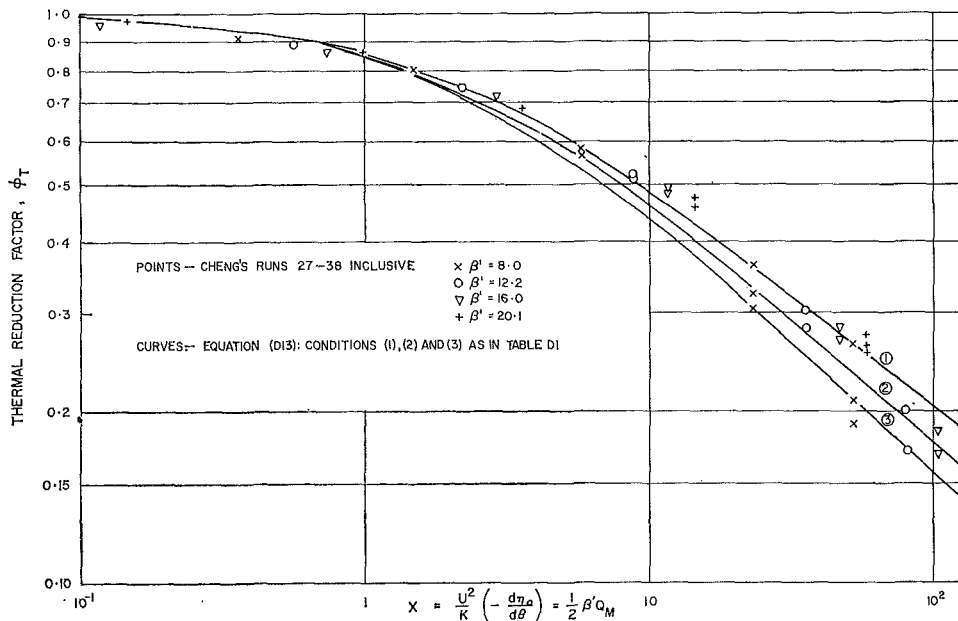


Fig. 7 Comparison between numerical results and analytical approximation

corresponding approximately to the limits of n that are likely to be encountered. The values chosen are given in Table 2.

Table 2 Selected values of constants

Condition	a	b	n	c	Error, % in coefficient of X^3
1	0.515	8/15	2.46	0.40	-4.5
2	0.458	$3/5 = \frac{9}{15}$	3.95	0.45	-3.8
3	0.412	$2/3 = \frac{10}{15}$	7.70	0.50	-3.7

The thermal reduction factor, ϕ_T , defined by Cheng⁵ as the ratio of the minimum film thickness, with thermal effects considered, to the isothermal value, is

$$\phi_T = (\bar{\alpha}/\alpha)^{3/4} \simeq (1 + \alpha X)^{-c} \quad (60)$$

where $c = 3b/4$, is also given in Table 2.

Cheng uses an Arrhenius relation to describe the variation of viscosity with temperature:

$$\frac{\eta}{\eta_0} = \exp \left[\alpha p + (\beta + \gamma p) \left(\frac{1}{T} - \frac{1}{T_0} \right) \right]$$

where T is the absolute temperature.

I see nothing in Cheng's treatment that specifies the zero of the temperature scale, however, and T may be replaced everywhere by $(T - T_0')$ where T_0' is any arbitrary constant temperature. The applicability of Cheng's treatment may therefore be extended to include a viscosity variation of the Vogel-Doolittle type, which describes the properties of practical lubricants over a much wider range than does the Arrhenius equation. For a direct comparison with the present treatment, Cheng's results have been expressed in terms of

$$X \equiv \frac{U^2}{K} \left(- \frac{d\eta_0}{d\theta} \right) = \frac{1}{2} Q_m \beta'$$

in Cheng's notation.

The comparison is shown in Fig. 7. The theoretical curves marked 1, 2, and 3 correspond to the entries in Table 2 for the constants a and c in equation (60). The marked points are for runs 27-38 inclusive of Cheng.⁵ These are all in pure rolling. The agreement is much better than could be expected from the rather drastic approximation that leads to equation (60).

Equation (60), with the constants shown in Table 2, is offered as a convenient analytical guide to thermal effects in film thickness in pure rolling, valid up to $X \simeq 100$, $\phi_T \simeq 0.2$.

K. L. Johnson⁶

The authors have made a most valuable contribution to the calculation of elastohydrodynamic film thicknesses, in providing a correction to the widely used isothermal formulae to allow for heating in the entry region. With common grades of oil the correction becomes important at high speed, but with very viscous oils, the term $d\eta/d\theta$ is usually large and I have found the correction to be significant at quite slow speeds.

