Determination of the rotation of Mercury from satellite gravimetry

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ABSTRACT
Space missions can have as a goal the determination of the interior structure of a planet: this is the case for the ESA Bevicolombo mission to Mercury. Very precise range and range-rate tracking from the Earth and onboard accelerometry will provide a huge amount of data, from which it will be possible to study the gravity field of Mercury and other parameters of interest. Gravity can be used to constrain the interior structure, but cannot uniquely determine the interior mass distribution. A much stronger constraint on the interior can be given by also determining the rotation state of the planet. If the planet is asymmetric enough, the gravity field as measured by an orbiting probe tracked from the Earth contains signatures from the rotation. Are these enough to solve for the rotation state, to the required accuracy, from tracking data alone, without measurements of the surface? In order to reach some result analytically, a simplified analytical model is developed, and the symmetry breaking, occurring when the shape of the planet deviates from spherical symmetry, is characterized by explicit formulae. Moreover, a full cycle numerical simulation of the Radio Science Experiment is performed, including the generation of simulated tracking and accelerometer data and the determination, by least-squares fit, of the Mercury-centric initial conditions of the probe, of Mercury’s gravity field and its rotation state, together with other parameters affecting the dynamics. The conclusion is that there is no reason of principle prohibiting the determination of the rotation from gravimetry, and the sensitivity of the measurements and the coverage are good enough to perform the experiment at the required level of accuracy. This will be important also in ensuring independent terms of comparison for the rotation experiment performed with a high-resolution camera. The mission is currently under development and much care has to be taken in guaranteeing the scientific goals even if there is some change in scenario.

Key words: methods: analytical – methods: data analysis – methods: numerical – techniques: radial velocities – celestial mechanics.

1 INTRODUCTION
According to science historians, Isaac Newton refused to publish his solution of the two-body problem until he had the proof that a spherically symmetric body generates, outside the body, the same gravity field as a point mass placed at the centre of mass. Unfortunately, this also proves that the inverse gravimetry problem is ill-posed: even a perfect knowledge of the gravity field outside a body does not allow us to solve for the internal mass distribution.

A question that often arises during the discussion on the scientific goals of planetary exploration missions is: can a planetary mission constrain the internal structure without landing on the planet? Newton’s classical result, and its modern versions, shows that even a perfect knowledge of the gravity field outside the surface of the planet does not constrain the concentration of the mass towards the centre, and thus does not allow us to constrain the size and density of the core. For example, six coefficients of the moment of inertia tensor are linearly related to the five harmonic coefficients of degree 2: if the latter are measured with remote gravimetry, one unconstrained parameter remains.

A solution of the problem could be to directly observe the rotation state of the planet. From a suitably defined obliquity of the rotation axis it is possible to estimate the absolute value of the principal moment of inertia, thus scaling correctly the moment of inertia tensor. It is also possible to measure the libration in longitude resulting from the coupling of the permanent equatorial ellipticity of Mercury with the Sun’s tidal pull, and from this to detect the presence of a decoupling (liquid layer) between core and mantle (see Peale 1988).

A first full analysis on this topic was done by Wu, Bender & Rosborough (1995), who investigated the possibility of measuring Mercury’s rotation by a lander–orbiter system. Under the hypothesis of very accurate range/range-rate tracking data from the Earth and good ranging of the lander from the orbiter, the results presented are very good, with total uncertainties for the libration in
longitude amplitude and the obliquity of 0.26 and 0.03 arcsec, respectively. For technical, and thus economic, reasons, for instance, the environment at the surface of Mercury makes it very difficult to design a lander which is able to stand for long enough to make the experiment successful, this experiment has never been performed in reality. However, it has been a strong source of inspiration for the ESA BepiColombo mission to Mercury.

In fact, other techniques for measuring the rotation state of a planet have been proposed later: e.g. by using a high-resolution camera (as will be attempted by the ESA BepiColombo mission to Mercury), or by using the radar images of the surface as done by the Cassini mission with Titan in Stiles et al. (2008), or by Margot et al. (2007) for the rotation of Mercury.

This rotation experiment proposed for BepiColombo imposes very tough constraints on the thermo-mechanical design of the orbiter, because of the need to measure spacecraft to surface directions in an absolute reference frame. Thus, it would have been desirable to measure the rotation from the gravity field above the surface. In principle, the time-dependent gravity field generated by a rotating planet, measured in an inertial frame, depends on the rotation state. Thus, by tracking a satellite for a long period of time and taking accurate measurements, it could be possible to measure the planetary rotation without looking at the planet.

A theoretical difficulty in determining the rotation state by using only measurements of the gravity field is the presence of an approximate symmetry in the orbit determination problem. The approximate symmetry results from the breaking of the exact symmetry applicable to a spherically symmetric planet: if it rotates, the gravity field does not change, thus the orbit of a satellite does not change and the tracking data do not contain any signal due to rotation.

This allows us to understand that the accuracy in the determination of the rotation state of Mercury from gravimetry is proportional to the magnitude of the harmonic coefficients of the planet's gravity field.

One of the main instruments of the ESA BepiColombo mission to Mercury is the so-called Mercury Orbiter Radio Science Experiment (MORE; see Benkhoff et al. 2010), which addresses scientific goals in geodesy, geophysics and fundamental physics. It consists of the determination of a large number of parameters characterizing the dynamics of the spacecraft tracked from the Earth, e.g. the spherical harmonic coefficients of the gravity field of Mercury $C_{lm}$ and $S_{lm}$, some general relativity parametrized post-Newtonian (PPN) parameters, the initial conditions of Mercury's centre of mass, the initial conditions of the spacecraft and several other quantities such as Mercury's rotation parameters (obliquity and libration in longitude), in a global least-squares fit of the observable tracking data and onboard camera images (see e.g. Milani et al. 2001, 2002).

In this work we analyse the problem of the determination of the rotation of Mercury, and the main issues on the state of its core, in the framework of a global parameter estimation process. The parameters describing the rotation of Mercury are determined by a global least-squares fit of the tracking data alone (the gravimetry experiment). This gives a contribution to the more complex experiment that includes the data from the high-resolution camera (the rotation experiment). The simulation and the data analysis are performed with a realistic measurement accuracy.

The paper is organized as follows. In Sections 2 and 3 a general description of the Mercury rotation theory is given. In Section 4 we describe the technique proposed for the determination of the rotation of Mercury and the state of its core. In Sections 5 and 6 we study this problem with an analytically simplified model, while in Section 7 a realistic model is presented. A full numerical simulation giving concrete and quantitative results about the feasibility of the gravimetry experiment is described in Sections 8 and 9.

2 SECULAR THEORY: LAPLACE PLANE AND CASSINI STATE

The planet Mercury follows a highly eccentric orbit around the Sun ($e \approx 0.206$), and it shows a particular spin–orbit resonance of 3:2 (Fig. 1). This resonance is explained by the theory of Colombo (Colombo & Shapiro 1966), which considers Mercury as a rigid body and combines the ellipticity of the orbit and of Mercury’s equator with the Sun’s gravitational torque acting on it in a secular perturbation approach. The exact resonance is a minimum energy condition of the system, and the longest axis of the elliptic equator has to be aligned with the Sun when Mercury is at perihelion.

A secular theory of the rotation of a rigid Mercury should also take into account the Cassini laws, by which the (averaged) spin axis has to belong to the Cassini plane, which is the one spanned by the orbital plane normal and the axis around which the orbital plane precesses because of secular planetary perturbations, averaged over long time-scales. Generalizing Cassini’s laws for the Moon, Colombo (Colombo & Shapiro 1966), and later Peale (Peale 1969), proposed a rotation model for Mercury according to which the planet actually occupies a Cassini state of type 1, meaning that the orbit normally lies between the spin axis and the precessional axis. This ‘precessional axis’ is also commonly called the Laplace pole $Z_L$, and it can be determined by averaging the perturbations over a suitable time-span (e.g. see Peale 2006; Yseboodt & Margot 2006; D’Hoedt et al. 2009). The precession period obtained from the theory is around $\sim 250000$ years, and the angle $\eta$ between the orbit normal $Z_1$ and the spin axis $V_1$ is called the obliquity.

We will assume the spin axis coincident with the maximum moment of inertia axis of Mercury $Z_2$ (no wobbling).

2.1 Peale’s experiment

With some approximations, in particular with Mercury in a permanent Cassini state with constant obliquity, Peale (1988) proposes that, with the global gravity field and the rotation state, it is possible to constrain the internal structure of the planet, determining if there is decoupling between a solid surface (mantle) and an inner core

![Figure 1. The 3:2 spin orbit resonance of Mercury. Shown is the elliptic equator of Mercury, whose longest axis is aligned towards the Sun only twice per orbit, resulting in a tidal torque with a period half of the orbital period of the planet.](https://academic.oup.com/mnras/article-abstract/427/1/468/1030686/4714681/100686/4714681/100686)
(presumably due to a liquid layer). Let us call \( A < B < C \) the principal moments of inertia of Mercury as a rigid body (rotation around the axis of maximum moment of inertia is assumed). From this secular theory, the obliquity \( \eta \) of Mercury occupying the Cassini state 1 equilibrium position is given by the following approximated formula (Peale 1988, 2006):
\[
\frac{1}{\eta} = \frac{1}{\sin i_{\ell}} \left( \frac{n J_2 f(e) MR^2}{w_1 - C} - \cos i_{\ell} \right),
\]
where \( M \) is the mass of Mercury, \( R \) is its mean radius, \( f(e) = G_{10}(e) + 2C_{20}(e)J_2 \) and \( J_2 \) are the degree 2 potential coefficients of Mercury’s gravity field (in the principal of inertia body-fixed reference system), \( G_{220}, G_{201} \) are eccentricity functions defined in Kaula (1966), \( n \) is the orbital mean motion, \( i_{\ell} \) is the inclination of the orbit with respect to the Laplace pole \( Z_L \), and \( w_1 \) is the nearly constant rate of the precession of the orbit around it (\( 2\pi/w_1 \approx 250000 \text{ yr} \)). Thus, the obliquity is directly related to the quantity \( CmR^2 \), called the concentration coefficient.

