Spatial Variations of the Order Parameter in Superconductors Containing a Magnetic Impurity

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By taking into account of the Kondo effect, the spatial variation of the order parameter in superconductors containing a single magnetic impurity is discussed. We use the method of the double-time Green function used by Zuckermann, and Takano and Matayoshi, and follow the methods of Zittartz, Müller-Hartmann and Kondo to solve the equation for the t-matrix.

The spatial variation of the order parameter is obtained by a method similar to that of Heinrichs. Only the case, in which the temperature is very close to $T_c$ and $\ln T_K/T = r$ ($T_K$: Kondo temperature) is very large, is considered. The spatial variation of the order parameter is considered up to the terms of the order of $(1/r)^3$. The main effect is to replace $r^2$ by $1/r^2$ is expression of the spatial variation of the order parameter and of the transition point. The term $1/r^3$ yields the asymmetry for $r>0$ and $r<0$, i.e. $T<K$ and $T>K$.

§ 1. Introduction

Abrikosov and Gor'kov$^1$ first investigated the properties of superconductors containing paramagnetic impurities. They treated the scattering effect due to the spin exchange interaction only within the Born approximation.

Since Kondo showed the occurrence of the anomalous scattering of conduction electrons by a magnetic impurity in the second Born approximation, many authors have investigated extensively this phenomenon. Recently, in normal metals Bloomfield and Hamann,$^2$ and Zittartz and Müller-Hartmann$^3$ (Z.M.) have found the exact solution of the integral equation for the scattering t-matrix, which has been derived by Nagaoka$^4$ with the decoupling method of the double-time Green function. Zuckermann,$^5$ Takano and Matayoshi$^6$ and Kondo$^7$ have extended this method to superconductors. In all these works the order parameter has been assumed to be spatially constant throughout the crystal.

Tsuzuki and Tsuneto$^8$ have discussed the spatial variation of the order parameter in superconductors containing a single magnetic impurity in the region $r > \xi_0$, where $\xi_0$ is the coherence length at 0°K which is defined by $v_F/\pi T_0$ ($v_F$: Fermi velocity and $T_0$: transition temperature). Their expression for the order parameter as a function of the distance from a single impurity lacks any mark of the behavior characteristic of the Ginzburg-Landau theory, proportional to $r^{-1}$ for $r > \xi_0$. Recently, Heinrichs$^9$ has solved the integral equation derived by Tsuzuki and Tsuneto, and found the term proportional to $r^{-1}$. However, both treatments$^8$,$^9$ have been made within the Born approximation.
In this paper the discussion of the spatial variation of the order parameter involving the Kondo effect will be made by using the method of double-time Green function as in the papers of Zuckermann\(^5\)) and of Takano and Matayoshi.\(^6\)) We follow the method by Z.M. and Kondo to determine the arbitrary function uniquely. For simplicity, we discuss only the case where the temperature is very close to \(T_c\). Even in this case, it is very difficult to get the exact solution. We therefore, consider the case in which \(|\ln T_K/T| > |\tau|\) (\(T_K\): Kondo temperature) is very large, and solve the problem exactly up to the order of \((1/\tau)^2\).

There are so many methods to treat the Kondo effect. One method is to sum up the most divergent terms in perturbation series and has been carried out by Abrikosov,\(^7\)) Daniach\(^8\)) and Kawamura.\(^9\)) As an example, we shall show the correspondence to the method of the multi-time Green function by Kawamura in Appendix A.

In § 2, we shall first derive the integral equation determining the spatial variation of the order parameter near \(T_c\) in the presence of a single paramagnetic impurity. This equation contains the scattering \(t\)-matrix which will be determined in § 3 by using the Z.M. and Kondo methods up to the order of \((1/\tau)^3\) for \(|\tau| \gg 1\). In § 4 we shall solve the integral equation for the order parameter by using a method similar to that of Heinrichs.

**§ 2. Formulation**

We consider the superconductor containing a magnetic impurity at the origin, and use the double-time Green function, of which equation of motion can be written as

\[
\left(\omega + \frac{p^2}{2m} \tau_3 + \Delta(r) \tau_1\right) \hat{G}_\omega(r, r') = \delta(r - r') - \frac{J}{2} \delta(r) \hat{F}_\omega(r, r'), \tag{1}
\]

\[
\left(\omega + \frac{p^2}{2m} \tau_3 + \Delta(r) \tau_1\right) \hat{F}_\omega(r, r') = -J \hat{H}(r) \hat{F}_\omega(0, r') + \frac{J}{2} \hat{R}(r) \hat{G}_\omega(0, r'), \tag{2}
\]

where \(\tau_1\) and \(\tau_3\) are Pauli matrices, and Green functions are defined as

\[
\hat{G}_\omega(r, r') = \begin{pmatrix}
\langle \psi_\tau(r); \psi_\tau^+(r') \rangle & \langle \psi_\tau(r); \psi_\tau^+ (r') \rangle \\
\langle \psi_\tau^+(r); \psi_\tau^+ (r') \rangle & \langle \psi_\tau^+(r); \psi_\tau^+ (r') \rangle
\end{pmatrix}, \tag{3}
\]

\[
\hat{F}_\omega(r, r') = \begin{pmatrix}
\langle \psi_\tau S_+ + \psi_\tau S_+; \psi_\tau^+ (r') \rangle & \langle \psi_\tau S_+ + \psi_\tau S_+; \psi_\tau^+ (r') \rangle \\
\langle \psi_\tau^+ S_+ - \psi_\tau^+ S_+; \psi_\tau^+ (r') \rangle & \langle \psi_\tau^+ S_+ - \psi_\tau^+ S_+; \psi_\tau^+ (r') \rangle
\end{pmatrix}. \tag{4}
\]

