The Accuracy of Determination of Seismic Interval Velocities from Variable Angle Reflection (Disposable Sonobuoy) Records

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' Eh bien! lui cria-t-elle, avais-je pas raison?
De quoi vous sert votre vitesse . . .'

La Fontaine

Summary

The accuracy of a new method of determining the seismic interval velocities from variable angle reflection (sonobuoy) records is derived. This new method, which uses the exact geometric ray path equations for a multilayered medium, taking account of refraction between layers and dipping interfaces, is then compared with the previous approximate method of Dix.

1. Introduction

A knowledge of seismic velocities is useful principally to the geologist, for whom seismic velocity is a fundamental parameter in the determination of the nature of rocks within the crust, and also to the seismologist for the dynamic correction of reflection data preparatory to common depth point stacking.

Recently, new experimental techniques have created the opportunity of obtaining good quality seismic data at sea using highly repetitive sources. Seismic techniques have advanced beyond the use of expensive refraction lines, giving a broad outline of the crustal structure to the possibility of cheaply obtaining more detailed knowledge of seismic velocities of sediments and of deeper structure within a very localized area. These new seismic studies involve the determination of seismic velocities from variable angle reflection data in the form of disposable sonobuoy or multi-trace streamer reflection records.

2.1 Previous work

Previous methods using variable angle reflection data to obtain seismic velocities have either ignored or included the refraction of rays at a seismic interface. In (a) a brief résumé is given of methods which have not taken account of the refraction of rays at an interface while in (b) mention is made of those methods which have.

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(a) No refraction of rays at an interface

The use of variable angle reflection data to obtain seismic velocities is not new. The most important exposés of earlier ray path methods are those of Dix (1955), Taner & Koehler (1969) and, with specific reference to marine sonobuoy work, Le Pichon, Ewing & Houtz (1968).

These methods all consist of obtaining a least squares fit of the data in $t_N^2$ and $x_N^2$ (Fig. 1; for a list of all mathematical symbols used in the text, see Appendix 1). The slope of the line obtained gives an apparent velocity squared, $\bar{v}_{a,N}^2$, from

$$t_{N,x}^2 = t_{N,0}^2 + \frac{x^2}{\bar{v}_{a,N}^2}. \quad (2.1)$$

This is not the interval velocity, but an average, normally called the root mean square (rms) velocity.

Alternatively, this velocity can be considered as a 'Pythagorean velocity' (since only the ray trajectories described by right-angled triangles are considered (Fig. 2) and not the refraction of rays at interfaces), or as an approximation in the polynomial expansion:

$$t_{N,x}^2 = a_1 x^2 + a_2 x^4 + a_3 x^6 + \ldots \quad (2.2)$$

(the coefficients $a_k$ are dependent on interval velocities and thicknesses) where only the first two terms in the expansion are used, i.e.

$$t_{N,x}^2 = a_1 x^2 + a_2 x^2 \quad (\text{Taner & Koehler 1969})$$

where

$$a_1 = t_{N,0}^2 = \tau_N^2, \quad a_2 = \frac{1}{\bar{v}_{a,N}^2}.$$
The derivation of the seismic interval velocity \( v_N \) from the apparent velocity \( \bar{v}_{aN^2} \), is given by Dix, the formula being:

\[
v_N^2 = \frac{\bar{v}_{aN^2} \tau_N - \bar{v}_{N-1}^2 \tau_{N-1}}{(\tau_N - \tau_{N-1})} 
\]  

(2.3)

It should be observed that the interval velocity obtained is dependent only on the root mean square velocities of the layers \( N \) and \( N-1 \), so that errors will not be propagated from upper to lower layers, a useful feature in velocity spectral studies (Taner & Koehler 1969).

The use of this formula assumes a small \( \chi \), and no dip. The distances obtained on sonobuoy records for example, are not necessarily small, so that Le Pichon \textit{et al.} (1968) used the fourth order term \(-a_3 x^4\). The treatment of dip has been more approximate and previous authors have used various methods. A most interesting solution is that of Dix (1955) and Taner \& Koehler (1969) where they use the symmetric properties of time–distance curves obtained from common reflection point data, that is, where source and receiver are equidistant from a fixed ‘Common Reflection Point’. Otherwise, in the sonobuoy method, where we do not have this configuration, the data have been corrected approximately layer by layer, the apparent dips being defined by a coincident normal incidence reflection profile (Le Pichon \textit{et al.} 1968).

\[(b) \text{ Refraction of rays at an interface}\]

A novel approach to the determination of seismic velocities was initiated by Sattlegger (1966). He considered the exact geometric ray paths generated in a multi-layered earth in which the layer velocities were homogeneous but where uniform dips were possible between layers. He then compared the theoretical and observed travel times to minimize:

\[
\sum (\delta t_{N,i})^2 = \sum (t_{N,i} - t_{N,j})^2 
\]

(2.4)

\[
\begin{align*}
\delta t_{N,i} &= \text{theoretical time} \\
t_{N,i} &= \text{observed time} 
\end{align*}
\]

in a least squares scheme, the unknowns being depth, velocity, and dip for each layer. In the scheme, each layer has to be calculated separately, starting from the top and working downwards.