Let us suppose that there is a core decoupled from a rigid mantle, and that the core does not follow the mantle over short time-scales (while it does over long time-scales; Peale 1988). This means that the moments of inertia reacting to the torques over short time-scales are the ones of the mantle alone \( A_m < B_m < C_m \). In particular, assuming rotation around principal axis of inertia, the moment which appears in the rotational kinetic energy is only \( C_m \), i.e. the one with respect to the spin axis. Then, if Mercury has a core decoupled from the rigid mantle, the ratio \( C_m/C \) is not equal to 1, and it is expected to be \( \sim 0.5 \) for a planet with an important liquid layer. Also the concentration coefficient \( CmR^2 \), which is 0.4 for a homogeneous planet, is expected to be significantly less for Mercury. Peale proposes to determine this ratio by the following relation:
\[
\frac{C_m}{C} = \frac{C_m}{B - A} \left( \frac{M R^2}{C} \right) \left( \frac{B - A}{M R^2} \right),
\]
where the first factor can be determined by measuring the short-term effects on the rotation of Mercury (librations in longitude; see Section 3), the second factor is measured by the obliquity and the third factor is measured by the harmonic coefficient \( C_{22} \):
\[
\frac{B - A}{M R^2} = 4 C_{22}.
\]
Thus, it should be possible, by measuring the rotation state of Mercury along with its gravity field, to constrain both the physical state and the state of Mercury’s core. An error budget for the study of the interior structure of Mercury would consist of measuring \( C_m/C \) with a relative accuracy of around 10 per cent and \( C/\mu R^2 \) with a relative accuracy of around 1 per cent (Milani et al. 2001).

Possible secular deviations of the spin axis \( Z_2 \) from the Cassini state equilibrium position can be modelled by suitable corrective constants.

### 3 SHORT-TERM PERTURBATION THEORY: THE LIBRATIONS IN LONGITUDE

We have seen that, according to Peale (1988), an interior model of Mercury consisting of a multi-layer, rigid mantle decoupled from the core implies that the short-term rotation of the planet refers to the rotation of the mantle alone. Thus, assuming no wobbling, the moment of inertia appearing in the kinetic energy is \( C_m \). A first general analysis of the short-term effects has been done considering only the torque of the Sun on the elliptic equator of Mercury. More recently, the indirect effects of planetary perturbations on the rotation of Mercury have been considered (e.g. see Peale, Yseboodt & Margot 2007, 2009; Dufey, Lemaitre & Rambaux 2008; Dufey et al. 2009; Yseboodt, Margot & Peale 2010).

#### 3.1 Libration effects without planetary perturbations

The gravitational torque from the Sun causes the phenomenon of libration in longitude, which is an oscillation around the secular equilibrium condition of the 3:2 spin–orbit resonance. Let us consider as a first approximation that the obliquity of Mercury is zero, with Mercury rotating around an axis parallel to the orbital angular momentum, and in a two-body approximation. Then, if \( \phi \) is the Mercury rotation angle, measured from the direction of pericentre to the zero meridian on the planet surface given by the axis of minimum moment of inertia (Mercury’s equatorial long axis), and introducing the libration angle \( \gamma = \phi - 3/2 \ell \) (\( \ell \) being the mean anomaly), it is possible to solve analytically for the small oscillations of \( \gamma \) around the equilibrium position \( \gamma = 0 \) (Fig. 2). For a more detailed discussion and for the following formulae (4) and (5), see Balogh & Giampieri (2002) and Jehn, Corral & Giampieri (2004).

Assuming that the free oscillations (which have a period of \( \sim 12 \) yr) have been damped by dissipation effects, the main oscillation term has the orbital period of Mercury (\( \sim 88 \) d). A good approximation of \( \phi \) is of the form:
\[
\phi = \frac{3}{2} \ell + \epsilon \sin \ell + \frac{\mu}{\mu} \sin 2\ell,
\]
where \( \mu \equiv -9.483 \) is a numerical constant depending only on the eccentricity of Mercury \( e \), and \( \epsilon \) is the libration in longitude amplitude.

It is also possible to find an approximate formula for \( \epsilon \). A spherical core implies \( B_m - A_m = B - A \), and an approximate formula for the libration amplitude \( \epsilon \) is
\[
\epsilon = \frac{3}{2} \frac{B - A}{C_m} \left( 1 - 11e^2 + \frac{959}{48} e^4 + \cdots \right).
\]
A reference value for \( \epsilon \) is \( \sim 35 \) arcsec [with \( (B - A)/C_m \equiv 2 \times 10^{-4} \)].

#### 3.2 Libration effects with planetary perturbations

A more complete rotation theory, including also planetary perturbations, has been developed by many authors. Among them, we recall again Dufey et al. (2009) and Yseboodt et al. (2010).

As we have already pointed out, the difficulty in finding a reliable model for the rotation of Mercury lies in the fact that over long time-scales Mercury should behave as a rigid body, and over short time-scales it should behave as a multi-layer body. One of the main properties of these two models is the proper free libration period and
free precession period, being respectively \( \sim 15.85 \) and \( \sim 1065 \) yr in the rigid case (D’Hoedt & Lemaître 2004), and \( \sim 12.06 \) and \( \sim 616 \) yr in the multi-layer model (Dufey et al. 2009), with a value for the ratio of the polar moments of inertia \( C_m/C = 0.579 \). The multi-layer model being the one commonly used for the librations in longitude, the \( \sim 12.06 \) yr free libration term represents a critical value for the amplitudes of planetary perturbation effects with periods close to it. In particular, the \( \sim 11.9 \) yr Jupiter period is evidently close to a resonance and the effect of its perturbation on Mercury’s libration could be large. This near resonance, ruled by the \( C_m \) value, plays a key role in all the studies about planetary perturbations on the rotation of Mercury.

In Yseeboodt et al. (2010), a completely analytical study of the forced planetary perturbations on the librations in longitude is performed, obtaining explicit formulae for the libration amplitudes and phases in an analogous way to the case without planetary perturbations (formula 4). An analytical formula for the rotation angle of Mercury, from an inertial direction to the axis of minimum moment of inertia, is given in the form (Yseeboodt et al. 2010, section 7)

\[
\phi = \alpha t + \sum_{i=1}^{\infty} \phi_i \cos (w_i t + \psi_i) + \phi_0, 
\]

where \( \alpha \) is the secular rate of the rotation angle, \( \phi_i, w_i, \) and \( \psi_i \) are the amplitudes, periods and phases of the periodic part, and \( \phi_0 \) is a constant. This formula is valid over a few tens of years around J2000. All the quantities appearing in (6) are computed assuming as reference epoch J2000, and as inertial reference direction the intersection between the Ecliptic and the orbital plane of Mercury at J2000. In this way an approximate value for \( \alpha \) is \( \alpha = 3/2 n + \sigma_t \), where \( n \) is the mean motion of Mercury at J2000 and \( \sigma_t \) is the mean angular velocity of the longitude of pericentre. The periodic part includes the main terms with period multiples of the orbital period of Mercury and the terms due to planetary perturbations, while the free libration is assumed to be damped. Moreover, all the amplitudes and phases can be expressed as functions of \( (B - A)/C_m \) (like formula 5) and a dissipation parameter.

Among the various trigonometric terms in the series, only a few of them are significant (above the \( \sim 1 \) arcsec level): at most four waves of basic period \( 2\pi n/(\text{arcsec}) \) (whose first and second amplitudes are the same as in Section 3.1 and the phases are zero with respect to the solar anomaly as in formula 4), and five terms with the basic periods depending on Venus, Jupiter, Earth and Saturn (see Table 1; and Yseeboodt et al. 2010, table 2). Besides the main \( \sim 88 \) d libration term, the largest perturbation is due to Jupiter and it is explained by the proximity to the resonance with the \( \sim 12.06 \) yr free libration period.

In Dufey et al. (2009), a comparative analytical and numerical study of the rotation of Mercury is performed. The dynamics are described in a Hamiltonian formalism, applying the Lie averaging process in the analytical study. The results presented are essentially in agreement with those of Yseeboodt et al. (2010).

However, discussing the limitations and the difficulties in the dynamical theory for the rotation of Mercury is beyond the purpose of this paper, which is dedicated to the feasibility of the gravimetry–rotation joint experiment. The aim of this work is not to describe how we can determine the rotation model, but to establish what can be determined by the experiments and thus to constrain the rotation model.

### 4 POSSIBLE METHODS FOR THE DETERMINATION OF THE ROTATION OF MERCURY AND THE STATE OF ITS CORE

As mentioned in Section 2.1, ‘Peale’s experiment’ proposes to constrain the state of Mercury’s core by determining Mercury’s rotation, through the obliquity \( \eta \) and the libration in longitude amplitude \( \epsilon \), and Mercury’s gravity field, through the potential coefficients \( C_{20} \) and \( C_{22} \). A more recent analysis includes the improved model of the obliquity and planetary perturbations (Section 3.2).

After Wu et al. (1995), where a possible experiment for the determination of the rotation of Mercury by a lander–orbiter system is investigated, a few other possible methods have been considered in the past decade:

(i) radar measurements of the surface of the planet from Earth stations;

(ii) onboard high-resolution camera imaging combined with satellite gravimetry by tracking of an orbiter around Mercury from the Earth;

(iii) pure satellite gravimetry by tracking of an orbiter around Mercury from the Earth.

Method number (i) was proposed by Margot et al. (2007), according to which it is possible to establish, from the analysis of the irregularities in the wavefront of the radar echoes from the planet surface, that Mercury occupies a Cassini state with obliquity of \( 2.11 \pm 0.1 \) arcmin and that it exhibits \( 88 \) d librations in longitude, as predicted by theory, of an amplitude of \( 35.8 \pm 2 \) arcsec.

Methods (ii) and (iii) instead were first proposed and studied in Milani et al. (2001) and Sánchez Ortíz, Belló Mora & Jeph (2006). Being part of a dedicated mission to Mercury, they should give tighter constraints to the determination of the obliquity and the librations in longitude.

The main point of this paper is to assess the determination of the rotation of Mercury, and of the state of its core, in case we could not use the direct observations of the surface provided by the onboard camera. The main features of method (iii) can be described as follows.