The quantities \(\hat{H}(r)\) and \(\hat{R}(r)\) are defined as

\[
\hat{H}(r) = \begin{pmatrix}
\langle \psi_\tau^+(0) \psi_\tau(r) \rangle & \langle \psi_\tau(0) \psi_\tau(r) \rangle \\
\langle \psi_\tau^+(0) \psi_\tau^+(r) \rangle & \langle \psi_\tau(0) \psi_\tau^+(r) \rangle
\end{pmatrix} - \frac{1}{2} \delta(r), \tag{5}
\]
Spatial Variations of the Order Parameter in Superconductors

\[ \tilde{\mathcal{M}}(r) = 3 \left( \langle \psi^+_t(0) \psi_t(r) S_+ \rangle \langle \psi^+_t(0) \psi^+_t(r) S_+ \rangle \right) - S(S+1) \delta(r). \]

If we introduce the following operator:

\[ \mathcal{F}_s(G(z)) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} f(z) \{ G(z + i\delta) - G(z - i\delta) \} dz. \]

Then Eqs. (5) and (6) can be expressed as

\[ \tilde{\mathcal{M}}(r) = \mathcal{F}_s[\tilde{\mathcal{G}_s}(r, 0)] - \frac{1}{2} \delta(r), \]
\[ \tilde{\mathcal{M}}(r) = 2\mathcal{F}_s[\tilde{\mathcal{F}_s}(r, 0)] - S(S+1) \delta(r). \]

The Green function \( \tilde{\mathcal{G}_s}(r, r') \) for the pure superconductor with the effective order parameter \( \Delta(r) \) satisfies the equation

\[ \left( \omega + \frac{p^2}{2m} + \Delta(r) \tau_3 \right) \tilde{\mathcal{G}_s}(r, r') = \delta(r-r'). \]

By using \( \tilde{\mathcal{G}_s}(r, r') \), we can rewrite Eqs. (1) and (2) as

\[ \tilde{\mathcal{F}_s}(r) = \mathcal{F}_s[\tilde{\mathcal{G}_s}(r, 0)] - \frac{1}{2} \delta(r), \]
\[ \tilde{\mathcal{F}_s}(r) = 2\mathcal{F}_s[\tilde{\mathcal{F}_s}(r, 0)] - S(S+1) \delta(r). \]

and Eq. (12) can be easily solved for \( \tilde{\mathcal{F}_s}(0, r') \), giving the results

\[ \tilde{\mathcal{F}_s}(0, r') = \frac{J}{2} \left\{ 1 + JN(\omega) + \frac{J^2}{4} \tilde{\mathcal{M}}(\omega) \tilde{\mathcal{G}_s}(0, 0) \right\}^{-1} \tilde{\mathcal{M}}(\omega) \tilde{\mathcal{G}_s}(0, r') \]
\[ = -\frac{2J}{J} \tilde{\mathcal{F}_s}(0, r'). \]

In this equation, we have used the following notations:

\[ \tilde{\mathcal{G}_s}(r, r') = \tilde{\mathcal{G}_s}(0, r') + \tilde{\mathcal{G}_s}(r, 0) \tilde{\mathcal{F}_s}(0, r'), \]
\[ \tilde{\mathcal{M}}(\omega) = \int \tilde{\mathcal{G}_s}(0, r) \tilde{\mathcal{F}_s}(r) d^3r \]
\[ = -\frac{4J}{J} \int \tilde{\mathcal{G}_s}(0, r) \tilde{\mathcal{F}_s}(\tilde{\mathcal{G}_s}(r, 0) \tilde{\mathcal{F}_s}(\omega')) d^3r - S(S+1) \tilde{\mathcal{G}_s}(0, 0), \]
\[ \tilde{N}(\omega) = \int \tilde{\mathcal{G}_s}(0, r) \tilde{\mathcal{F}_s}(r) d^3r \]
\[ = \int \tilde{\mathcal{G}_s}(0, r) \tilde{\mathcal{F}_s}[\tilde{\mathcal{G}_s}(r, 0) + \tilde{\mathcal{G}_s}(r, 0) \tilde{\mathcal{F}_s}(\omega')] \tilde{\mathcal{G}_s}(0, 0) d^3r - \frac{1}{3} \tilde{\mathcal{G}_s}(0, 0). \]
Next we consider the relations

\[ \int \tilde{G}_w(0, r) \left( \omega' + \frac{\sqrt{2}}{2m} \tau_3 + A(r) \tau_1 \right) \tilde{G}_w^0(r, 0) d^3r = \tilde{G}_w(0, 0), \quad (17) \]

\[ \int \tilde{G}_w(0, r) \left( \omega' + \frac{\sqrt{2}}{2m} \tau_3 + A(r) \tau_1 \right) \tilde{G}_w^0(r, 0) d^3r = \tilde{G}_w(0, 0). \quad (18) \]

If we subtract Eq. (18) from Eq. (17), we obtain

\[ \int \tilde{G}_w(0, r) \tilde{G}_w^0(r, 0) d^3r = \frac{1}{\omega - \omega'} [\tilde{G}_w^0(0, 0) - \tilde{G}_w^0(0, 0)]. \quad (18) \]

By using Eq. (19), Eqs. (15) and (16) lead to

\[ \tilde{M}(\omega) = -4 \int \tilde{F}_w \left[ \frac{\tilde{G}(\omega') - \tilde{G}(\omega)}{\omega - \omega'} \right] - S(S + 1) \tilde{G}(\omega), \quad (20) \]

\[ \tilde{N}(\omega) = \int \tilde{F}_w \left[ \frac{\tilde{G}(\omega') - \tilde{G}(\omega)}{\omega - \omega'} \right] + \int \tilde{F}_w \left[ \frac{\tilde{G}(\omega') - \tilde{G}(\omega)}{\omega - \omega'} \tilde{t}(\omega') \right] - \frac{1}{2} \tilde{G}(\omega), \quad (21) \]

\[ \tilde{t}(\omega) = \left\{ \tilde{A}(\omega) + J \int \tilde{F}_w \left[ \frac{\tilde{G}(\omega') - \tilde{G}(\omega)}{\omega - \omega'} \tilde{t}(\omega') \right] \right\}^{-1} \times \left\{ \frac{J^2}{4} S(S + 1) \tilde{G}(\omega) + J \int \tilde{F}_w \left[ \frac{\tilde{G}(\omega') - \tilde{G}(\omega)}{\omega - \omega'} \tilde{t}(\omega') \right] \right\}, \quad (22) \]

where we have used the following notations:

\[ \tilde{G}(\omega) = \tilde{G}_w^0(0, 0), \quad (23) \]

\[ \tilde{A}(\omega) = 1 - \frac{J}{2} \tilde{G}(\omega) - \frac{J^2}{4} S(S + 1) \tilde{G}(\omega) + J \int \tilde{F}_w \left[ \frac{\tilde{G}(\omega') - \tilde{G}(\omega)}{\omega - \omega'} \right]. \quad (24) \]

We can use the above equation in the whole temperature region. However, we are interested only in the behavior of the order parameter \( \tilde{A}(\omega) \) near the transition temperature \( T_c \), and we can restrict our consideration to the linear term of the order parameter.