Unfortunately, however, he did not consider the extra dip information given by the normal incidence reflection profile, unlike Le Pichon \textit{et al.} (1968). Neither did he consider the propagation of errors from one layer to the next, assuming the \( N-1 \) upper layers to be determined without error in the calculation of the \( N \)th layer.

\[2.2 \text{ New method}\]

Recently an exact geometric ray path method has been developed at the Institut Français du Pétrole to process variable angle reflection records in deep water regions where we have deep penetration normal incidence reflection profiles. This method, which will be detailed elsewhere (Patriat & Limond, in preparation) uses the equations for dipping layers in which the interval velocities are homogeneous (Slotnick 1959). This is similar to Sattlegger’s method (1966), but makes use of the extra dip information from the normal incidence profile.

Two typical records are shown juxtaposed in Fig. 3, the one to the left is a variable angle reflection profile, and to the right is a normal incidence reflection profile record from exactly the same location. In a normal incidence reflection profile the dips and the initial depth \( P_N \) are defined in terms of two-way travel times \( \tau_{N,a} \) and \( \tau_{N,b} \) (Fig. 4). The use of the two records together facilitates the recognition of seismic discontinuities or interfaces.
FIG. 3. A typical variable angle reflection record is shown on the left beside the corresponding normal incidence reflection profile. The latter defines the dips of the interfaces used in the interpretation of the variable angle reflection record. (See Fig. 4.)

The method consists of determining the arrival time of a reflection from an interface $N$, defined by a characteristic correlation of phases across the record, and using these arrival times $t_{N,i}$ at distance $x_{N,i}$ to obtain the seismic interval velocity $v_N = f(t_N, x_N)$ of the layer $N$, where the velocities, depths and dips of the upper layers 1 to $N-1$ are already defined.

Each layer is treated in turn, starting from the top layer and working down to the deepest. We determine for an initial trial velocity $v_N$ the theoretical values $t_{N,p}, x_{N,p}$ for the raypaths defined by the ray parameters, $p = \tan \theta_0$. The theoretical or calculated value $t_{N,j}$ at $x_{N,j}$ is now calculated, either by iteration or by interpolation, and the time difference

$$\delta t_{N,i} = t_{N,i} - t_{N,j}$$  \hspace{1cm} (2.5)

between the observed and calculated times at distance $x_{N,i}$ is deduced. We minimize this time difference or more exactly, the variance $\sigma^2(\delta t) = \langle \delta t^2 \rangle - \langle \delta t \rangle^2$, varying the velocity $v_N$ to obtain a close fit of the theoretical curve to the observed data.

It should be noted that the only parameter varied in this minimization process is the velocity $v_N$, since the other variables—depth and dip are controlled by the values of time measured on the normal incidence profile, $\tau_{N,a}$ and $\tau_{N,b}$ (Fig. 4).

2.3 Calculation of $t_{N,p}$ and $x_{N,p}$ in the new method

The first step in the determination of the theoretical times and distances is to calculate the apparent dip, $\theta_N$, of the interface $N$ and the depth $P_N$ to this interface.
for the assumed velocity, $v_N$. The apparent dip, $\phi_N$ on the time section (Fig. 4) is given by: $\phi_N = \tan^{-1}[(\tau_{N, a} - \tau_{N, b})/\rho]$. To calculate the apparent dip, $\theta_N$, we use

$$\gamma_{N, 0} = \sin^{-1}[(\tau_{N, a} - \tau_{N, b}) v_1/\rho]$$

which is the angle in the first layer of the ray perpendicular to layer $N$ (Fig. 5). (Brewitt-Taylor & Wright, private communication). By using the Snell's Law relationship and the fact that, by geometry $\theta_N = \gamma_{N, N-1}$ (Fig. 5), we find

$$\frac{\sin (\theta_N - \theta_{N-1})}{v_N} = \frac{\sin (\gamma_{N, N-2} - \theta_{N-1})}{v_{N-1}}$$

assuming a velocity $v_N$. $\gamma_{N, N-2}$ is derived from $\gamma_{N, 0}$ by using Snell's Law for this ray in the layers 1 to $N-1$.

The depth $P_N$ is determined from:

$$P_N = XD \tan (\theta_N) + D_N (v_N/\cos \theta_N) + ZD$$

$$D_N = \tau_{N, a}/2 - SD$$

where $XD$, $SD$, $ZD$ are the calculated horizontal distance, one way travel time and depth respectively of the intersection with interface $N-1$ of the ray perpendicular to interface $N$ (Fig. 5).
Fig. 5. Ray trajectory (thick-dashed line) for wave reflected at interface $N$ where source and receiver are both located at $O$ (normal incidence or vertical reflection).

Fig. 6. Ray trajectory (thick-dashed line) in layer $N$, for ray with parameter $p = \tan \alpha_0$ which is reflected ($RX$) and refracted ($Ra$) at interface $N$. $R_N$ is the reduced or one-way downward distance travelled by the ray in the layer. The terms are explained in the text and Appendix I.
The next step is to calculate the horizontal distance $X_{N,p}$ and depth $Z_{N,p}$ of the intersection of a ray $p$ with the interface $N$, where $Z_{N-1,p}$ and $X_{N-1,p}$ are known, and $\alpha_{N-1}$, $\theta_N$ and $P_N$ are deduced from the velocity $v_N$. The appropriate parameters are displayed in Fig. 6.