The calculation of film thickness response to the temperature rise, given in the second part of the paper, is quite complicated; it would seem that a similar result can be obtained by a simple approximate treatment, which helps to reveal the essential mechanism of the process.

Isoviscous theory results in a temperature rise at the center of the film $\Delta\theta$ given by equation (18) when $\eta_s = \eta_0$, i.e.,

$$\Delta\theta = \frac{3}{4} \frac{\eta_0 U^2}{K} \left(1 - \frac{h_0}{h} \right)^2$$

⁵ Cheng, H. S., "Calculation of Elastohydrodynamic Film Thickness in High Speed Rolling and Sliding Contacts," Mechanical Technology Incorporated, Report MTL-67 TR24, May 1967.

⁶ Engineering Laboratory, Cambridge University, Cambridge, England.

The bracket varies from 1 to 0 throughout the entry region so that we might take 0.5 as an average value. The temperature profile for an isoviscous lubricant is parabolic so that the average temperature rise across the film is 2/3 that at the center. Hence a rough average value for the temperature rise in the film is:

$$\overline{\Delta\theta} \simeq \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \frac{\eta_0 U^2}{K} = \frac{\eta_0 U^2}{4K}$$

To a first approximation this rise in temperature will reduce the viscosity of the oil at entry from η_0 to η_0' , where

$$\eta_0' = \eta_0 \left(1 + \frac{d\eta_0}{d\theta} \overline{\Delta\theta} \right) = \eta_0 \left(1 + \frac{U^2}{4K} \frac{d\eta_0}{d\theta} \right)$$

The film thickness is influenced by the viscosity according to equation (34) i.e.,

$$\begin{aligned} \frac{h'}{h} &\simeq \left(\frac{\eta_0'}{\eta_0} \right)^{3/4} \\ &\simeq \left[1 + \frac{U^2}{4K} \frac{d\eta_0}{d\theta} \right]^{3/4} \\ &\simeq 1 + \frac{3}{16} \frac{U^2}{K} \frac{d\eta_0}{d\theta} + \dots \end{aligned}$$

This result is close to that of the authors expressed in equation (37).

Authors' Closure

We are most grateful to Mr. Dyson for his contribution, and we must congratulate him on his ingenious extrapolation formula which agrees so well with Professor Cheng's numerical solutions. This adds enormously to the value of the paper by providing a simple equation with which one can estimate thermal effects. We might perhaps emphasize that the equation is an extrapolation, and not a direct result of Mr. Dyson's success in obtaining an exact series solution in place of our approximate solution. A similar extrapolation can be applied to our equation (46) and leads to a very similar result (Dyson, private communication). Although it should certainly be kept in mind that some features of the numerical solutions (e.g., thermal conduction within the disks) are not included in the parameter $X \equiv \frac{U^2 |d\eta|}{K |d\theta|}$ and will therefore lead to variations from the existing results, it seems that these variations may be minor and that for a wide range of conditions Dyson's expression can be used to predict the thermal reduction factor.

Dr. Johnson's contribution is most helpful in bringing out the mechanism of the thermal effect: it is no complicated differential action such as is needed to explain the lubrication of contra-rotating disks⁷ but a direct result of the reduced viscosity in the inlet. This suggests that while we were mathematically correct to regard the effect as a reduction in the value of α , it would have been better to regard it as a reduced value of η_0 . It also suggests that we can estimate the importance of thermal effects in other cases. For example if we have one disk moving with velocity U_1 while the other is stationary, both disks being thermal conductors, the maximum temperature rise across the film with an isoviscous lubricant is

$$\frac{\eta U_1^2}{16K} \left[9 - 10 \frac{h_0}{h} + 3 \left(\frac{h_0}{h} \right)^2 \right]$$

varying from $\frac{9}{16} \frac{\eta U_1^2}{K}$ at infinity to $\frac{1}{8} \frac{\eta U_1^2}{K}$ when $h = h_0$. The appropriate average is probably about $\frac{3}{8} \frac{\eta U_1^2}{K}$ compared to

⁷ Dyson, A., and Wilson, A. R., "Film Thicknesses in EHL at High Slide/Roll Ratios," *Proc. I. Mech. E.*, Vol. 183, Part 3P, 1968-1969, pp. 81-97.

$\frac{3}{8} \cdot \frac{\eta U^2}{K}$ for pure rolling; and we may expect to get the thermal reduction factor for pure sliding from that in pure rolling by replacing U by U_1 . A similar calculation may be made for a

conducting disk rolling or sliding over a stationary nonconducting disk (glass or sapphire) and suggests that thermal effects will then appear at much lower speeds.