In order to derive the linearized self-consistent equation for the order parameter \( \tilde{A}(\omega) \), we consider the 12-component of Eq. (14),

\[ \tilde{G}_w(r, r') = \tilde{G}_w^0(r, r') + \tilde{G}_w^0(r, 0) \delta_{12}(\omega) \tilde{G}_w(0, r') + \tilde{G}_w^0(r, 0) \delta_{22}(\omega) \tilde{G}_w^0(0, r') + \tilde{G}_w^0(r, 0) \delta_{33}(\omega) \tilde{G}_w^0(0, r'). \quad (25) \]

As far as we consider the linear term of \( \tilde{A}(\omega) \), we can replace the diagonal components by the Green function for the normal state, and expand the off-diagonal Green functions up to the linear terms of \( \tilde{A}(\omega) \). If we define the Green function for the pure normal state as \( \tilde{F}_w(r, s) \), we obtain

\[ \tilde{G}_w(\omega)(r, s)_{12} = \int \tilde{F}_w(\omega, l) \tilde{A}(\omega) \tilde{F}_w^*(l, s) d^3l. \quad (26) \]
From Eqs. (25) and (26) we can obtain the self-consistent equation

\[
\mathcal{A}(r) = -\frac{g}{4\pi i} \int \frac{d\omega}{2} \left[ \tilde{G}_+(r, r) - \tilde{G}_-(r, r) \right] d\omega
\]

\[
= -\frac{g}{4\pi i} \left[ \int \frac{d\omega}{2} \left( K_\omega^+(|r-s|, \omega) - K_\omega^-(|r-s|, \omega) \right) \mathcal{A}(s) d^3 s d\omega 
- \int \frac{d\omega}{2} \left( t_{11}^+(\omega) K_\omega^+(r, \omega) - t_{12}^+(\omega) K_\omega^-(r, \omega) \right) d\omega 
+ \int \frac{d\omega}{2} \left( t_{11}^+(\omega) + t_{12}^+(\omega) \right) K_\omega^+(r, \omega) \mathcal{A}^+(\omega) 
- t_{11}^-(\omega) + t_{12}^-(\omega) \right) K_\omega^-(r, \omega) \mathcal{A}^-(\omega) \right] d\omega \right],
\]

where we have defined

\[
F^\pm(\omega) = F_{\omega}^\pm(r, r),
\]

\[
K_\omega^\pm(|r-s|, \omega) = F_{\omega}^\pm(r, s) F_{-\omega}^\pm(r, s),
\]

\[
\mathcal{A}^\pm(\omega) = \int K_\omega^\pm(r, \omega) \mathcal{A}(r) d^3 r.
\]

Since we are interested in the variation of the order parameter over the distance much larger than \( \rho_F^{-1} \), we have neglected the terms oscillating with \( \rho_F \), and we have used the expression for the Green function for \( |r-r'| \cdot \rho_F \gg 1 \) in \( F_{\omega}(r, r') \).

### § 3. Calculation of the \( t(\omega) \)-matrix

To use Z.M's method, we rewrite the \( t(\omega) \)-matrix. Since the \( \tilde{G}(\omega) \)-matrix is \( \begin{pmatrix} a & b \\ b & a \end{pmatrix} \), we can consistently assume that the \( t(\omega) \)-matrix is \( \begin{pmatrix} a' & b' \\ b' & a' \end{pmatrix} \). If we use the following identity, following Takano and Matayoshi,\(^6\)

\[
\begin{pmatrix} 1 & \pm1 \\ \pm1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix} = (a \pm b) \begin{pmatrix} 1 & \pm1 \\ 1 & 1 \end{pmatrix},
\]

we obtain

\[
t(z) = \frac{J^2}{4} S(S+1) G(z) + J^2 \mathcal{F}_a \left[ \frac{G(\omega) - G(z)}{z - \omega} t(\omega) \right] \\
\times \left\{ A(z) + J^2 \mathcal{F}_a \left[ \frac{(G(\omega) - G(z))^2 \mathcal{F}_a t(\omega)}{z - \omega} \right] \right\}^{-1},
\]

where \( t(z) \) and \( G(z) \) are defined by \( t_{11}(z) \pm t_{13}(z) \) and \( G(z)_{11} \pm G(z)_{13} \), respectively. In the following, we consider the case \( t(z) = t_{11}(z) + t_{13}(z) \).