By geometry,

\begin{align}
a_{N,p} &= P_N - Z_{N-1,p} - X_{N-1,p} \tan \theta_N \\ b_{N,p} &= a_{N,p} / (1 + \tan \alpha_{N-1} \tan \theta_N) \\ Z_{N,p} &= Z_{N-1,p} + b_{N,p} \\ X_{N,p} &= X_{N-1,p} + b_{N,p} \tan \alpha_{N-1} \\ \tan \beta_N &= \tan (2\theta_N - \alpha_{N-1})
\end{align}

and finally the refraction angle for the next layer $\alpha_N$ can be calculated from

\begin{equation}
\frac{\sin (\alpha_N - \theta_N)}{v_{N+1}} = \frac{\sin (\alpha_{N-1} - \theta_N)}{v_N} \quad \text{(Snell's Law)}
\end{equation}

![Fig. 7. Ray trajectory (solid line with arrows) for wave reflected at interface 2. This interface can be considered as a mirror so that the ray trajectory, $R_3 + R_4$, is equivalent to the upward one-way distance travelled by the ray (after Sattlegger 1966).](https://academic.oup.com/gji/article-abstract/43/3/905/821087/fig/102)
The distance travelled by the downgoing ray in the layer $N$:

$$R_{N, p} = b_{N, p}/\cos \alpha_{N-1}$$  \hspace{1cm} (2.16)

The distance of the upward travelling ray is found by a reversal of the ray tracing process. In Fig. 7, which shows a two-layer case, the reversal is equivalent to tracing the ray from $A_2$ to $A_3$ taking account of the change of co-ordinate system at interface 2. The total horizontal distance, $x_{N, p}$, is obtained directly from $X_{2N, p}$.

The full distance travelled by ray $p$ in the layer $N$ for interfaces with constant dip can also be given by:

$$s_{N, p} = \frac{2a_N \cos \theta_N \cos (\theta_{N-1} - \theta_N)}{\cos (2\theta_N - \theta_{N-1} - \alpha_{N-1})}$$  \hspace{1cm} (2.17)

(from geometry), so that the total theoretical value of two-way travel time, $t_{N, p}$, used for comparison with the data is now calculated from:

$$t_{N, p} = s_{N, p}/v_N + t_{N-1, p},$$  \hspace{1cm} (2.18)

where $t_{N-1, p}$ is the two-way travel time for the ray $p$ reflected at interface $N-1$. This is derived by considering the downward and upward passage of the ray $p$ through the upper $N-1$ layers.

We are now in a position to calculate the theoretical travel time $t_{N, j}$ at observed distance $x_{N, i}$, and hence

$$\delta t_{N, j} = t_{N, i} - t_{N, j}. $$  \hspace{1cm} (2.5)

The variance of these differences is minimized to fit the observed data to the time-distance curve defined by the seismic velocity in the layer, $v_N$.

3. Uncertainty and errors

We have carried out a study of the accuracy of the new method, and of the effect of introducing various controlled errors into the calculation. This study has taken the form of creating models from which we derived exact time values at certain distances, and also out-putting time–distance plots which were then used as normal records to obtain time–distance values. The differences between the two model studies arise from the interpretation errors in marking off times in the second case. The basic model 1 used is shown in Table 1. Variations of this model have been used with varying dips and thicknesses. For example, model 3P is exactly the same as model 1 except that there is a dip of $+2^\circ$ in layers 5 and 6.

Table 1

<table>
<thead>
<tr>
<th>Layer $N$</th>
<th>Velocity $v_N$ (km s$^{-1}$)</th>
<th>Depth $P_N$ (km)</th>
<th>Thickness $h_N$ (km)</th>
<th>Dip $\theta_N$</th>
<th>Model 3P $\theta_N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.51</td>
<td>4.70</td>
<td>4.70</td>
<td>0$^\circ$</td>
<td>0$^\circ$</td>
</tr>
<tr>
<td>2</td>
<td>1.60</td>
<td>4.90</td>
<td>0.20</td>
<td>0$^\circ$</td>
<td>0$^\circ$</td>
</tr>
<tr>
<td>3</td>
<td>1.80</td>
<td>5.10</td>
<td>0.20</td>
<td>0$^\circ$</td>
<td>0$^\circ$</td>
</tr>
<tr>
<td>4</td>
<td>2.10</td>
<td>5.50</td>
<td>0.30</td>
<td>0$^\circ$</td>
<td>0$^\circ$</td>
</tr>
<tr>
<td>5</td>
<td>2.50</td>
<td>5.90</td>
<td>0.40</td>
<td>2$^\circ$</td>
<td>2$^\circ$</td>
</tr>
<tr>
<td>6</td>
<td>3.00</td>
<td>6.20</td>
<td>0.30</td>
<td>2$^\circ$</td>
<td>2$^\circ$</td>
</tr>
<tr>
<td>7</td>
<td>4.00</td>
<td>6.70</td>
<td>0.50</td>
<td>0$^\circ$</td>
<td>0$^\circ$</td>
</tr>
<tr>
<td>8</td>
<td>5.40</td>
<td>7.20</td>
<td>0.50</td>
<td>0$^\circ$</td>
<td>0$^\circ$</td>
</tr>
<tr>
<td>9</td>
<td>6.70</td>
<td>10.00</td>
<td>2.80</td>
<td>0$^\circ$</td>
<td>0$^\circ$</td>
</tr>
</tbody>
</table>
Determination of seismic interval velocities

These models have also been used to compare this new method with the more approximate Dix's method. The first step, however, is to calculate the theoretical uncertainty of a velocity determination.