This method is based on the classical tools of satellite geodesy (Kaula 1966), applied on a satellite orbiting around another planet instead of the Earth (see also Milani & Gronchi 2010, chapter 17).

A static rigid mass distributed in a region \( W \) generates outside \( W \) a potential \( V \). With the centre of mass at the origin of the adopted reference frame, using spherical coordinates \((r, \theta, \lambda)\), \( V \) can be expanded in a spherical harmonics series:

\[
V(r, \theta, \lambda) = \frac{GM}{r} + \sum_{l=2}^{\infty} \sum_{m=0}^{l} \frac{G M R^l}{r^{l+1}} P_l^m (\sin \theta) [C_{lm} \cos m\lambda + S_{lm} \sin m\lambda],
\]

where

<table>
<thead>
<tr>
<th>Period ( 2\pi n/(\text{arcsec}) )</th>
<th>( \phi_i ) (arcsec)</th>
<th>( \psi_i ) (°)</th>
<th>( \phi_i w_i ) (arcsec ( \text{yr}^{-1} ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jupiter</td>
<td>11.86</td>
<td>40.25</td>
<td>-8.43</td>
</tr>
<tr>
<td>Jupiter</td>
<td>5.93</td>
<td>1.37</td>
<td>-175.85</td>
</tr>
<tr>
<td>Venus</td>
<td>5.66</td>
<td>3.59</td>
<td>-92.84</td>
</tr>
<tr>
<td>Earth</td>
<td>6.57</td>
<td>0.58</td>
<td>152.14</td>
</tr>
<tr>
<td>Saturn</td>
<td>14.72</td>
<td>1.6</td>
<td>34.90</td>
</tr>
</tbody>
</table>

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where $P_{22}$ are the Legendre associated functions and $M$, $R$ are the planet’s mass and mean radius, respectively. $C_{10}$, $S_{10}$ are the potential coefficients, whose values depend on the choice of the body-fixed reference system. In this way, the orbit of a satellite around the body contains information about $C_{10}$, $S_{10}$, and measuring the orbit accurately enough, it is possible to solve for them by a least-squares fit. Notice that the orbit determination and the gravity field estimation are in general not independent, and that other effects such as non-gravitational perturbations may have to be considered. This means that, in general, the orbit determination and parameter estimation must be done by a global least-squares fit (Milani & Gronchi 2010, chapter 17).

However, when the rotation state of the body is not well known, the (time-dependent) rotation matrix $R = R(t)$ converting from an inertial reference system to a body-fixed one contains uncertain parameters. For example, the approach described in Sections 2 and 3 gives a rotation matrix that depends on the obliquity $\eta$ and the libration in longitude amplitude $\epsilon$: $R = R(\eta, \epsilon, t)$. In this case the orbit of a satellite around the body contains information about the rotation parameters. A global orbit determination and parameter estimation process could solve for these unknown parameters. The feasibility and the quality of this parameter estimation are highly affected by the quality of the experimental data and by the intrinsic rank deficiencies that can occur, depending on the interactions between the dynamical and the observation model.

In order to simulate an orbit determination and parameter estimation experiment, we need to set up an observation model along with a dynamical model. Then we can compute the predicted values for the observations to compare with the simulated observational data in a least-squares fit (see Section 8.2). In the case of a satellite orbiting around another planet, such as in the MORE experiment, the observational technique is complicated by many factors, but it can be simply considered as a tracking from an Earth-based station, giving range and range-rate information (see Iess & Boscaglì 2001).

An overview of the dynamical model used to compute the observables is given in Fig. 3. To compute the range distance from the ground station on the Earth to the spacecraft (s/c) around Mercury, and the corresponding range-rate, we need the following state vectors, each one with its own dynamics:

(i) the Mercury-centric position of the s/c $x_{\text{sat}}$;
(ii) the Solar system barycentric positions of Mercury and of the Earth–Moon barycentre (EMB) $x_M$ and $x_{\text{EM}}$;
(iii) the geocentric position of the ground antenna $x_{\text{ant}}$;
(iv) the position of the Earth barycentre with respect to the EMB $x_E$.

For a discussion on these dynamical and observation models we refer to Milani et al. (2010) and Tommei, Milani & Vokrouhlický (2010).

When the orbit determination of an object orbiting around another planet is performed by radial and radial velocity observations, there is an important symmetry responsible for the weakness of the orbit determination that is an approximate version of the exact symmetry found in Bonanno & Milani (2002). In our case, if the Mercury-centric orbit is rotated around an axis $\hat{p}$ in the direction from the Earth to the centre of Mercury, then there would be an exact symmetry in the range and range-rate observations if $\hat{p}$ were constant and Mercury spherical. Given that $\hat{p}$ changes with time, the small parameter in the approximate symmetry is the displacement angle by which $\hat{p}$ rotates (in an inertial reference system) during the observation arc time-span. The weak directions $\hat{u}_{dp}$ and $\hat{u}_{dv}$ of the orbit determination in the three-dimensional subspaces of the s/c initial position $r_0$ and velocity $v_0$, respectively, are given by

\[ \hat{u}_{dp} = \frac{\hat{p} \times r_0}{|\hat{p} \times r_0|}, \quad \hat{u}_{dv} = \frac{\hat{p} \times v_0}{|\hat{p} \times v_0|}. \]  

Different solutions can be adopted to stabilize the solution (see Milani & Gronchi 2010, chapter 17 and Section 8.3).

**5 GRAVITY FIELD AND ROTATION STATE: SYMMETRIES AND UNDETERMINED COEFFICIENTS**

Besides the harmonic coefficients, the internal mass distribution of a planet also defines the inertia tensor $T = (I_{ij})$. Let us consider, at the same time and in the same reference system, the inertia tensor and a set of five potential coefficients of degree $l = 2$. It can be shown (Milani & Gronchi 2010, chapter 13) that the following relations hold:

\[ C_{20} = \frac{1}{MR^2} \left( \frac{I_{11} + I_{22}}{2} - I_{33} \right), \quad C_{22} = \frac{(I_{22} - I_{11})}{4MR^2}, \]
\[ C_{21} = -\frac{I_{13}}{MR^2}, \quad S_{21} = -\frac{I_{23}}{MR^2}, \quad S_{22} = -\frac{I_{12}}{2MR^2}. \]

Thus, knowing the potential coefficients is not enough to know the inertia tensor, because there are five equations in six unknowns. That is, there is a one-parameter symmetry group, by which a change in the inertia tensor of the form $T \rightarrow s \text{Id} + T$, where $\text{Id}$ is the unit matrix, does not change the potential coefficients. If $A$, $B$, $C$ are the principal moments of inertia of the body, the symmetry changes $A$, $B$, $C$ by the same additive constant: $A \rightarrow A + s$, $B \rightarrow B + s$, $C \rightarrow C + s$.

In general, the one-parameter symmetry can be broken if we find another independent equation for $A$, $B$, $C$. In the case of Mercury, if we consider decoupling between core and mantle, a further equation is needed to find the parameter $C_{10}$. According to Peale (1988) this can be done if we know the rotation state of the body and the torque acting on it. The problem is then that of determining the rotation state from the gravity field.

In fact, knowing the gravity field is enough to determine the principal axes of inertia, because they are the same for $T$ and $s \text{Id} + T$. At a fixed time, we can diagonalize the tensor $T$ and find the directions of the principal axes of inertia. In the case of a triaxial

![Figure 3](https://example.com/f3.png)

**Figure 3.** Multiple dynamics for the tracking of the s/c around Mercury from the Earth: $x_{\text{sat}}$ is the Mercury-centric position of the s/c, $x_M$ and $x_{\text{EM}}$ are the Solar system barycentric positions of Mercury and of the EMB, $x_{\text{ant}}$ is the geocentric position of the ground antenna and $x_E$ is the position of the Earth barycentre with respect to the EMB.
body ($A < B < C$), this defines instantaneously the rotation state of the body. In contrast, in the case of some rotational symmetry, ($A = B < C$) or ($A = B = C$), it would clearly be impossible to discriminate a rotation around the symmetry axis (or axes) since the gravity field does not change. As discussed in Milani & Gronchi (2010, chapter 6), when the rotation symmetry is broken for a realistic planet, there is still an approximate symmetry that could be significant.

6 ANALYTICAL STUDY: THE MODEL PROBLEM

In this section we set up a simplified model to analytically obtain some results on the determination of the rotation of Mercury from the gravity field.

6.1 Rotation of the gravity field

The spherical harmonics development (7) describes the gravity field generated by a rigid mass with respect to a body-fixed reference frame. If we consider the same development with respect to a reference frame in which the mass is rigidly rotating, formula (7) is valid instantaneously, and we can use it by introducing time dependence in the coefficients:

$$V(r, \vartheta, \lambda, t) = \frac{GM}{r} + \sum_{l=2}^{+\infty} \frac{GM R^l}{r^{l+1}} \sum_{m=0}^{l} P_{lm}(\sin \vartheta) \{ \tilde{C}_{lm}(t) \cos m\lambda + \tilde{S}_{lm}(t) \sin m\lambda \}. \quad (9)$$

If the mass is rotating around a fixed axis $\lambda$, whose inertial direction is given by two angles ($\eta$, $\xi$) (e.g. $\eta$ colatitude), and $\phi = \phi(t)$ is the rotation angle, then the rotation from the inertial to the body-fixed system is the composition of a rotation of $\eta$ around axis-2, a rotation of $\xi$ around axis-1 and a rotation of $\phi$ around axis-3. If $\tilde{C}_{lm}$, $\tilde{S}_{lm}$ are the potential coefficients in a body-fixed system (uniquely determining the gravity field), then the explicit formulae for $\tilde{C}_{lm}(t)$, $\tilde{S}_{lm}(t)$ as functions of $C_{lm}$, $S_{lm}$ can be obtained either by the methods of spherical trigonometry, e.g. Kaula (1966), or from the representation theory of the rotation group $SO(3)$ into the spherical harmonic space of degree $l$ (Wigner 1959).