The next step is to determine the Green function for the normal state. If we take Lorentzian form for the state density
we obtain

\[ F^\pm (\omega) = \int \frac{\rho (\xi)}{\omega \pm iD} \frac{d\xi}{\omega \pm iD}, \quad (34) \]

where \( \rho \) is the state density at Fermi surface, and \( D \) is the band width. \( F(\omega) \) has the branch cut on the real axis, as shown in Fig. 1. We define the function \( F_r(\omega) \) as being equal to \( F^+(\omega) \) in the upper half plane and equal to its analytical continuation in the lower half plane. Similarly, \( F_a(\omega) \) is defined as equal to \( F^-(\omega) \) in the lower half plane and its analytical continuation in the upper half plane. The off-diagonal component of \( G_{r,a}(\omega) \) is \( \Delta^\pm (\omega) \). Then, we have

\[ G_{r,a}(\omega) = F_{r,a}(\omega) + \Delta_{r,a}(\omega). \quad (35) \]

Various functions involving the operator \( \bar{T}_a \) can be written explicitly as

\[ A(z) = 1 + R(z) - \frac{J^z}{4} S(S+1) G^2(z), \quad (36) \]

where

\[ R(z) = \frac{J}{4\pi i} \int_{-\infty}^{\infty} \frac{\text{th}(\beta \omega/2)}{z-\omega} (G_r(\omega) - G_a(\omega)) d\omega - \frac{J^z}{4} S(S+1) G^2(z), \quad (37) \]

and

\[ \bar{T}_u \left[ \frac{G(\omega) - G(z)}{z-\omega} t(\omega) \right] = I_z(z) - F(z) I_t(z), \quad (38) \]

\[ \bar{T}_u \left[ \frac{(G(\omega) - G(z))^2}{z-\omega} t(\omega) \right] = I_z(z) - 2G(z) I_t(z) + F^2(z) I_v(z), \quad (39) \]

where \( I_v(z) \) is defined by

\[ I_v(z) = \frac{1}{4\pi i} \int_{-\infty}^{\infty} \frac{\text{th}(\beta \omega/2)}{z-\omega} \left[ G_r^u(\omega) t_r(\omega) - G_a^u(\omega) t_a(\omega) \right] d\omega. \quad (40) \]

\( t_r(\omega) \) and \( t_a(\omega) \) are defined by Eq. (32) with \( G(\omega) \) replaced by \( F_r(\omega) \) and \( F_a(\omega) \), respectively. Using the simple algebra, we can write \( t_r(\omega) \) and \( t_a(\omega) \) as

\[ 1 \pm (G_r(z) - G_a(z)) t_{r,a}(z) = \frac{C(z)}{D_{r,a}(z)}, \quad (41) \]

where
Spatial Variations of the Order Parameter in Superconductors

\[ C(z) = 1 + R(z) - \frac{J^2}{4} S(S+1) G_r(z) G_a(z) \]

\[ + J \left[ I_0(z) - (G_r(z) - G_a(z)) I_1(z) + G_r(z) G_a(z) I_2(z) \right], \tag{42} \]

\[ D_{r,a}(z) = 1 - \frac{J^2}{4} S(S+1) G_{r,a}^2(z) + R(z) \]

\[ + J \left[ I_0(z) - 2G_{r,a}(z) I_1(z) + G_{r,a}^2(z) I_2(z) \right]. \tag{43} \]

By using the similar argument of Z.M., Eq. (43) can be written as

\[ D_r^+(z) = e^{-Q(z)}, \quad D_a^-(z) = e^{-Q(z)}, \tag{44} \]

where

\[ Q(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{d\omega}{z - \omega} \ln \left[ C^+(z) C^-(z) - \frac{J^2 S(S+1)}{4} (G_r(z) - G_a(z))^2 \right]. \tag{45} \]

In Eq. (42), the terms containing \( I_n(z) \) are of higher order with respect to \( J \) compared with other terms and may be neglected. Thus, we have the expression of \( C(z) \) retaining terms up to linear to \( A(r) \):

\[ C^\pm(z) = 1 + \frac{J}{4\pi i} \int_{-\infty}^{+\infty} \frac{\rho(\omega)}{z - \omega} \ln \left[ C^+(z) C^-(z) - \frac{J^2 S(S+1)}{4} (G_r(z) - G_a(z))^2 \right] d\omega \]

\[ - \frac{J^2 S(S+1)}{4} (F_r(z) A^- (z) + F_a A^+(z) + F_r(z) F_a(z)) \]

\[ = X^\pm(z) + Y^\pm(z) - \frac{J^2 S(S+1)}{4} (F_r(z) A^- (z) + F_a(z) A^+ (z)), \tag{46} \]

where

\[ X^\pm(z) = 1 - \frac{J}{2} \int_{-\infty}^{+\infty} \frac{\rho(\omega)}{z - \omega} \ln \left[ C^+(z) C^-(z) - \frac{J^2 S(S+1)}{4} (G_r(z) - G_a(z))^2 \right] d\omega \]

\[ Y^\pm(z) = \frac{J}{4\pi i} \int_{-\infty}^{+\infty} \frac{\rho(\omega)}{z - \omega} (A^+ (\omega) - A^- (\omega)) d\omega. \tag{47} \]

If \( \omega < D \), the state density \( \rho(\omega) \) changes little from \( \rho \). The contribution from the region \( \omega \gtrsim D \) is very small owing to the rapid phase modulation. Then we can use the following approximation:

\[ X^\pm(z) = \gamma \left[ \ln \frac{T_K}{T} - g^\pm \left( \frac{z}{2\pi T} \right) \right], \tag{49} \]

where

\[ g^\pm(x) = \Psi' \left( \frac{1}{2} \mp ix \right) - \Psi \left( \frac{1}{2} \right), \tag{50} \]

\( \gamma = \rho J \), \( T_K \) is the Kondo temperature and \( \Psi(x) \) is a digamma function.

For simplicity, we consider the case in which \( |r| = |\ln T_K/T| >> 1 \), and expand Eq. (39) in power series of \( 1/r \), and retain the terms up to \( (1/r)^3 \). \( Q^\pm(z) \) is
expressed as the sum of the terms of the zeroth and the first order with respect to the order parameter \( \Delta(r) \). If we denote these terms by \( K^\pm(z) \) and \( L^\pm(z) \), respectively, we obtain

\[
K^\pm(z) = \mp \ln \frac{\Delta^\pm(z)}{\Delta(z)} = \pm \frac{\pi^2 S(S+1)}{2 \tau^3} \pm \frac{g^\pm(z)}{2 \tau^3} + \frac{g^\pm \pi^2 S(S+1)}{3 \tau^3},
\]

\[
L^\pm(z) = \frac{1}{i \tau} \left( \mp \frac{Y^\pm(z)}{2} + \frac{i \pi J_T S(S+1)}{2} \Delta^\pm(z) \right)
\]