3.1 Velocity determination— theoretical uncertainties

Since in the new method we minimize the variance of the time differences for all the theoretical raypaths \( j \) in the layer \( N \), the seismic velocity of the layer can be considered as an average velocity,

\[
v_N = \langle v_{N,j} \rangle = \left\langle \frac{s_{N,j}}{t_{N,j} - t_{N-1,j}} \right\rangle
\]  

(3.1)

This formula permits the calculation of the theoretical uncertainty in \( v_N \) from the uncertainties of the quantities \( s_{N}, t_{N}, t_{N-1} \). The formulæ, derived fully in Appendix II, are as follows:

The velocity uncertainty:

\[
\sigma^2(v_N) = \langle (dv_N)^2 \rangle 
\]

\[
\approx \langle (dv(M)_N)^2 \rangle + \langle (\frac{\partial s_N}{\partial v_N})^2 (dv(M)_N)^2 \rangle 
\]

\[
= \sigma^2(v(M)_N) + \frac{\sigma^2(v(M)_N)}{\langle \Delta t_N^2 \rangle} \left\langle \left( \frac{\partial s_N}{\partial v_N} \right)^2 \right\rangle 
\]  

(AII.13)

where the variance of the reduced velocity, \( v(M)_N \) (Appendix II),

\[
\sigma^2(v(M)_N) = \langle (dv(M)_N)^2 \rangle
\]

\[
= \left\langle \left( \frac{ds(M)_N}{\Delta t_N} \right)^2 \right\rangle + v_N^2 \left[ \left\langle \left( \frac{dt_N}{\Delta t_N} \right)^2 \right\rangle + \left\langle \left( \frac{dt_{N-1}}{\Delta t_{N-1}} \right)^2 \right\rangle \right]
\]

+ \[2v_N \left[ \frac{ds(M)_N}{\langle \Delta t_N \rangle} \cdot dt_{N-1} - v_N \left\langle \frac{dt_{N-1}}{\langle \Delta t_N \rangle} \right\rangle - \left\langle \frac{ds(M)_N dt_N}{\langle \Delta t_N \rangle^2} \right\rangle \right] 
\]

(AII.8)

This reduced velocity \( v(M)_N \) has been introduced to clear terms in \( dv_N \) from the right-hand side of equation (AII.18) (Appendix II).

It is apparent that each term in this equation contains \( (1/\Delta t_{N,j}) \) where,

\[
\Delta t_{N,j} = t_{N,j} - t_{N-1,j}
\]

(3.2)

the total time spent by ray \( j \) in the layer \( N \). The uncertainty is therefore dependent both on the length of the profile and on the thickness of the layer. Figs 8 and 9, which are plots of the variation of standard deviations \( \sigma(t_N) \approx \sigma(\delta t_N) \) with velocity \( v_N \), show the effect of the length of the profile. These models use the calculated values of time so that there are no interpretation errors. A comparison of Fig. 8(a) and (b) shows that the minimum standard deviation, \( \sigma^2(t_N) = \langle (dt_N)^2 \rangle \) is not necessarily changed with the length of the profile, but that the shorter profile curves are considerably broadened. The effect of adding interpretation errors (Fig. 10) is to increase \( \sigma(t_N) \) in the neighbourhood of the minimum, but not to broaden the curve.

3.2 Random errors in the layer

Consider the standard deviation of the velocity in the layer, due uniquely to the scatter of \( t_{N,j} \), neglecting the effects of upper layers 1 to \( N-1 \), that is:

\[
\sigma(v_N) = \sqrt{v_N^2 \left\langle \left( \frac{dt_N}{\Delta t_N} \right)^2 \right\rangle} 
\]  

(3.3)
Determination of seismic interval velocities

For a trial model 1 with exact data values we find the minima of $\sigma(v_N)$ of layer 6 for the two different lengths $D = 5$ km and $D = 9$ km, to be $4.9$ s$^{-1}$ and $2.8$ s$^{-1}$ whereas the minima of $\sigma(t_N)$ minimum values are $0.268$ ms and $0.243$ ms respectively (Figs 8(a) and (b)). Thus a weighting has been introduced for the length of the profile in the determination of velocity uncertainty, $\sigma(v_N)$. Evidently, to reduce this uncertainty one must try to use long profiles. For the same reasons, since $\Delta t_N$ is dependent on the thickness of the layer $N$ and also $\sigma^2(v_N) \propto \langle (\Delta t_N)^2 \rangle$ it is advisable not to treat layers 3.3(a) which are too thin.

The importance of the need for long profiles emphasizes one of the weaknesses in Dix's method (1955), where the approximation of small $x$ becomes untenable and the systematic error due to this is aggravated.

In equation (3.2) we take account only of the uncertainty and random scatter of data values $t_{N,j}$ for the layer. Factors influencing this uncertainty are the precision of the theoretical calculation which is limited only by the precision of the calculating machine (Patriat & Limond, in preparation) and the accuracy of determination of the time values, where we have to consider the scale of the records, the noise level, the picking of phases, the accuracy of digitization, and the geology itself.

In order to improve the statistics of the method it is necessary to have as many good data points as possible. A selection procedure could be established using confidence limits to reject bad data (Wolberg 1967, p. 68), but we do not consider this problem here.

3.3 Propagation of random errors

Up to this point only the uncertainty in $t_{N,j}$, $\sigma^2(t_{N,j})$ has been considered, that is, the quantity which defines the lower interface of the layer. In order to calculate the velocity $v_N$, however, the method needs the velocities, depths, and dips of the preceding $N-1$ layers. If there are errors or scatter in these there will be a corresponding uncertainty in the determination of $v_N$. These uncertainties form the remainder of equation (11.8).

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Fig. 8. Model 1 (no dip). Variation of standard deviation $\sigma(t)$, with velocity $v_N$, for calculated data values. $N =$ no. of data points. $D =$ length of profile (km).