In the following we present the explicit formulae for degree $l = 2$, assuming also $\xi = 0$ and $\eta$ small:

$$\tilde{C}_{20} = C_{20} - 3\eta S_{21} \cos \phi + 3\eta S_{21} \sin \phi + \mathcal{O}(\eta^2),$$
$$\tilde{C}_{21} = \eta C_{20} - 2\eta C_{22} \cos \phi + 2\eta S_{22} \sin \phi + C_{21} \cos \phi + S_{21} \sin \phi + \mathcal{O}(\eta^2),$$
$$\tilde{C}_{22} = C_{22} \cos 2\phi - S_{22} \sin 2\phi + \frac{1}{2} \eta C_{21} \cos \phi + \frac{1}{2} \eta S_{21} \sin \phi + \mathcal{O}(\eta^2),$$
$$\tilde{S}_{21} = -2\eta C_{22} \sin 2\phi - 2\eta S_{22} \cos 2\phi + C_{21} \sin \phi + S_{21} \cos \phi + \mathcal{O}(\eta^2),$$
$$\tilde{S}_{22} = C_{22} \sin 2\phi + S_{22} \cos 2\phi + \frac{1}{2} \eta C_{21} \sin \phi + \frac{1}{2} \eta S_{21} \cos \phi + \mathcal{O}(\eta^2). \quad (10)$$

We will assume as rotation angle

$$\phi = \frac{3}{2} \ell + \varepsilon \sin \ell. \quad (11)$$

where $\ell$ is the mean anomaly of Mercury and $\varepsilon$ the libration in longitude amplitude. For the purpose of the analytic study, we have neglected the $-\sin 2\ell$ term appearing in (4) because it is a small effect not affecting the result qualitatively. In this way $\varepsilon$ appears in equations (10) through the angle $\phi$. A first-order approximation in the parameters $\eta, \varepsilon$ is also used.

6.2 Reference systems and angles

Let us denote by $(X_0, Y_0, Z_0)$ the Ecliptic J2000 inertial reference frame, and use it for the description of Mercury’s rotation. We also associate an inertial reference frame with the orbit of Mercury at J2000: $(X_1, Y_1, Z_1)$, where $Z_1$ is the orbital plane normal and $X_1 = -X_{\text{peri}}$, with $X_{\text{peri}}$ the pericentre direction at J2000.

Finally, let $(X_2, Y_2, Z_2)$ be the Mercury body-fixed principal of inertia reference frame. $X_2$ is the unit vector along the longest axis of the equator of Mercury (minimum moment of inertia), assumed as rotational reference meridian.

6.3 Simplifying assumptions

In order to be able to handle the problem analytically, we must make some approximations.

Inertial reference system. We use as inertial reference system $(X_1, Y_1, Z_1)$ defined in Section 6.2.

Rotation state. Mercury is rotating around an inertially fixed axis $V_\phi$ by the rotation angle $\phi$. The rotation from the inertial to the body-fixed system is given by a rotation of an angle $\eta$ (the obliquity) around axis $-2$ and a rotation of $\phi$ around axis $-3$. Consistently with Section 3.1, the formula for the angle $\phi$ is given by (11). We also assume that the body-fixed reference system is principal of inertia.

Gravity field. We consider the spherical harmonics series of the gravity potential truncated to degree $l = 2$. Given that the body-fixed reference frame is principal of inertia, the only parameters appearing in the right-hand side of equations (10) are $C_{20}$ and $C_{22}$ (which represent the oblateness and the equatorial ellipticity), the obliquity $\eta$ and the libration amplitude $\varepsilon$ through the angle $\phi$.

Reference satellite orbit. Finally, in order to describe analytically the perturbed orbit of the satellite by a first-order theory, we define a reference unperturbed orbit. We define a circular reference orbit of radius $r_0$, inclination $\pi/2$ and longitude of the ascending node $\Omega$ with respect to the frame $(X_1, Y_1, Z_1)$. We indicate with $\nu$ the true anomaly of the satellite, $t = 0$ is the time of perihelion passage and we assume also $\nu_0 = 0$, the initial time of the satellite orbit. We also indicate with $n$ the satellite mean motion while $n$ is the mean motion of Mercury.

Notice that the hypothesis of a nearly polar orbit is a very good approximation of the expected inclination of the BeptColombo orbit around Mercury. The hypothesis of a circular orbit is a less accurate approximation, since the real orbit is expected to have an eccentricity of $\sim 0.16$.

6.4 Hill's equations

Let $(r_0, \dot{r}_0)$ be a reference satellite in a circular orbit with radius $r_0$, with local coordinates in the radial $\mathbf{F}_0$, along-track $\mathbf{v}_0$ and out of plane $\mathbf{u}_0 = r_0 \times \mathbf{v}_0$ directions, respectively (RTW coordinates). Let the perturbing potential be $\mathcal{D} = V = GM/r$, small compared to the monopole term $GM/r$. We can describe the perturbed satellite relative motion $(\mathbf{r} - r_0, \mathbf{v} - \mathbf{v}_0)$, assuming $|\mathbf{r} - r_0| \ll r_0$, through
the so-called Hill’s equations (Dunning 1973):

\[
\begin{aligned}
\ddot{u} - 2n_v \dot{v} - 3n_u^2 u &= f_u, \\
\ddot{v} + 2n_u \dot{u} &= f_v, \\
\ddot{w} + n_u^2 w &= f_w,
\end{aligned}
\]

where \([r]_{KW} = (r_u, v, w), [v]_{KW} = (\dot{u} - n_v v, \dot{v} + n_v (r_0 + u), \dot{w}), n_v = \sqrt{GM/r_0}\) is the reference mean motion and \((f_u, f_v, f_w)\) is the perturbing force, i.e. the gradient of the perturbing potential \(D\) calculated in the reference satellite position.

Combining formulae (9), (10), (11) and the simplifying above assumptions, we have

\[
D = A_1 + \sum_{i=2}^{11} (A_i \cos (\sigma_i v) + B_i \sin (\sigma_i v)),
\]

where \(\sigma_i \in \{0, 2\}\) and \(\beta_i \in \{0, \pm 2, \pm 3, \pm 4\}\) in all the possible combinations with \(\sigma_i, \beta_i\) not simultaneously zero. \(A_i, B_i\) are known functions of \(C_{20}, C_{22}, n, e, r_0\) and \(\Omega\), while the frequencies \(\sigma_i = \alpha_i n_v + \beta_i n, \) including also the zero frequency, are summarized in Table 2.

Finally, the perturbing force is

\[
f_u = \frac{\partial D}{\partial r_0}; \quad f_v = \frac{1}{r_0} \frac{\partial D}{\partial v}; \quad f_w = -\frac{1}{r_0 \cos v} \frac{\partial D}{\partial \Omega}.
\]

6.5 Rotation from gravimetry analytical results

From a comparison of the expected accuracies for the range and range-rate measurements of the BepiColombo mission (see Section 8.1), it turns out that the gravimetry experiment is performed mainly with the range-rate, being more accurate than the range in measuring short-term changes. Thus, we focus only on the perturbed velocity \(v - v_0\) and in particular on the along-track component \(\dot{v} + n_u\). From formulae (13) we deduce that \(f_u, f_v\) are of the form

\[
f_u = A_1 + \sum_{i=2}^{11} (A_i \cos (\sigma_i t) + B_i \sin (\sigma_i t)),
\]

\[
f_v = \sum_{i=2}^{11} (C_i \cos (\sigma_i t) + D_i \sin (\sigma_i t)),
\]

where \(A_i, B_i, C_i, D_i\) are given by

\[
A_i = \frac{\partial A_i}{\partial r_0} = -\frac{3}{r_0} A_i, \quad B_i = \frac{\partial B_i}{\partial r_0} = -\frac{3}{r_0} B_i,
\]

\[
C_i = \frac{1}{r_0} \alpha_i B_i, \quad D_i = -\frac{1}{r_0} \alpha_i A_i.
\]

Because Hill’s equations are linear, it is possible to obtain a particular solution of the form:

\[
\begin{aligned}
u &= a_1 + \sum_{i=2}^{11} (a_i \cos (\sigma_i t) + b_i \sin (\sigma_i t)), \\
v &= \sum_{i=2}^{11} (c_i \cos (\sigma_i t) + d_i \sin (\sigma_i t)),
\end{aligned}
\]

\[
\dot{u} = \sum_{i=2}^{11} (\dot{a}_i \cos (\sigma_i t) + \dot{b}_i \sin (\sigma_i t)),
\]

\[
\dot{v} = \sum_{i=2}^{11} (\dot{c}_i \cos (\sigma_i t) + \dot{d}_i \sin (\sigma_i t)),
\]

where

\[
\begin{aligned}
a_1 &= \frac{1}{r_0 n_v^2} A_i, \\
b_1 &= c_1 = d_1 = \dot{a}_1 = \dot{b}_1 = \dot{c}_1 = \dot{d}_1 = 0 \quad (20)
\end{aligned}
\]

and \(\forall i > 1:\)

\[
a_i = -3\sigma_i + 2n_v \alpha_i A_i, \\
\frac{b_i}{r_0 n_v \sigma_i} = \frac{3n_v^2 \alpha_i}{r_0 n_v \sigma_i} - \frac{3n_v^2 \alpha_i}{r_0 n_v \sigma_i} B_i, \\
c_i = \frac{n_v^2 \alpha_i}{r_0 n_v \sigma_i} - \frac{6n_v \sigma_i}{r_0 n_v \sigma_i} B_i, \\
d_i = \frac{3n_v^2 \alpha_i + \alpha_i \sigma_i^2 - 6n_v \sigma_i}{r_0 n_v \sigma_i} A_i, \\
\dot{a}_i = \sigma_i b_i, \quad \dot{b}_i = -\sigma_i a_i, \quad \dot{c}_i = \sigma_i d_i, \quad \dot{d}_i = -\sigma_i c_i.
\]

The particular solution has to be summed to the homogeneous solution \((f_u = f_v = f_w = 0):\)

\[
\begin{aligned}
u &= -\frac{2}{3n} z_2 \cos n_v t - \frac{1}{2} z_2 \sin n_v t, \\
v &= z_1 + z_2 t + z_3 \sin n_v t + z_4 \cos n_v t
\end{aligned}
\]

to obtain a general solution. However, by definition, the homogeneous solutions do not contain the signal of the perturbation \(D\); it only contains the signal of the variations of the actual orbit with respect to the circular reference orbit. In particular, focusing on the along-track component, it contains the signal due to a variation in mean motion, initial phase angle, eccentricity and longitude of pericentre \(e \sin \omega, e \cos \omega\).