\[
+ \frac{1}{i \tau^3} \left\{ -i \pi J_T S(S+1) \Delta^\pm(z) \mp g^\pm(z) Y^\pm(z) - \frac{i \pi J_T S(S+1)}{4} \right\}
\]

\[
\times \left( -g^\pm(z) \Delta^\mp(z) - g^\mp(z) \Delta^\pm(z) + 2 \pi i T \sum_{n>0} \left( \frac{1}{z-i \omega_n} - \frac{1}{z+i \omega_n} \right) \Delta(n) \right)
\]

\[
+ \frac{1}{i \tau^3} \left\{ \mp \pi^2 S(S+1) Y^\pm(z) \mp g^\pm(z) Y^\pm(z) + i \pi J_T S(S+1) \right\}
\]

\[
\times \left( -g^\pm(z) \Delta^\mp(z) - g^\mp(z) \Delta^\pm(z) + 2 \pi i T \sum_{n>0} \left( \frac{1}{z-i \omega_n} - \frac{1}{z+i \omega_n} \right) \Delta(n) \right)
\}

\[
(52)
\]

where \( \Delta(n) \) is the function obtained by replacing \( \omega \) by \( \pm i \omega = \pm i(2n+1)\pi T(n>0) \) in \( \Delta^\pm(\omega) \), respectively.

Substituting Eqs. (49), (51) and (52) into Eqs. (45) and (46), we get

\[
\ell_{11, r, a}(z) = \pm \frac{S(S+1)}{\rho^2} \left( \frac{\pi \rho}{4 i \tau^3} + \frac{\pi \rho g^\pm(z)}{2 i \tau^3} \right),
\]

\[
\ell_{12, r, a}(z) = \pm \frac{S(S+1)}{\rho^2} \left[ \left( \Delta^\pm(z) - \Delta^-(z) \right) \left( \frac{1}{8 \tau^3} + \frac{g^\pm(z)}{4 \tau^3} \right) \right.
\]

\[
- \left( \frac{\tau}{8 \tau^3} + \frac{\tau}{8 \tau^3} g^\pm(z) \right) \left( \Delta^\pm(z) - \Delta^-(z) \right) \mp \frac{\tau}{4 \tau^3} \Delta^\pm(z) \mp \frac{1}{2 \tau^3} \Delta^\pm(z)
\]

\[
\mp \frac{\tau}{8 \tau^3} \left\{ -g^\pm(z) \Delta^\mp(z) - g^\mp(z) \Delta^\pm(z) + 2 \pi i T \sum_{n>0} \left( \frac{1}{z-i \omega_n} - \frac{1}{z+i \omega_n} \right) \Delta(n) \right\}
\]

\[
+ \frac{\pi \rho}{2 i \tau^3} Y^\pm(z) \pm \frac{1}{2 \tau^3} \left\{ -g^\pm(z) \Delta^\mp(z) - g^\mp(z) \Delta^\pm(z) \right.
\]

\[
\left. + 2 \pi i T \sum_{n>0} \left( \frac{1}{z-i \omega_n} - \frac{1}{z+i \omega_n} \right) \Delta(n) \right\} \}. \]

\[
(54)
\]

§ 4. Solution of the integral equation for \( \Delta(r) \)

To solve the integral equation (27), we introduce an integral operator \( L(r) , \)
Spatial Variations of the Order Parameter in Superconductors

\[ L(r) \Delta(r) = \frac{1}{g \rho} \left( \Delta(r) - \frac{g}{4 \pi i} \int \text{th} \frac{\beta \omega}{2} (K_0^+ (|r-s|, \omega) - K_0^- (|r-s|, \omega)) \Delta(s) \, d^3 s \right). \]  

(55)

If we denote the second and third terms in Eq. (27) by I and II, respectively, we obtain

\[
I = \frac{g S (S+1)}{\rho^3} \left[ - \left( \frac{\gamma}{8} - \frac{1}{8 \tau^2} \right) T \sum_n \Delta(n) K_0(n) \right.
\]

\[
- \left( \frac{\gamma}{8} - \frac{1}{4 \tau^2} \right) T \sum_n g(n) \Delta(n) K_0(n) \]

\[
- \left( \frac{\gamma}{8} - \frac{1}{4 \tau^2} \right) T \sum_n T \sum_{n=0}^\infty \Delta(n) K_0(n) \]

\[
+ \left( \frac{\gamma}{2 \tau^2} - \frac{1}{2 \tau^2} \right) T \sum_n g(n) \Delta(n) K_0(n) \]

\[
+ \left( \frac{\gamma}{8 \tau^2} - \frac{1}{4 \tau^2} \right) \frac{1}{4 \pi i} \int \text{th} \frac{\beta \omega}{2} (g^- (\omega) \Delta^+ (\omega) K_0^- (\omega) - g^+ (\omega) \Delta^- (\omega) K_0^+ (\omega)) \, d\omega \]

\[
- \left( \frac{\gamma}{8 \tau^2} - \frac{1}{2 \tau^2} \right) T \sum_n \Delta(n) K_0(n) \]

\[
\left. \left( \frac{\gamma}{8 \tau^2} - \frac{1}{4 \tau^2} \right) \frac{1}{4 \pi i} \int \text{th} \frac{\beta \omega}{2} \sum_{n>0} \left( \frac{1}{\omega - i \omega_n} - \frac{1}{\omega + i \omega_n} \right) \Delta(n) (K_0^+ (\omega) - K_0^- (\omega)) \, d\omega \right]. \]

(56)

\[
II = - \frac{g S (S+1)}{\rho^3} T \sum_n \left( \frac{1}{4 \tau^2} + \frac{g(n)}{2 \tau^2} \right) \Delta(n) K_0(n), \]

(57)

where \( g(n) \), \( Y(n) \) and \( K_0(n) \) are the functions obtained by replacing \( \omega \) by \( \pm i \omega_n = \pm i (2n+1) \pi T \) \( (n \geq 0) \) in \( g^\pm (\omega) \), \( Y^\pm (\omega) \) and \( K_0^\pm (r, \omega) \), respectively. For example, \( g(n) \) is given by \( g(n) = \Psi(n+1) - \Psi \left( \frac{n}{2} \right) \).