First four layers input. Velocities determined for layers 5, 6 are:

(a) $v_5 = 2.500 \pm 0.002$ km s$^{-1}$ 
$v_6 = 3.000 \pm 0.005$ km s$^{-1}$ 
\begin{align*} 
\sigma(t) &= 0.25 \text{ ms;} & \sigma(v) &= 1.80 \text{ m s}^{-1} \\
\sigma(t) &= 0.27 \text{ ms;} & \sigma(v) &= 3.56 \text{ m s}^{-1} \\
\sigma(v) &= 4.90 \text{ m s}^{-1}
\end{align*}

(b) $v_5 = 2.500 \pm 0.001$ km s$^{-1}$ 
$v_6 = 3.000 \pm 0.003$ km s$^{-1}$ 
\begin{align*} 
\sigma(t) &= 0.14 \text{ ms;} & \sigma(v) &= 0.78 \text{ m s}^{-1} \\
\sigma(t) &= 0.24 \text{ ms;} & \sigma(v) &= 2.20 \text{ m s}^{-1} \\
\sigma(v) &= 2.80 \text{ m s}^{-1}
\end{align*}

(c) $v_5 = 2.500 \pm 0.001$ km s$^{-1}$ 
$v_6 = 3.000 \pm 0.003$ km s$^{-1}$ 
\begin{align*} 
\sigma(t) &= 0.22 \text{ ms;} & \sigma(v) &= 1.20 \text{ m s}^{-1} \\
\sigma(t) &= 0.25 \text{ ms;} & \sigma(v) &= 2.40 \text{ m s}^{-1} \\
\sigma(v) &= 3.20 \text{ m s}^{-1}
\end{align*}

NB The first standard deviations $\sigma(v)$ are quoted for the time data scatter in the layer.

\[
\sigma^2(v) = \frac{\nu^2}{\langle \Delta t^2 \rangle} \sigma^2(t_N).
\]

In layer 6 the second $\sigma(v)$ considers the scatter in the layer above.

\[
\sigma^2(v) = \frac{\nu^2}{\langle \Delta t^2 \rangle} \left[ \sigma^2(t_N) + \sigma^2(t_{N-1}) \right]
\]
Fig. 9. Model 3P. Variation of $\sigma(t)$ with $v$ using calculated data values. $N =$ no. of data points; $D =$ length of profile (km). First four layers input. Velocities determined for layers 5 and 6.

(a) $v_5 = 2.500 \pm 0.002 \text{ km s}^{-1}$ \hspace{1cm} $\sigma(t) = 0.56 \text{ ms}$; \hspace{1cm} $\sigma(\sigma) = 1.64 \text{ m s}^{-1}$

$\nu_6 = 3.000 \pm 0.004 \text{ km s}^{-1}$ \hspace{1cm} $\sigma(t) = 0.22 \text{ ms}$; \hspace{1cm} $\sigma(\sigma) = 3.06 \text{ m s}^{-1}$

$\nu(\nu) = 3.84 \text{ m s}^{-1}$

(b) $v_5 = 2.500 \pm 0.003 \text{ km s}^{-1}$ \hspace{1cm} $\sigma(t) = 0.50 \text{ ms}$; \hspace{1cm} $\sigma(\sigma) = 2.56 \text{ m s}^{-1}$

$\nu_6 = 3.000 \pm 0.004 \text{ km s}^{-1}$ \hspace{1cm} $\sigma(t) = 0.38 \text{ ms}$; \hspace{1cm} $\sigma(\sigma) = 2.96 \text{ m s}^{-1}$

$\sigma(\sigma) = 4.30 \text{ m s}^{-1}$

See NB in caption of Fig. 8.
The variance,
\[ \sigma^2(t_{N-1}) = \langle (dt_{N-1})^2 \rangle \]  
(3.4)
is the easiest to understand since it expresses the uncertainty in the time values \( t_{N-1} \), defining the upper interface of the layer \( N \).

The variance,
\[ \sigma^2(s(M)N) = \langle ds(M)N \rangle^2 \]  
(3.5)
is however more complicated. In equation (II.6), we see that the uncertainty of \( s_N \) is dependent on the known or calculated uncertainties of several other quantities:
\[ \sigma^2(v_{N-1}), \sigma^2(\theta_{N-1}), \sigma^2(x_{N-2}), \sigma^2(Z_{N-1}), \sigma^2(X_{N-1}), \sigma^2(SD), \sigma^2(XD), \sigma^2(ZD), \]
which are calculated from the previous layer;
\[ \sigma^2(\tau_N), \] the uncertainty reading time values from the record;
\[ \sigma^2(\gamma_{N,N-2}), \] the uncertainty in the angle to the vertical in layer \( N-1 \) of the ray which is perpendicular to interface \( N \), and is calculated, assuming \( \sigma^2(\tau_N) \) and \( \sigma^2(\rho) \), the uncertainty in the length of the profile.

Theoretically, we are now prepared to predict the uncertainty in the velocity \( v_N \), \( \sigma^2(v_N) \), apart from systematic errors, and providing that we have a reasonable estimate of \( \sigma^2(\tau_N) \) and \( \sigma^2(\rho) \). This has been carried out for the same models used previously, and the results are given in Tables 2 and 3. The variation of \( \sigma(v_N) \) with velocity \( v_N \) is represented in Figs 11-13.