To justify the validity of Hill’s equations for a long time-span (e.g. one Mercury orbital period), we assume that we have a priori observed the orbit of the s/c for long enough to constrain the reference mean motion \(n_v\) and the initial phase to the real mean motion and initial phase. Moreover, we assume that the true orbit does not deviate too much from a circular one \(|v - v_0|/n_v r_0 \ll 1\). Thus, we do not consider the homogeneous solution because it does not contain the signal of the perturbation, and to reach some analytical conclusions we concentrate our analysis on the solution \(\dot{v} + n_u\).

We assume that it is possible to measure the corresponding signal
with an accuracy $\delta_i$, for a time-span long enough to determine independently all the amplitudes $\hat{e}_i + n_i a_i, \hat{d}_i + n_i b_i$, with the same accuracy $\delta_n$ (actually, the constant term $n_a a_i$ with accuracy $\delta_n/2$). Then it is possible to compute the uncertainties in the determination of the parameters $C_{20}, C_{22}, \eta, \epsilon$ from the observation of the amplitudes of the signal $\hat{v} + n_u$. The explicit formulae for the amplitudes $\hat{e}_i + n_i a_i, \hat{d}_i + n_i b_i$, are given in Table 3, where we have also used the following approximation ($n_s \gg n$):

\[
\frac{3n_i^2 a_i + \sigma_i \sigma_i^2 - 6n_s \sigma_i}{\sigma_i (n_i^2 - \sigma_i^2)} \approx \frac{1}{3n_s} \quad \text{for } i = 2, \ldots, 8;
\]

\[
\frac{3n_i^2 a_i + \sigma_i \sigma_i^2 - 6n_s \sigma_i}{\sigma_i (n_i^2 - \sigma_i^2)} \approx \frac{6}{n_s} \quad \text{for } i = 9, 10, 11.
\]

We immediately deduce from Table 3 that the rotation parameters appear in the potential always multiplied by the potential coefficients. This is the approximate symmetry that we have introduced in Section 5 by which the smaller the potential coefficients, the weaker the rotation of the spin axis) and in Sections 3.1 and 3.2 (librations in longitude).

7 REALISTIC MERCURY-CENTRIC DYNAMICS

Let $\mathbf{r}$ be the Mercury-centric position of the s/c and $a$ its acceleration with respect to an inertial reference system (e.g. the Ecliptic J2000). Then $a = \mathbf{F}/m$ where $\mathbf{F}$ is the force exerted on the s/c and $m$ is its mass. The orbit of an s/c around Mercury has an equation of motion $\ddot{r} = a$ which must contain: the spherical harmonics of the Mercury gravity field; the solar and planetary differential attractions and tidal perturbations; the non-gravitational perturbations, possibly replaced by onboard accelerometer readings; and relativistic corrections.

Let $\mathbf{r}_{IN}$ and $\mathbf{r}_{BF}$ be the s/c coordinates with respect to the inertial and the body-fixed reference system, respectively. The coordinates of the acceleration due to the static gravity field of Mercury in the body-fixed system are $\mathbf{a}_{BF} = \nabla V(\mathbf{r}_{BF})$, where $V$ is the gravitational potential given by formula (7). If $\mathbf{R}$ is the rotation matrix that converts from the inertial to the body-fixed coordinates, we have: $\mathbf{r}_{BF} = \mathbf{R} \mathbf{r}_{IN}$, $\mathbf{a}_{BF} = \mathbf{R} \mathbf{a}_{IN}$ and $\mathbf{a}_{IN} = \mathbf{R}^T \nabla V(\mathbf{R} \mathbf{r}_{IN})$.

7.1 Rotational dynamics

For the computation of the rotation matrix $\mathbf{R}$, we use a semi-empirical model: an explicit analytical formula for the rotation of Mercury is obtained combining the discussions in Section 2 (direction of the spin axis) and in Sections 3.1 and 3.2 (librations in longitude).

We use the reference systems defined in Section 6.2, where the body-fixed system considered $\mathbf{r}_{BF}$ is the one with principal axes. The principal axis $X_2$ is assumed as zero meridian. Let $\mathbf{R}_0$ be the rotation matrix converting from the inertial Ecliptic J2000 ($X_0, Y_0, Z_0$) to the inertial orbital reference system ($X_1, Y_1, Z_1$).

Then we need to compute the rotation from the orbital to the body-fixed system (for time-scales of the order of a few years).

As a first approximation, we define the direction of the spin axis with respect to the orbital reference system by two constant angles $(\delta_1, \delta_2)$:

\[
[Z_2]_1 = (\sin \delta_1 \cos \delta_2, -\sin \delta_2, \cos \delta_1 \cos \delta_1)^T.
\]

The obliquity $\eta$ is simply given by

\[
\cos \eta = \cos \delta_2 \cos \delta_1.
\]

In this way, the spin direction is ‘model independent’, meaning that we solve for its direction without assuming the Cassini state. In our

| $n_a a_i$ | $\frac{1}{3} C_{20}$ |
| $v_2 + n_a a_2$ | $-\frac{1}{3} C_{20}$ |
| $v_3 + n_a a_3$ | $-\frac{1}{3} \epsilon C_{20} \cos 2\Omega$ |
| $v_4 + n_a a_4$ | $\frac{1}{3} C_{20} \sin \Omega + \frac{1}{3} C_{20} \sin 2\Omega$ |
| $v_5 + n_a a_5$ | $\frac{1}{3} \epsilon C_{20} \sin 2\Omega$ |
| $v_6 + n_a a_6$ | $-\frac{1}{3} C_{20} \cos 2\Omega$ |
| $v_7 + n_a a_7$ | $-\frac{1}{3} \epsilon C_{20} \cos 2\Omega$ |
| $v_8 + n_a a_8$ | $\frac{1}{3} C_{20} \cos 2\Omega$ |
| $v_9 + n_a a_9$ | $-\frac{1}{3} \epsilon C_{20} \cos 2\Omega$ |
| $v_{10} + n_a a_{10}$ | $\frac{9}{3} C_{20} \cos 2\Omega$ |
| $v_{11} + n_a a_{11}$ | $\frac{9}{3} C_{20} \cos 2\Omega$ |

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simulation we chose the reference values for the direction of the spin axis in an arbitrary way, without assuming any knowledge of the Cassini plane from the theory. If \( \phi \) is the rotation angle, we define the rotation matrix \( \mathbf{R} \) as

\[
\mathbf{R} = \mathbf{R}_z(\phi) \mathbf{R}_y(\phi_0) \mathbf{R}_y(\phi_1) \mathbf{R}_z(\phi_2),
\]

where \( \mathbf{R}_x(\alpha) \) is the matrix associated with the rotation by an angle \( \alpha \) about the \( i \)th axis (\( i = 1, 2, 3 \)). The explicit formula for the rotation angle \( \phi \) can be obtained from formula (6), keeping only the terms significant over a 1 year mission time-span. In general, it is the sum of a secular term, some periodic libration terms and a constant which depends on the choice of the zero meridian.

At this point, we need to specify what exactly we want to determine and what we do not. We have seen that, if the threshold of accuracy of the experiment is of the order of 1 arcsec, we would need to keep about 10 trigonometric terms in the formula for \( \phi \).

However, the largest effects are by far the following three:

(i) the main \( \sim 88 \) d term due to Mercury: \( \epsilon_1 \sin(n t) \);
(ii) the second harmonic term \( \sim 44 \) d due to Mercury: \( \epsilon_1 \mu \sin(2 n t) \) where \( \mu = -9.483 \) from Jehn et al. (2004) depends only on the eccentricity and it is considered here as a known constant;
(iii) the \( \sim 11.9 \) yr near resonant term due to Jupiter: \( \epsilon_2 \cos(w_j t + \psi_j) \);

while the other effects due to Venus, Earth and Saturn are smaller effects that are impossible to determine independently in a 1 year mission time-span, because of the closeness of their periods and the smallness of their amplitudes. Since they are far from the resonance, a good nominal value for their amplitudes can be provided by the theory, and a possible small error in the nominal amplitudes can be absorbed by a constant (see the following discussion about the \( S_{22} \) coefficient). In order to simplify the analysis in this simulation, we have chosen not to add these effects at all.

The libration amplitudes that we want to determine in the simulated experiment are just \( \epsilon_1 \) and \( \epsilon_2 \). However, it is not obvious a priori under which conditions \( \epsilon_2 \) can be determined with only a few months observation time-span, depending strongly on the magnitude of its amplitude and on the phase of the signal at the time of the simulation (see Section 9). The libration term with a period of \( \sim 88 \) d instead should be always detectable.

Finally, the analytical formula adopted for \( \phi \) is

\[
\phi = \frac{3}{2} n (t - t_p) + \epsilon_1 \sin(n (t - t_p))
+ \frac{\epsilon_2}{\mu} \sin(2 n (t - t_p)) + \epsilon_2 \cos(w_j (t - t_p) + \psi_j),
\]

(30)

where we use as reference values for the frequencies \( n, w_j \) and the time of perihelion \( t_p \) the ones at epoch J2000. Because of the different reference time, the phase \( \psi_j \) above is shifted from the one given in Table 1 by

\[
\psi_j \rightarrow \psi_j + w_j (t - t_{2000}) \approx -11.97.
\]

It is important to notice that we have not added any constant phase lag \( \phi_0 = \phi(t_0) \) to be determined in the experiment. This is done on purpose because of a rank deficiency that occurs when we try to solve also for the gravity field coefficient \( S_{22} \). Being, by our definition, the rotation matrix \( \mathbf{R} \) converting to the principal of inertia reference system \( \Psi_{NI} \), the simulated values for the gravity field coefficients \( C_{lm}^{PI} \) and \( S_{lm}^{PI} \) have to be such that \( C_{21}^{PI} = S_{21}^{PI} = S_{22}^{PI} = 0 \). Let us suppose we are solving simultaneously for \( \phi_0 \) and \( S_{22} \), and let \( \xi \) be an error in \( \phi_0 \). In this case, the coefficients \( S_{22} \) and \( C_{22} \) with respect to the reference system rotated by \( \phi_0 \) are

\[
S_{22} = C_{22}^{PI} \sin 2 \xi, \quad C_{22} = C_{22}^{PI} \cos 2 \xi.
\]

(31)

Thus, if \( \xi \) is small, as it should be if we are performing convergent differential corrections, we will have \( C_{22} \approx C_{22}^{PI} \) and \( S_{22} \approx 2 \xi C_{22}^{PI} \), giving a correlation between \( S_{22} \) and \( \phi_0 \), equal to 1. The coefficient \( S_{22} \) and the constant phase angle \( \phi_0 \) cannot be determined independently.