We can conveniently write \( I + II \) as \( -g T \sum_n \mathbb{E}(\omega_n) K_0(r, \omega_n) \langle K_0(r, \omega_n) \Delta(r) \rangle \), where \( \langle f(r) \rangle = \int f(r) \, d^3 r \). If we introduce a constant \( z \), to satisfy

\[
\langle (L(r) + z) \Delta(r) \rangle = -g T \sum_n \mathbb{E}(\omega_n) \langle K_0(r, \omega_n) \rangle \langle K_0(r, \omega_n) \Delta(r) \rangle + z \langle \Delta(r) \rangle = 0, \]

(58)

we obtain an equation to determine the transition temperature \( T_c \):

\[
\ln \frac{T_{c0}}{T_c} = z, \]

(59)

where \( T_{c0} \) is the transition temperature for the pure state. If we consider the
system containing $N$ magnetic impurities we may only multiply $z$ by $N$, and
\[ \ln \frac{T_{e0}/T_e}{1} \text{ becomes proportional to the concentration of impurities, as shown in Appendix B.} \]

Using Eq. (57), we get
\[ \mathcal{A}(r) = A_0 - \frac{T}{\rho} \sum_n \mathcal{E}(\omega_n) \frac{1}{L(r) + z} K_0(r, \omega_n) \langle K_0(r, \omega_n) \mathcal{A}(r) \rangle + \frac{z}{L+z} \mathcal{A}(r). \] (60)

We now solve Eqs. (58) and (60) by iteration. To do so, we introduce
\[ \mathcal{A}(r) = (1 + f(r)) A_0, \quad A_0 = A(\infty), \] (61)
and we may replace $\mathcal{A}(r)$ in Eqs. (58) and (60) by $A_0$ as the first approximation. Thus we obtain
\[ z = g T \sum_n \mathcal{E}(\omega_n) \langle K_0(\omega_n) \rangle^2, \] (62)
\[ f(r) = -\frac{T}{\rho} \sum_n \mathcal{E}(\omega_n) \frac{1}{L(r) + z} K_0(r, \omega_n) \langle K_0(r, \omega_n) \rangle, \] (63)
where we have neglected the terms of order $1/V$ in Eq. (60) compared with the second term. If we substitute $A_0$ into $\mathcal{A}(r)$ in Eqs. (51) and (57), we have
\[ I + II = -\frac{g S(S+1)}{\rho^2} \left( \frac{1}{2z^2} T \sum_n \mathcal{A}(n) K_0(n) + \frac{3}{2z^2} T \sum_n g(n) \mathcal{A}(n) K_0(n) \right). \] (64)

From Eqs. (62), (63) and (64), we get the transition temperature $T_e$ and the spatial behaviour of the order parameter near $T_e$,
\[ \ln \frac{T_{e0}}{T_e} = \frac{\pi^2 S(S+1)}{V \rho} \sum_n \left( \frac{1}{2z^2} + \frac{3g(n)}{2z^5} \right) \frac{1}{|\omega_n|^2}, \] (65)
\[ f(r) = \frac{\pi S(S+1)}{\rho^2} CT \sum_n \left( \frac{1}{2z^2} + \frac{3g(n)}{2z^5} \right) \frac{1}{|\omega_n|}
\times \left\{ \frac{\nu_F}{|\omega_n|} \frac{1}{r} \left( \exp \left( -\frac{2|\omega_n|}{\nu_F} \right) - 1 \right) - E_1 \left( \frac{2|\omega_n|}{\nu_F} \right) \right\}, \] (66)
where $C = (12/(2\pi)^7 \zeta(3)) \cdot (m^2/\xi^3)$, $E_1(x) = \int_x^{\infty} (e^{-t}/t) \, dt$ and $\zeta(x)$ is the $\zeta$-function.

Equation (65) does not take account of the effect of the spatial variation of $\mathcal{A}$. We can write down the expression for the transition temperature in which the effect of the spatial variation is taken into account as follows:
\[ \ln \frac{T_{e0}}{T_e} = \frac{\pi^2 S(S+1)}{V \rho} \left[ T_e \sum_n \left( \frac{1}{2z^2} + \frac{3g(n)}{2z^5} \right) \frac{1}{|\omega_n|^2} + 2\pi^2 T_e \frac{\xi}{\rho} S(S+1) \right]
\times \sum_n \left( \frac{1}{r^2} + \frac{3g(n)}{r^5} \right) \frac{1}{|\omega_n|} \delta_{\omega_n} \left[ \frac{1}{|\omega_n|} \ln \frac{|\omega_n|}{|\omega_n| + |\omega_n'|} + \frac{1}{|\omega_n|} \ln \frac{|\omega_n'|}{|\omega_n| + |\omega_n'|} \right]. \] (67)
If we expand $\tau$ with respect to $\gamma$, we obtain

$$
\tau = \frac{1}{\gamma} - \left( \ln \frac{\pi T}{D} \right) + \text{const}
$$

as Z.M.\textsuperscript{19} have obtained. We may rewrite Eq. (68) as

$$
\tau = \frac{1}{\gamma} - \left( \ln \frac{\omega_n}{D} - g(n) + \text{const} \right).
$$

If we substitute Eq. (69) in Eqs. (65), (66) and (67), and expand them with respect to $\gamma$, it is seen that Eq. (65) coincides with the expression of Abrikosov and Gor'kov\textsuperscript{19} up to terms of $\gamma^2$, and that of Liu\textsuperscript{19} up to $\gamma^3$, and that Eqs. (66) and (67) coincide with the expressions of Heinrichs up to $\gamma^3$.

The discussion of these results will be given in the next section.