It should be remembered, however, that the equations derived in Appendix II are valid only when the deviations \( d(t_N), d(t_{N-1}), dv_N \) are small (Wolberg 1967), so that too much reliance should not be placed on these variation curves.

The graphs in Figs 11 and 12(a) show that although dip does not greatly change the uncertainty the velocity determination may be not as good, as demonstrated by the slight flattening out of the curve in Fig. 12(a) compared to Fig. 11. There is a similar effect of narrowing the curve when the profile is lengthened (Fig. 12(b)), but surprisingly the error in layer 6 has been substantially increased. This increase is understandable when we consider the contributions of the various terms to the uncertainty. These are represented in Table 2(a) and (b) for the two different profile lengths of 5 and 9 km, for the layer 6. When the profile length is increased to 9 km, the uncertainty in the distance \( \sigma^2(s(M)N)/\langle \Delta t^2 \rangle \) increases substantially, even though the uncertainty in the times, \( \sigma^2(t_N)(v^2/\langle \Delta t^2 \rangle) \) has been slightly reduced (the scatter \( \sigma(t_N) \) for model 3P at \( D = 9 \) km was much greater than for \( D = 5 \) km (Fig. 9), thus explaining the small reduction). The greatest changes come from the augmentation of the factors \( (\partial s_N/\partial s_{N-2}) \) and \( (\partial s_N/\partial v_{N-1}) \) which both affect the ray angle \( \alpha_{N-1} \). Thus although the minimization of \( \sigma^2(t_N) \) may be greatly improved by lengthening the profile (Figs 8 and 9) this does not necessarily mean that the final uncertainty will be less, since the uncertainty in the raypath \( s_N \) is increased.

It is noticeable from the curves (Figs 11 and 12) that the minimum of \( \sigma^2(v_N) \) does not always define the best velocity estimate. All these effects are, however, rather artificial since we are only dealing with ideal calculated data, and the term \( \sigma^2(s(M)N) > \sigma^2(t_N) \).

The results of adding digitization errors to the data can be seen by comparing Figs 12(a) and 13 and Tables 2(a) and 3(a); all the uncertainties have been substantially increased. If we now consider the difference between the uncertainties when the velocities of only layers 5 and 6 have to be determined (Table 3(a)) and the uncertainties when the velocities of layers 2 onwards have to be determined (Table 3(c)), it is observed that there is another increase in all the uncertainties. This shows the effect of the propagation of random errors from more than one layer. A study of Table 3(b) and a comparison with Table 1, shows that the final velocity uncertainties thus determined are reasonable.
The effect of using a longer profile with the digitized data values can be seen by comparison of Table 3(a) and (d). The uncertainties in \( t_N \) and \( t_{N-1} \) have been reduced, whereas the uncertainty in \( s_N \) has been increased sufficiently to raise the uncertainty in velocity, \( \sigma(v_N) \), from 0.030 for \( D = 5 \text{ km} \) to 0.044 for \( D = 9 \text{ km} \) for layer 6.

3.4 Systematic errors and their propagation

The preceding discussion of the equation (11.8) treats errors of a random nature, but we must also take into account errors of a systematic and interpretational nature, which can be serious since they are difficult to quantize. These may be classified into two groups; those relevant to any method of velocity determination, and those which are specific to the method described in this paper. Later, we compare this method with Dix's method, to show that, as expected there is a systematic error in the latter, associated with the approximation used.

If there are ill-defined distance or time scales, the values of time and distance \( t_{N-1} \) and \( x_{N-1} \) will be systematically wrong leading to an erroneous velocity. This is a very grave danger in the sonobuoy method where distances have to be calculated from the arrival times of the direct water wave, or the wave reflected from the sea bottom (Patriat & Limond, in preparation). An allowance can be made for this error in the calculation by using a large uncertainty in the time \( \sigma^2(t_N) \).

At the interpretational stage, if false phase correlations are picked then the velocity will be correspondingly wrong. This is possible where the records are noisy. A related problem is which part of the phase to pick, since the signal might involve several phases (bubble pulse), or there may be complicated interference effects at greater distances between the time distance curves of layers with different velocities (varying 'moveout'). Furthermore, it is possible to pick a multiple reflection as a first arrival or even to miss out a layer.

Apart from these general errors there are errors associated with the new method. As explained, this uses a combination of the variable angle reflection profile and the corresponding normal incidence reflection profile in order to define the apparent dip, \( \phi_N = \tan^{-1}[(\tau_{N,a} - \tau_{N,b})/\rho] \). If the first portion of the variable angle reflection profile is missing, it is possible to make a false correlation between the phases on the two different profiles.

![Diagram](https://academic.oup.com/gji/article-abstract/43/3/905/821087)
Fig. 10. Model 3P using data digitized from time distance plots (thus including random interpretational errors). $N =$ no. of data points. $D =$ length of profile (km).

(a) First four layers input. Layers 5 and 6 determined. (cf. Table 1.)

(i) Before (change of $\tau_{W, x}$) (dashed line):

\begin{align*}
 v_5 &= 2.580 \pm 0.069 \text{ km s}^{-1} & \sigma(t) &= 1.19 \text{ ms}; & \sigma(v) &= 69.47 \text{ m s}^{-1} \\
v_6 &= 3.080 \pm 0.117 \text{ km s}^{-1} & \sigma(t) &= 3.06 \text{ ms}; & \sigma(v) &= 116.76 \text{ m s}^{-1} \\
\end{align*}

See NB in caption of Fig. 8.