It follows that the value of \( S_{22} \) represents the offset of the axis of minimum moment of inertia of Mercury with respect to the zero meridian defined by \( \phi_0 = \phi(t_0) \). In other words, if there is a constant (or quasi-constant) error \( \xi \) in the libration with respect to our model, then the value of \( S_{22} \) obtained by fitting the data should be different from zero consistently with formula (31). Moreover, all the other spherical harmonic coefficients should be consistent with the following formula (Milani & Gronchi 2010, chapter 13):

\[
\begin{bmatrix} C_{lm}^{PI} \\ S_{lm}^{PI} \end{bmatrix} = \begin{bmatrix} \cos(m \xi) & -\sin(m \xi) \\ \sin(m \xi) & \cos(m \xi) \end{bmatrix} \begin{bmatrix} C_{lm}^{PI} \\ S_{lm}^{PI} \end{bmatrix}.
\]

(32)

The case in which the error \( \xi \) is not constant but significantly changes in time, as would be an error in the libration term due to Jupiter \( \epsilon_2 \), is a little bit different and will be discussed at the end of Section 9.

7.2 Sun, planetary and tidal perturbations

The solar and planetary perturbative acceleration, \( \mathbf{a}_p \), on a satellite orbiting around Mercury, consists of a 'third-body' relative term due to the Sun, Venus, Earth–Moon, Mars, Jupiter, Saturn, Uranus and Neptune:

\[
\mathbf{a}_p = \sum_{\text{bodies}} G M_b \left( \frac{d_{bs}}{d_{bs}^3} - \frac{x_b}{x_b^3} \right),
\]

(33)

where \( d_{bs} \) is the position of a body (of mass \( M_b \)) with respect to the satellite and \( x_b \) is its position with respect to Mercury (see e.g. Roy 2005).

So far, we have considered the rotation of a rigid body. If we introduce an elastic component and the body is subject to some external force, then it could be deformed. This is the case for Mercury under the tidal field of the Sun. The effect is a classic tidal bulge oriented, at each instant, in the direction of the Sun. This deformation changes the expression of the Newtonian potential \( V \) (formula 7) by a quantity \( V_L \), called the Love potential (Kozai 1965):

\[
V_L = \frac{G M_s k_2 R^5}{r_s^3} \left( \frac{3}{2} \cos^2 \psi - \frac{1}{2} \right),
\]

(34)

where \( M_s \) is the Sun’s mass, \( r_s \) is the Mercury–Sun distance and \( \psi \) is the angle between the s/c Mercury-centric position \( r \) and the Sun Mercury-centric position. The Love number \( k_2 \) is the elastic constant that characterizes the effect, and for the simulation we have used a value 0.25.

7.3 Non-gravitational perturbations and accelerometer

Mercury being so close to the Sun, the solar radiation pressure at the planet is very high. Thus, it must be considered as a source of perturbation to the orbit of the s/c, not only its direct component on the satellite, but also the reflected radiation from the planet’s surface (albedo radiation pressure).

The direct radiation pressure \( a_{rad} \) is modelled assuming a spherical satellite with coefficient 1 (i.e. neglecting the diffusive term).
shadow of the planet is computed accurately, taking into account the penumbra effects. The albedo radiation pressure $a_{ab}$ is described assuming a zero relaxation time for the thermal re-emission on Mercury. In this simple setting we do not need information about the distribution of the albedo on Mercury’s surface since it is effectively equal to 1 (this model has been supplied by D. Vokrouhlicky, Charles University of Prague). We are not including thermal thrust and other indirect radiation pressure effects (Milani, Nobili & Farinella 1987, chapter 5).

In general, modelling these non-gravitational effects $a_{ng} = a_{rad} + a_{ab}$ is difficult and the determination of the unknown parameters appearing in the equations is a tough problem. Trying to determine these parameters could degrade the results of the whole experiment, the non-gravitational effects being poorly modelled.

It is possible to overcome this problem by using an onboard accelerometer, which measures differential accelerations between a sensitive element and its rigid frame (cage) and gives accurate information on the non-gravitational accelerations (Iafolla & Nozzoli 2001). A fundamental issue to consider is the so-called calibration of the accelerometer: since it measures only differential accelerations, the zero of the measurement scale is a critical point. In other words, there are sources of error in the accelerometer measurements that inevitably shift the zero of the scale. Let us call them $c$, so that $a_{acc} \rightarrow a_{acc} + c$. In particular, the thermal effects are very important, because the accelerometer is sensitive to the temperature and it acts also as a thermometer (Iafolla & Nozzoli 2001). This error is not included in the dynamical model and so it can be a source of systematic errors in the orbit determination fit.

In order to absorb the error $c$, we must calibrate the accelerometer reading $a_{acc}$ with an additional term $c$, which is in principle a function of time and of a certain number of unknown parameters depending on the model. As a first approximation we could consider $c$ constant for time-scales of the order of $10^4$ s (Milani et al. 2001). With this approach $c(t)$ is approximated as a discontinuous piecewise constant function. A more refined approach consists of representing $c(t)$ as a Hermite cubic spline. For the whole interval going from the central time $t_{k-1}$ of the arc $k-1$ to the central time $t_k$ of the arc $k$ (see Section 8.3), we represent $c(t)$ with a cubic polynomial $c_{k-1,k}(t)$, such that $c_{k-1,k}(t_k) = c_{k,k+1}(t_k) = \xi_k$ and $c_{k-1,k}(t) = c_{k,k+1}(t_k) = \xi_k$. The quantities to be determined are $\xi_k$ and $c_k$, for each arc $k$, and the boundary conditions, six associated with the first observation time and six with the last one.

Eventually, we have $a_{ng} = -(a_{acc} + c)$, and the parameters to solve for become only the ones contained in the calibration $c$.

### 7.4 Manoeuvres

Additional sources of perturbation on the orbit of the s/c around Mercury are the manoeuvres performed on it. In particular, the reaction wheels desaturation manoeuvres. A detailed discussion on this problem is given in Alessi et al. (2012), according to which, under suitable hypothesis, it is possible to add the $\Delta v$ of each manoeuvre to the list of solve-for parameters. We will assume as a general scenario to have one dump manoeuvre during tracking and one dump manoeuvre in the ‘dark’ periods without tracking. The presence of orbital manoeuvres is not considered here. The values for the $\Delta v$ used in the simulation, along with all the details on the modellization and implementation of the manoeuvres scenario, are given in Alessi et al. (2012).

### 7.5 Relativistic corrections

The main relativistic correction in the Mercury-centric orbit is due to the need to use proper time, that is a Mercury Dynamical Time (TDM) affected by the gravitational potential at Mercury (see Milani et al. 2010).

### 8 Numerical Simulation

In this section we will describe the main features and the results of a full cycle numerical simulation, aimed at testing the feasibility of the determination of the rotation of Mercury from remote gravimetry. Results are computed by using the orbit determination and parameter estimation software ORBIT4, developed by the Celestial Mechanics group of the University of Pisa, Department of Mathematics, under ASI contract.

The main programs used to perform the numerical simulation belong to two categories: data simulator and differential corrector. The simulator generates simulated observables (range and range-rate, accelerometer readings) and preliminary orbital elements. The corrector solves for all the parameters which can be determined by a least-squares fit (possibly constrained and decomposed in a multi-arc structure; see Section 8.3).

Because of the multiple dynamics upon which the observables depend, the main programs need to have the propagated state available for each dynamics (for the list of dynamics, see Section 4). This is obtained in different ways, depending on the dynamics. For the dynamics that have to be propagated by numerical integration, that is the Mercury-centric orbit of the s/c and the Solar system orbits of Mercury and the EMU, we call a propagator which solves the equation of motion, for the requested time interval. The states (time, position, velocity, acceleration, etc.) are stored in a memory stack, from which interpolation is possible with the required accuracy. Then, when the state is needed to compute the observables range/range-rate, the dynamics stacks are consulted and interpolated. In the case of the Earth rotational dynamics and of the planetary ephemerides, an interpolation table is already available from external sources (IERS, JPL).

### 8.1 Error models

The error models for the measurements are provided by the University of Rome ‘La Sapienza’ (Professor L. Iess and his group) for the range/range-rate measurements and by the Italian Spring Accelerometer team (ISA, V. Iafolla and his group) for the accelerometer measurements.

The error models contain not only random errors, but also systematic errors, the latter being more important to determine the true accuracy of the results (as opposed to the formal accuracy; see Milani et al. 2001).

The reference values for the white noise (standard deviations) of the range/range-rate measurements are based on Iess & Boscagli (2001). At 1000 s integration time we have

$$
\sigma_{r}^{\text{1000 s}} = 10 \, \text{cm}, \quad \sigma_{rr}^{\text{1000 s}} = 3 \times 10^{-4} \, \text{cm s}^{-1}
$$

for a $K$-band tracking, and

$$
\sigma_{r}^{\text{1000 s}} = 100 \, \text{cm}, \quad \sigma_{rr}^{\text{1000 s}} = 3 \times 10^{-3} \, \text{cm s}^{-1}
$$

for an $X$-band tracking. If the integration time $\Delta t$ is less than 1000 s, the standard deviation associated with the measurements is obtained by

$$
\sigma = \sigma_{r}^{\text{1000 s}} \sqrt{\Delta t / 1000}.
$$
From a comparison of the accuracies for the range and the range-rate it turns out that $\sigma_{r_0} / \sigma_{r_0} = 3 \times 10^4$, which implies that the range-rate measurements are more accurate than the range when we are observing phenomena with period shorter than $3 \times 10^4$. Since the s/c orbital period, and then the periods related to the gravity field perturbations, is less than $10^4$, the gravimetry experiment is performed mainly with the range-rate tracking data, while the opposite is true in the Relativity Experiment context (Milani et al. 2002).