§ 5. Conclusion

We have obtained the spatial variation of the order parameter in which the Kondo effect is taken into account. When $|\tau| = \ln |T_K/T| \gg 1$, we include the terms up to the order of $(1/\tau)^3$. In this region, the terms $1/\tau^2 + 3g(n)/\tau^3$ in Eq. (66) determine the main behavior of the spatial variation. If we expand $1/\tau^2 + 3g(n)/\tau^3$ in power series of $\gamma$, we obtain essentially the same expression as that of Heinrichs which is obtained in the first Born approximation. Equation (66) contains the dynamical character of the impurity spin as expressed $3g(n)/\tau^3$, which may have a large effect on the spatial variation of the order parameter. For very large $\tau$, i.e. when $T \gg T_K$ or $T < T_K$, the only effect is to replace $\gamma^3$ in the expression in the first Born approximation by $1/\gamma^2$. Thus, the region where $A(r)$ is very small shrinks as $|\tau|$ becomes large, and its size will be maximum for $T \sim T_K$. In other words, we may say that the coherence length of the system becomes shorter as $|\tau|$ increases. In the case of the positive $J > 0$, $\tau$ is always positive and larger than $1/\gamma$, and the decrease of $A(r)$ in the vicinity of the magnetic impurity becomes smaller when the Kondo effect is taken into account. For $J < 0$ and $\tau < 0$, $|\tau|$ is smaller than $|1/\gamma|$, and the decrease of $A(r)$ becomes larger when the Kondo effect is taken into account. For $J < 0$ and $\tau > 0$, $|\tau|$ may become either larger or smaller than $1/\gamma$, and the decrease of $A(r)$ may also be smaller or larger. The main effect of the term $3g(n)/\tau^3$ is to give the asymmetry for $\tau > 0$ and $\tau < 0$, i.e. $T \gg T_K$ and $T \ll T_K$.

The transition temperature is expected to behave similarly. The main effect is to replace $\gamma^3$ in the expression of the first Born approximation by $1/\tau^2$, which is symmetric for $\tau > 0$. The decrease of $T_c$ is expected to be maximum when $T \sim T_K$. The effect of the next term $3g(n)/\tau^3$ yields the asymmetry for $\tau > 0$ and $\tau < 0$. These qualitative behaviors agree with the results of Zuckermann.\textsuperscript{19}

Our consideration in this paper is restricted to the case in which the temperature $T$ is very close to $T_c$ and the distance $r$ is much larger than the
coherence length $\xi_0$. In order to investigate the case of lower temperature, we must generally solve the nonlinear integral equation for the order parameter, which seems very difficult to solve. In the case of shorter distance, on the other hand, we can show in a similar way to that of Heinrichs that, as long as $r$ is larger than $1/p_F$, the spatial variation of the order parameter has the long range spatial oscillations which occur owing to the sharpness of the Fermi surface.

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Appendix A

In this appendix, we show that the most divergent terms in perturbation series can be summed up by using the method of the multi-time Green functions by Kawamura, and the same equation for the spatial variation of the order parameter can be obtained. Kawamura's multi-time Green function has been defined as

$$\langle T\{A(t_{i-1}), B(t_{i-1}), C(t_{i-2}), \ldots, D(t)\} \rangle$$

$$= \int_{-\beta}^{\beta} \delta(t_i - t_{i-1}) \langle T\{A(t_i), B(t_{i-1}), C(t_{i-2}), \ldots, D(t)\} \rangle.$$  \hfill (A.1)

By using this multi-time Green function, the Green function $\hat{F}(r, r')$ in Eq. (4) can be written as

$$\hat{F}(r, r') = T \sum_{\omega_1} \hat{F}(\omega_1, \omega; r, r'),$$

where $\hat{F}(\omega_1, \omega; r, r')$ is formally expressed as

$$\hat{F}(\omega_1, \omega; r, r') = \begin{pmatrix} \langle \phi_t(r) \rangle - \langle \phi_t^+(r) \rangle & \langle \phi_{t-1}^+(r) \rangle - \langle \phi_{t-1}^+(r) \rangle \end{pmatrix}_{\omega_1} \begin{pmatrix} S_\uparrow ; \phi_t \rangle & S_\downarrow ; \phi_t \rangle \\ -S_\downarrow ; \phi_t^+ \rangle & -S_\uparrow ; \phi_t^+ \rangle \end{pmatrix}_{\omega_1 - \omega_1 - \omega}.$$  \hfill (A.2)

If we pursue iteratively only the motion of the electron field operators in the left-hand side matrix in Eq. (A.3), we finally obtain the general expression

$$\left(i\omega_t + \frac{p_t^2}{2m} + A(r) \tau_3 \right) \hat{F}(\omega_1, \omega_{t-1}, \ldots, \omega_1, \omega; r, r')$$

$$= -\hat{S}(r - r') \hat{S}(\omega_1, \omega_{t-1}, \ldots, \omega_1, \omega) - \frac{f}{2} \hat{S}(r) T \sum_{\omega_1} \hat{F}(\omega_{t+1}, \omega_1, \ldots, \omega; r, r').$$  \hfill (A.4)

Here

$$\hat{S}(\omega_1, \omega_{t-1}, \ldots, \omega_1, \omega)$$

$$= -\beta \hat{S}_{\omega_1, \omega} \begin{pmatrix} \langle S_z \rangle & \langle S_+ \rangle \\ \langle S_z \rangle & -\langle S_+ \rangle \end{pmatrix}_{\omega_{t-1} - \omega_t} \begin{pmatrix} S_\uparrow ; S_\downarrow \rangle & S_\downarrow ; S_\uparrow \rangle \\ -S_\downarrow ; S_\uparrow \rangle & S_\uparrow ; S_\downarrow \rangle \end{pmatrix}_{\omega_{t-1} - \omega_t - \omega_1 - \omega_t - 1}.$$
Spatial Variations of the Order Parameter in Superconductors

\[ \begin{pmatrix} S_z - S_r \\ -S_+ - S_z \end{pmatrix}_{a_2 \cdots a_1} \begin{pmatrix} S_{+2} \\ -S_{-1} - S_- \end{pmatrix}_{a_2 \cdots a_1}, \]  

where \{i\} can couple to only \{t\}, and \{i \cdots t\} = 0. Using Eq. (10), we can rewrite Eq. (A·4) as

\[ \tilde{F}(\omega_t, \omega; r, r') = -\hat{G}_{\omega}^\delta (r, 0) \]

\[ \times \sum_{i=1}^{l-1} \left( \prod_{k=2}^{i-1} T \sum_{\omega_k} \left( -\frac{J}{2} \right) \hat{G}_{\omega_k}^\delta (0, 0) \right) T \sum_{\omega_l} \left( -\frac{J}{2} \right) \hat{G}_{\omega_l}^\delta (0, r') \hat{S}(\omega_t, \omega_{l-1}, \cdots, \omega_{l}, \omega). \]  