(ii) After:

\begin{align*}
 v_5 &= 2.520 \pm 0.007 \text{ km s}^{-1} & \sigma(t) &= 1.19 \text{ ms}; & \sigma(v) &= 6.69 \text{ m s}^{-1} \\
v_6 &= 3.000 \pm 0.029 \text{ km s}^{-1} & \sigma(t) &= 3.05 \text{ ms}; & \sigma(v) &= 26.95 \text{ m s}^{-1} \\
\end{align*}

(b) Only first layer input. Layers 2–8 determined.
Table 2(a)

Model 3P. Layer 6, \( np = 5, D = 5 \text{ km} \). Calculated data values. Partial derivatives multiplied by standard deviations. (See Appendix II).

<table>
<thead>
<tr>
<th>( X_{N,1} ) (km)</th>
<th>( \frac{\partial \psi_N}{\partial \psi_{N-1}} )</th>
<th>( \frac{\partial \psi_N}{\partial \theta_{N-1}} )</th>
<th>( \frac{\partial \psi_N}{\partial X_{N-1}} )</th>
<th>( \frac{\partial \psi_N}{\partial X_{N-2}} )</th>
<th>( \frac{\partial \psi_N}{\partial Z_{N-1}} )</th>
<th>( \frac{\partial \psi_N}{\partial Y_{N-1}} )</th>
<th>( \frac{\partial \psi_N}{\partial Y_{N-2}} )</th>
<th>( \frac{\partial \psi_N}{\partial Z_{N-2}} )</th>
<th>( \frac{\partial \psi_N}{\partial Y_{N-2}} )</th>
</tr>
</thead>
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<td>0.018465</td>
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<td>0.0015063</td>
<td>0.000150</td>
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<td>2</td>
<td>0.000620</td>
<td>0.000449</td>
<td>0.018465</td>
<td>0.001703</td>
<td>0.001100</td>
<td>0.014755</td>
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<td>0.002501</td>
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<tr>
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<td>0.000680</td>
<td>0.018465</td>
<td>0.002561</td>
<td>0.001730</td>
<td>0.014419</td>
<td>0.000227</td>
<td>0.004150</td>
<td>0.002366</td>
</tr>
<tr>
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<td>0.003426</td>
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</tr>
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<td>0.018466</td>
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<td>0.003480</td>
<td>0.013616</td>
<td>0.000355</td>
<td>0.000805</td>
<td>0.001948</td>
</tr>
</tbody>
</table>

\[
\frac{\sigma^2(s(M)_N)}{\langle \Delta t^2 \rangle} = 5 \cdot 10^{-6}
\]

\[
\sigma^2(t_N) \left( \frac{\sigma^2}{\langle \Delta t^2 \rangle} \right) = 9 \cdot 36 \cdot 10^{-6}
\]

\[
\sigma^2(t_{N-1}) \left( \frac{\sigma^2}{\langle \Delta t^2 \rangle} \right) = 5 \cdot 418 \cdot 10^{-6}
\]

NB In \( ds(c) \) we have collected the terms \( d r_N, d(SD), d(ZD), d(SD) \).
### Table 2(b)

**Model 3P. Layer 6, np = 5, D = 9 km. Calculated data values. Partial derivatives multiplied by standard deviations. (See Appendix II)**

<table>
<thead>
<tr>
<th>X_{N-1} (km)</th>
<th>( \frac{\partial v_{N}}{\partial v_{N-1}} )</th>
<th>( \frac{\partial v_{N}}{\partial t_{N-1}} )</th>
<th>( \frac{\partial v_{N}}{\partial N_{N-2}} )</th>
<th>( \frac{\partial v_{N}}{\partial N_{N-3}} )</th>
<th>( \frac{\partial v_{N}}{\partial X_{N-1}} )</th>
<th>( \frac{\partial v_{N}}{\partial X_{N-1}} )</th>
<th>( \frac{\partial v_{N}}{\partial Z_{N-1}} )</th>
<th>( \frac{\partial v_{N}}{\partial Z_{N-1}} )</th>
<th>( \frac{\partial v_{N}}{\partial t_{N}} )</th>
<th>( \frac{\partial v_{N}}{\partial t_{N}} )</th>
</tr>
</thead>
<tbody>
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<td>-0.000978</td>
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</tr>
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<td>0.009509</td>
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<td></td>
</tr>
<tr>
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<td>-0.025159</td>
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<td>-0.007051</td>
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<tr>
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<td>-0.009297</td>
<td>0.059874</td>
<td>-0.015157</td>
<td>-0.001092</td>
<td>-0.003380</td>
<td>0.001427</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \frac{\sigma^2(d(M)_{d})}{\sigma^2(d(M)_{d})} = 3 \cdot 10^{-4} \]

\[ \sigma^2(t_{N}) \left( \frac{u^2}{\sigma^2(t_{N})} \right) = 8.739 \cdot 10^{-6} \]

\[ \sigma^2(t_{N-1}) \left( \frac{u^2}{\sigma^2(t_{N-1})} \right) = 9.77 \cdot 10^{-6} \]