8.2 Least squares

In the following we will apply the least-squares method to process the (simulated) observations (Milani & Gronchi 2010, chapters 5 and 6). Here we introduce the main definitions and notations. Let $\xi(X) = O - C(X)$ be the residuals, i.e. the differences between the observations and the corresponding computed values, and let $X$ be the parameters which affect the dynamical and observation model and that we want to determine. The target function $Q$ to minimize is a quadratic form in $\xi$:

$$Q(\xi(X)) = \frac{1}{m} \xi^T \mathbf{W} \xi = \frac{1}{m} \sum_{i=1}^{m} \sum_{k=1}^{m} w_{i,k} \xi_i \xi_k,$$

where $m$ is the number of observations, $\mathbf{W} = (w_{i,k})$ is the weight matrix, a symmetric matrix with non-negative eigenvalues used to weight the residuals. The minimum $X^*$ of the target function is then found by an iterative differential corrections method:

$$X_{k+1} = X_k - C^{-1} \xi^* \mathbf{W} \xi^*,$$

where $C = \xi^* \mathbf{W} \xi^*$ is the Normal matrix and $\xi^* = \partial \xi / \partial X$. We always use the probabilistic interpretation of its inverse $\Gamma = C^{-1}$, as the covariance matrix of the vector $X$, considered as a multivariate Gaussian distribution with mean $X^*$ in the space of parameters.

8.3 Pure and constrained multi-arc strategy

We call an observed arc each set of range and range-rate tracking data, separated by several hours because of the visibility conditions of the s/c from the Earth (typically one set per day). Between two subsequent observed arcs we have a ‘dark’ period without tracking. We define an extended arc as an observed arc extended from half of the dark period before it to half of the dark period after it. In this way two subsequent extended arcs have one connection time (Fig. 4). Finally, an orbital arc is a sequence of causally connected subsequent extended arcs. Two different orbital arcs are considered as belonging to different objects.

As in Alessi et al. (2012, section 2.3), we can adopt a multi-arc strategy to process the data, and we can classify the solve-for parameters in different categories: global parameters, local parameters and local external parameters, depending on the arc they affect.

In the case of the BeppoColombo mission to Mercury, the problem is affected by the symmetries and rank deficiency described in Bonanno & Milani (2002), thus considering the observed arcs as not causally connected would lead to a weak and unstable orbit determination. However, exploiting the fact that each observed arc belongs to the same object (the s/c), we can add information by considering that at the connection times between two subsequent extended arcs the orbits should coincide. This technique is defined in Alessi et al. (2012, section 4), and it is called the constrained multi-arc strategy. In particular, in this paper we use the a priori constrained technique, which consists of constraining by a fixed quantity the discrepancies, in positions and velocities, of the propagation of two subsequent arcs at their connection time.

The total target function is defined as

$$Q = \frac{1}{m+6(n-1)} \sum_{k=1}^{n} \xi_k^T \mathbf{W} \xi_k + \frac{1}{m+6(n-1)} \sum_{k=1}^{n} d_{k,k+1}^T \mathbf{C}_{k,k+1} d_{k,k+1},$$

where $m$ is the total number of observations, $n$ is the total number of observed arcs, $\mathbf{W}_k$ are the weight matrices of the observations, assumed to be diagonal with entries of the form $1/\sigma^2$, where $\sigma$ is given for each observation by the random error model of Section 8.1. Finally, $d_{k,k+1}$ are the discrepancy vectors defined in Alessi et al. (2012, section 4) along with their weight matrices

$$C_{k,k+1} = \mu^{-1} \text{diag}(1, 1, 1, 10^6, 10^6, 10^6).$$

The value of $\mu$ can be chosen in order to have more smoothness at the connection times.

9 RESULTS AND STATISTICAL ANALYSIS

Here in the following is defined the list of assumptions for the numerical simulation.

9.1 Assumptions

(i) Two ground stations are available for tracking, one in X band (in Madrid, Spain) and one in Ka band (in Goldstone, CA). Range measurements are taken every 120 s while range-rate measurements are taken every 30 s.

(ii) Range-rate is measured with top accuracy. Random and systematic errors in range-rate for Ka-band tracking, random only for X-band tracking, are added to the simulated observables, as described by the error model in Section 8.1.

(iii) Range is measured with top accuracy, but, as discussed in Section 8.1, this does not matter since range measurements are not important.

(iv) The observation time-span covers 88 arcs (about one Mercury orbital period).

(v) Gravity field spherical harmonics up to degree $l_{\text{max}} = 25$, plus tidal effects represented by the Love number $k_2$. The nominal values for the normalized gravity field coefficients, with respect to the body-fixed principal of inertia reference system, used in the simulator, are

(a) degree 2: $C_{20} = -2.7 \times 10^{-5}$, $C_{22} = 1.6 \times 10^{-5}$ from Anderson et al. (1987), $C_{21} = S_1 = S_2 = 0$.

(b) degree $>2$: roughly scaled from those of the Earth (EGM96) by the ratio between the gravitational accelerations on the surfaces of the Earth and of Mercury, which is $\approx -2.65$ (in Milani et al. 2001, they use a factor of 3). In this way we have a gravity field following Kaula’s rule $\approx -2.65 \times 10^{-7}/l^2$ for the rms of the normalized coefficients of degree $l$ (Kaula 1966);

(c) $k_2 = 0.25$. 

Figure 4. Temporal structure of the observed arcs and extended arcs.
(vi) The rotation model assumed is the semi-empirical one, described in Section 7.1. The nominal values for the rotation parameters assumed in the simulator are:

\[
\begin{align*}
\delta_1 &= 4.3 \text{ arcmin}, & \delta_2 &= 0, \\
\epsilon_1 &= 35 \text{ arcsec}, & \epsilon_2 &= 40 \text{ arcsec}.
\end{align*}
\]

(vii) Solar radiation pressure and indirect Mercury albedo radiation pressure are present in the simulation of the observables.

(viii) The accelerometer is always on, and is operated in such a way that its error model is as given in 8.1; when there is no tracking, the data are stored and retransmitted later. The calibration is modelled using a cubic spline.

(ix) Two dump manoeuvres are performed for each arc, one during tracking and one in the dark period between two subsequent arcs, as described in Section 7.4. Notice that they are assumed not to affect the accelerometer readings.

(x) The strategy used to process the simulated observations is the a priori correlated-constrained multi-arc method, as defined in Section 8.3, with \( \mu = 1 \) cm.

(xi) The s/c initial conditions with respect to the reference system \((X_1, Y_1, Z_1)\) are the following [in classic Keplerian elements, values from García et al. (2009)] at epoch 2020 September 1:

\[
\begin{align*}
a &= 3.394 \times 10^7 \text{ km}; \quad e = 0.16; \quad t = 90^\circ; \\
\Omega &= 67.7^\circ; \quad \omega = 16^\circ; \quad \nu_0 = 0.
\end{align*}
\]

The correction step is defined by the list of solve-for parameters:

(i) **Global dynamical**: coefficients of the (normalized) spherical harmonics of the gravity field of Mercury, static part; of degrees from 2 to 25, all possible orders.

(ii) **Global dynamical**: dynamical Love number \( k_2 \).

(iii) **Global dynamical**: rotation parameters \( \delta_1, \delta_2, \epsilon_1, \epsilon_2 \).

(iv) **Global dynamical**: six accelerometer calibration constants \((\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6)\), for each arc, plus 6+6 boundary conditions.

(v) **Local dynamical**: six initial conditions, Mercury-centric position and velocity in the Ecliptic J2000 inertial frame, for each arc.

(vi) **Local dynamical**: three dump manoeuvre components, taking place during tracking, for each observed arc.

(vii) **Local external dynamical**: three dump manoeuvre components, taking place in the dark period between each pair of subsequent observed arcs.

We are assuming that an arc is terminated when either the station to s/c (up-leg) radio wave path or the s/c to station (down-leg) path has an elevation at the station with respect to the station horizon below a critical value \( E_{l0} \); the default used is \( E_{l0} = 20^\circ \). Note that both the down-leg and the up-leg conditions have to be verified, since the two are separated by about 1/4 of an hour. The arc is also terminated if the angle between the spacecraft and the Sun, as seen from the ground station, is less than a critical value \( \psi_0 \). As default we use \( \psi_0 = 2^\circ \). In practice we terminate the arc whenever there is an interruption of the range-rate observation longer than \( t_{\text{gap}} = 1 \) h: this interval is longer than the longest possible interruption due to occultation of the s/c by Mercury (and by the s/c itself). Thus the observed arc can contain shorter gaps due to occultations. We also discard arcs with total duration below a minimum (2 h), because the initial conditions would be too poorly determined; this indeed occurs for some arcs near the superior conjunctions.

### 9.2 Results

All the results presented in the following are intended to be at convergence of the differential corrections. The analysis is performed both on formal statistics (standard deviations and correlations), as given from the formal covariance matrix \( \Gamma = C^{-1} \), and on the actual (true) errors between the parameter’s value at convergence and the value used in the simulation of the observables. Note that, once a formal standard deviation (formal uncertainty) is associated with each parameter from the matrix \( \Gamma \) (Section 8.2), the convergence of the differential corrections can be tested by adding an error of a few times (e.g. five times) the formal uncertainty to each parameter first guess.

A detailed analysis on the convergence basin of the differential corrections and on the implications and results regarding the s/c initial conditions determination, the accelerometer calibrations and the dump manoeuvres is beyond the purpose of this paper. This analysis will be presented in a future work.

Here we focus the analysis on the gravity field and rotation state determination.

**Gravity field.** The main goal of the gravimetry experiment is to solve for the harmonic coefficients of the gravity field of Mercury. The results can be effectively summarized by Fig. 5, showing the gravity field signal as simulated, the Kaula’s rule approximation, together with the formal error and the actual error including possible systematic effects. The error is given as the rms value for all the normalized harmonic coefficients of a given degree.