The arrow \( \rightarrow \) means that we arrange terms in the direction of the arrow. Next we investigate the spin operator \( \hat{S}(\omega_t, \cdots, \omega) \). If we are concerned with the most divergent terms and take the initial condition as

\[ \hat{S}(\omega_t, \omega_{l-1}, \cdots, \omega, \omega) = -\beta^2 S(S+1) \hat{\delta}_{\omega_0, \omega}, \]  

we get

\[ \hat{S}(\omega_t, \omega_{l-1}, \cdots, \omega_1, \omega) = -\beta^2 \hat{\delta}_{\omega_0, \omega} 2^{l-2} S(S+1) \sum_{k=1}^{l-1} \left( \frac{1}{i\omega_k - i\omega_{l-1}} \right) \hat{\delta}_{\omega_k, \omega}. \]  

From Eqs. (11), (14), (A·6) and (A·8), we obtain

\[ \lambda_{11}(\omega) = F(\omega) \gamma_{\omega}^2, \quad \lambda_{12}(\omega) = \gamma_{\omega}^2 \int K(r, \omega) \mathcal{A}(r) d^3r, \]  

where

\[ \gamma_{\omega}^2 = \frac{S(S+1)J^2}{4} \frac{1}{(1 - Jh(\omega))^2} \]  

and

\[ h(\omega_0) = T \sum_{\omega_0} \frac{F(i\omega_0)}{i\omega_0 - i\omega_0} = -\rho \ln \frac{D}{|\omega_0|}. \]  

The last equation is valid when \(|\omega| \ll D\). Thus we obtain

\[ z = \frac{\pi^2 S(S+1) \gamma^2 C T}{2 V \rho} \sum_{n} \frac{\gamma^2}{|\omega_n|^2} \left( 1 + \gamma \ln D/|\omega_n|^2 \right)^{\frac{1}{2}}, \]  

\[ f(r) = \frac{\pi S(S+1) \gamma^2 CT}{2 \rho^2} \sum_{n} \frac{1}{(1 + \gamma \ln D/|\omega_n|^2)^{\frac{1}{2}}} \frac{1}{|\omega_n|} \times \left\{ \begin{array}{l} \frac{v_{\rho} \rho}{2|\omega_n|} \exp \left( -\frac{2|\omega_n|}{v_{\rho} \rho} r \right) - 1 \end{array} \right\}. \]  

Appendix B

Here we investigate the transition temperature of the superconductors in the presence of a small concentration of impurities, following the method of Tsuzuki.
If we denote the position of $N$ magnetic impurities by $R_1, \ldots, R_N$, we obtain in the form of Eq. (14)

$$\tilde{G}_\omega(r, r') = \tilde{G}_\omega^*(r, r') + \sum_j \tilde{G}_\omega^*(r, R_j) \tilde{t}_j(\omega) \tilde{G}_\omega^0(R_j, r')$$  \hspace{1cm} (B.1)

A linearized self-consistent equation is given by taking 12-component in Eq. (B.1):

$$\Delta(r, R_1, \ldots, R_N) = -gT \sum_{\omega_n} \sum_j \left\{ K_6(|r-R_j|, \omega_n) \tilde{t}_{11}(\omega) + \frac{\tilde{t}_{12}(\omega) + \tilde{t}_{13}(\omega)}{2F(\omega_n)} K_6(|r-R_j|, \omega_n) \right\}$$

\begin{align*}
&\times \left\{ \int K_6(|R_j-s|, \omega_n) \Delta(s, R_1, \ldots, R_N) \, d^3s \right\}. \hspace{1cm} (B.2)
\end{align*}

When we derive Eq. (B.2), we use the same approximation as in Eq. (26). If we introduce $f(r, R_1, \ldots, R_N)$ as Eq. (61),

$$\Delta(r, R_1, \ldots, R_N) = (1 + f(r, R_1, \ldots, R_N)) \Delta_0$$  \hspace{1cm} (B.3)

since the concentration of impurities is small, $f(r, R_1, \ldots, R_N)$ is written as the follows:

$$f(r, R_1, \ldots, R_N) = \sum_j f(r, R_j).$$  \hspace{1cm} (B.4)

Thus, $\frac{\gamma^N}{}$ to determine the transition temperature is given by

$$\frac{\gamma^N}{\theta} = \frac{T}{\theta} \sum_{\omega_n} \sum_j \langle K_6(\omega) \rangle \left( \tilde{t}_{11}(\omega) + \frac{\tilde{t}_{12}(\omega)}{F(\omega)} \right) \int K_6(|R_j-s|, \omega_n) (1 + \sum_k f(s, R_k)) \, d^3s$$

\begin{align*}
&\quad - \frac{\gamma^N}{T} \sum_j \langle f(r, R_j) \rangle . \hspace{1cm} (B.5)
\end{align*}

where $\tilde{t}_{12}(\omega) \int K_6(|R_j-s|, \omega_n) \Delta(s) \, d^3s = \tilde{t}_{12}(\omega)$. Since $K_6(r, \omega) \sim 0$ for $\xi_0 < r$, the integrand in Eq. (B.5) is written as

$$\int K_6(|R_j-s|, \omega_n) (1 + \sum_k f(s, R_k)) \, d^3s = \int K_6(|R_j-s|, \omega_n) (1 + f(s, R_j)) \, d^3s.$$

Equation (B.5) leads to

$$\frac{\gamma^N}{T} = N\nu.$$

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Spatial Variations of the Order Parameter in Superconductors

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