The signal to error is still above 1 at degree 25, and the decrease in the error is roughly by a factor of 10 over the entire spectrum of degrees up to 25.

The results for the determination of the Love number \( k_2 \), describing the non-static gravity field of Mercury (equation 34), are shown in Table 4. A formal relative accuracy of \( \sim 0.36 \) per cent, much

![Figure 5](https://academic.oup.com/mnras/article-abstract/427/1/468/1030686)

**Table 4. Results for the determination of the Love number \( k_2 \).**

<table>
<thead>
<tr>
<th>True value</th>
<th>Formal Sigma</th>
<th>True error</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_2 )</td>
<td>0.25</td>
<td>( 9.2 \times 10^{-4} )</td>
</tr>
</tbody>
</table>
We show here only the results of the orbit determination of each arc’s initial conditions in the weak direction component defined by formula (8) (see Fig. 6). The initial conditions determined for each arc are not affected by large systematic errors. The weakness of the orbit determination is not uniform over the simulation time-span and depends upon many factors, including the duration of the observing session (a seasonal effect) and the angle between the orbit plane and the direction of the Earth. Anyway, the initial conditions are determined with an accuracy of a few centimetres in the weak direction, and with an accuracy of a few centimetres in directions orthogonal to the weak one (e.g. radial $\hat{r}$ and Earth-Mercury $\hat{p}$).

This result is a direct consequence of the constraints used in the multi-arc method, formula (37). An analogous experiment without constraints, i.e. a pure multi-arc strategy, would have given a much worse orbit determination, with standard deviations in the weak direction of the order of ~50 m or even worse.

Accelerometer calibration. The most critical issue about the accelerometer calibration is that the accelerometer measures only relative values of the non-gravitational perturbations, and we need to derive the average of these values from the tracking data. If this determination is inaccurate, this directly impacts the accuracy of the orbit and propagates, through the correlations present in the solution for all parameters, to the gravity field solution. In our simulation, thanks to the assumed moderate accelerometer error and the efficiency of the spline calibration model, this problem still exists but it is not too severe. The calibration spline transversal component is determined to an accuracy of a few times $10^{-8}$ cm s$^{-2}$, while the radial and out-of-plane components are determined less accurately ($10^{-6}$ cm s$^{-2}$).

It is clear that possible improvements both in the reduction of the accelerometer error and in its calibration model would be desirable for a better success of the Radio Science Experiment.

Desaturation manoeuvres. The results for the formal accuracies of the determination of the desaturation manoeuvres are analogous to Alessi et al. (2012, section 4.1) (a priori case). In our case, these quantities are the most affected by systematic errors coming from the lack/inaccuracy of the accelerometer calibration model. The true error dominates the formal uncertainty up to an order of magnitude in several cases. However, even in the worst cases, the error does not exceed $\sim 10^{-2}$ cm s$^{-1}$ and a relative accuracy of the order of a few per cent of the nominal value.

Rotation from gravimetry. The results for the determination of the rotation state of Mercury, in terms of the semi-empirical rotation model defined in Section 7.1, are gathered in Table 5. These results show a very good accuracy in the determination of both the direction of the spin axis and the amplitude of the libration in longitude. The formal relative accuracies for the obliquity $\delta_1$ and the libration in longitude amplitude $\epsilon_1$ are

$\Delta \delta_1 \approx 0.07$ per cent, $\Delta \epsilon_1 \approx 2.7$ per cent,

giving the same relative accuracies for the quantities $CMR^2$ and $C_{\omega}(B - A)$, respectively (from the approximate formulæ 1 and 5).

Peale’s experiment, as introduced in Section 2.1, needs also a good determination of the gravity field degree 2 spherical harmonics, and this is expressed by Fig. 5. The details of the five degree 2 coefficients are presented in Table 6.

The relative accuracy in the determination of the ratio $C_{\omega}/C$ from formula (2) is then the following:

$\Delta(C_{\omega}/C) \approx 2.7$ per cent,

which is better than the required accuracy of 10 per cent proposed in Milani et al. (2001).

These accuracies appear to be competitive with respect to the ones expected from the rotation experiment performed with the onboard camera. This should not be interpreted as a replacement of the camera experiment: the fact that the results are comparable at least in terms of the order of magnitude allows us to cross-check the two independent experiments to confirm their correctness. The presence of any form of inconsistency between them can be exploited to seek for sources of systematic errors.

Moreover, there could also be room for improvement in the experiment with the camera independently from the gravimetry...
experiment and the mix of the two techniques must give better and more robust results than both methods alone.

Another important thing to underline is that all these results are obtained with only three months of observations (one Mercury orbit). This is encouraging in the concrete terms of the BepiColombo mission. The expected time-span of the mission is around 1 year, but having the possibility to obtain valuable scientific results just from one-fourth of the mission is certainly an insurance against the possible shortening of the mission due to difficulties resulting from the extreme Mercurian environment.

Finally, a comparison between these numerical results and formulae (27) and (28) is presented in Table 7, where we have considered \( \eta = \delta_1 \) and a value of \( \Omega = 67.7 \). It turns out that at least the order of magnitude is respected, but the two results are different by a factor of \( \sim 3 \). To explain clearly such a difference in terms of the much more complicated features of the numerical simulation, compared to the simplifications of the analytical model, is a tough problem. Indeed, the main reason why numerical simulations are needed is to take into account the full complexity of the solutions, with thousands of correlated parameters to be solved for. The relevance of each single parameter in isolation could be discussed with some analytical theory, or at least by order of magnitude arguments; the complexity of all the interactions is beyond the reach of any analytical argument.

The Jupiter term. A few considerations must be made about the determination of the libration amplitude \( \epsilon_2 \) due to Jupiter. As we anticipated in Section 7.1, the determination of \( \epsilon_2 \) is quite unstable and its success is strongly affected by the duration of the mission and by the phase of the periodic signal at the observation times. Additional experiments have been performed in order to test this problem. With a period of \( \sim 11.9 \) yr, the signal of \( \epsilon_2 \cos (\omega_2(t - t_p) + \varphi_j) \) over \( \sim 88 \) d can be seen as a linear variation in \( \phi \) of a few arcsec, depending on the size of \( \epsilon_2 \) and on the phase of the cosine at the time of the mission. For example, in the numerical simulation above, the choice of the initial condition is such that the variation due to the Jupiter term is about 5 arcsec/88 d. In this case the conditions are good enough to determine the rotation from gravimetry in a satisfactory way.

Other tests have been made in the worst cases, in which the variation is \( \sim 10^{-1} \) arcsec/88 d. In these cases the differential corrections give a solution for the gravity field orders of magnitude less accurate than the one in Fig. 5, a completely wrong solution for \( \epsilon_2 \), and very high formal correlations (e.g. the correlation between \( S_{22} \) and \( \epsilon_2 \) is \( \sim 0.9999 \), while in the previous favourable case it was around 0.8).

The reasons for this instability lie in the fact that the Jupiter term contribution is a sinusoid whose behaviour oscillates between a quasi-constant shift in \( \phi \), if the time of the mission is close to a stationary point, and a small secular variation, if the time of the mission is close to a stationary point of its derivative. If the secular variation is too small, the formal correlation between \( \epsilon_2 \) and \( S_{22} \), and in general with the whole gravity field, becomes very close to 1 because of formulae (31) and (32). If we try to solve for both of them simultaneously, the global differential corrections can have numerical instabilities and even diverge, or they can give inaccurate results. Notice also that trying to absorb the Jupiter term by the constant \( S_{22} \) without determining \( \epsilon_2 \) would introduce high systematic errors if the variation of \( \phi \) is too large (e.g. more than 1 arcsec).

The conclusion is that either we are in favourable conditions with the phase of Jupiter to have a large variation of \( \epsilon_2 \cos (\omega_2(t - t_p) + \varphi_j) \), e.g. > 5 arcsec, or we should consider a different model for the libration in longitude, such as the semi-analytic ones described in Dufey et al. (2009) and Yseboodt et al. (2010), which contain only the dynamical parameter \( (B - A)/C\eta \) to be determined.

### 10 CONCLUSIONS

The problem of the determination of the rotation state of Mercury from remote gravimetry has been described and analysed in detail. Even though not all the complicated aspects of the theory of the rotation of Mercury have been included in the model, we have shown that the main quantities, obliquity and 88 d forced libration in longitude, can be determined processing three months of range and range-rate tracking data in a global least-squares fit. Moreover, under suitable conditions, a significant libration term due to the Jupiter perturbation on Mercury’s orbit can also be determined. In this context, a global least-squares fit means that a large number of solve-for parameters are determined all together:

(i) the coefficients of the spherical harmonics of the static gravity field of Mercury of degrees from 2 to 25, all possible orders;
(ii) the tidal Love number \( \kappa_2 \);
(iii) the rotation parameters \( \delta_1, \delta_2 \) for the direction of the spin axis of Mercury, the 88 d forced libration in longitude amplitude \( \epsilon_1 \) and the 11.9 yr forced libration in longitude amplitude \( \epsilon_2 \) due to Jupiter;
(iv) the accelerometer calibration parameters;
(v) the Mercury-centric initial conditions in some inertial frame for each arc;
(vi) the dump manoeuvre components, taking place during tracking in the dark periods between each pair of subsequent arcs.

Some new features in the problem of the orbit determination of an s/c around Mercury have also been considered.

Already introduced in Alessi et al. (2012), a new method to stabilize the orbit determination decomposed in a multi-arc structure has been successfully applied. In this way it has been possible to partially remove the rank deficiency due to the orbit determination around another planet (Bonanno & Milani 2002). Moreover, special care has been taken in including dump manoeuvres in the dynamics and in calibrating the onboard accelerometer by a model based on cubic splines.

All the results turned out to be consistent with the standard requirements of the MORE experiment. In particular, the results regarding the rotation of Mercury were particularly good and encouraging, also in terms of the understanding of the interior of Mercury. Nevertheless, this work needs to be continued with the goal of obtaining better results. For example, the modelling of the accelerometer calibration is not complete because of the lack of information on the in-flight behaviour of the instrument.
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