**NB.** In \( ds(c) \) we have collected the terms \( d(r_{N}), d(SD), d(ZD), d(XD) \).
Table 3(a)

Model 3P. Layer 6, \( n_p = 11, D = 5 \) km. Digitized data values. (First four layers input.) Partial derivatives multiplied by standard deviations. (See Appendix II)

\[
\begin{array}{cccccccccccc}
X_{N,1} & \left( \frac{\partial \theta N}{\partial \phi N_{-1}} \right) d\theta N_{-1} & \left( \frac{\partial \phi N}{\partial \phi N_{-1}} \right) d\phi N_{-1} & \left( \frac{\partial \theta N}{\partial \phi N} \right) d\phi N & \left( \frac{\partial \theta N}{\partial \phi N_{-2}} \right) d\phi N_{-2} & \left( \frac{\partial \theta N}{\partial Z} \right) d\phi N & \left( \frac{\partial \theta N}{\partial X N} \right) d\phi N & \left( \frac{\partial \theta N}{\partial \theta N} \right) d\phi N & \left( \frac{\partial \phi N}{\partial \theta N} \right) d\phi N & \left( \frac{\partial \phi N}{\partial \phi N} \right) d\phi N & \left( \frac{\partial \phi N}{\partial Z} \right) d\phi N & \left( \frac{\partial \phi N}{\partial X N} \right) d\phi N \\
0 & -0.000001 & -0.000000 & 0.054736 & 0.000001 & -0.044829 & -0.000034 & 0.006543 & 0.012060 & 0.006543 & 0.012060 \\
0.5 & -0.000099 & 0.000316 & 0.054737 & -0.001247 & 0.006000 & -0.044390 & -0.000348 & 0.003889 & 0.012029 & 0.003889 & 0.012029 \\
1.0 & -0.000394 & 0.000633 & 0.054736 & -0.002496 & 0.012066 & -0.043945 & -0.000379 & 0.013047 & 0.011936 & 0.013047 & 0.011936 \\
1.5 & -0.000898 & 0.000951 & 0.054736 & -0.003746 & 0.01826 & -0.043490 & -0.000426 & 0.008881 & 0.011781 & 0.008881 & 0.011781 \\
2.0 & -0.001634 & 0.001272 & 0.054737 & -0.004998 & 0.02469 & -0.043021 & -0.000484 & 0.011919 & 0.011564 & 0.011919 & 0.011564 \\
2.5 & -0.002628 & 0.001596 & 0.054737 & -0.006254 & 0.03148 & -0.042529 & -0.000551 & -0.000379 & 0.011286 & -0.000379 & 0.011286 \\
3.0 & -0.003926 & 0.001925 & 0.054736 & -0.007514 & 0.03876 & -0.042012 & -0.000627 & 0.021502 & 0.010949 & 0.021502 & 0.010949 \\
3.5 & -0.005601 & 0.002260 & 0.054737 & -0.008780 & 0.04673 & -0.041463 & -0.000711 & 0.075572 & 0.010553 & 0.075572 & 0.010553 \\
4.0 & -0.007739 & 0.002603 & 0.054738 & -0.010053 & 0.05561 & -0.040880 & -0.000803 & 0.032928 & 0.010100 & 0.032928 & 0.010100 \\
4.5 & -0.010471 & 0.002956 & 0.054738 & -0.011334 & 0.06573 & -0.040258 & -0.000903 & 0.015998 & 0.009593 & 0.015998 & 0.009593 \\
5.0 & -0.013982 & 0.003322 & 0.054737 & -0.012624 & 0.07754 & -0.039599 & -0.001011 & 0.005379 & 0.009034 & 0.005379 & 0.009034 \\
\end{array}
\]

\[
\frac{\sigma^2(s(M))}{\langle \Delta t^2 \rangle} = 5 \times 10^{-5}
\]

\[
\frac{\sigma^2(t_{N-1}) v^2}{\langle \Delta t^2 \rangle} = 7.263 \times 10^{-4}
\]

\[
\frac{\sigma^2(t_{N-2}) v^2}{\langle \Delta t^2 \rangle} = 1.215 \times 10^{-4}
\]

NB. In \( ds(c) \) we have collected the terms \( d\theta N, d(SD), d(ZD), d(XD) \).
The program uses $\tau_{N,a}$ to calculate the models used for comparison and fits a curve parallel to the observed data curve, since we minimize the variance $\sigma^2(\delta t_n)$. The correction $\Delta T_{N,a} = \langle \delta t_{N,a} \rangle$ can thus be automatically applied to fit the theoretical curve more closely to the data (Figs 10 and 14), i.e. by reducing $\sigma^2(\delta t_n)$. There is a danger, however, that only one of the points $\tau_n$ used in the dip calculation may be in error, leading to a false dip and hence erroneous velocity. This error will affect the layer in question, and also the following layer similar to the effect of having an erroneous upper layer velocity.

To calculate the velocity in the $N$th layer, the method requires the velocities, dips and depths of the upper $N-1$ layers. Should there be a systematic error in one of these layers there will be a propagation of this systematic error into the $N$th layer.

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Layer & $v$ & $\sigma_v$ & $\theta$ & $\sigma_\theta$ \\
\hline
1 & 1.51 & & & \\
2 & 1.62 & 0.022 & 4.90 & 0.002 \\
3 & 1.82 & 0.014 & 5.10 & 0.004 \\
4 & 2.12 & 0.009 & 5.95 & 0.004 \\
5 & 2.48 & 0.017 & 5.90 & 0.006 & 1.982 & 0.057 \\
6 & 2.98 & 0.031 & 6.19 & 0.011 & 1.982 & 0.058 \\
7 & 4.02 & 0.052 & 6.70 & 0.020 & 0.010 & 0.078 \\
\hline
\end{tabular}
\caption{Model 3P, using digitized data values. Layer 1 known, 2–7 unknown. (cf. Table 1)}
\end{table}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig11}
\caption{Model 1. Full error calculation, using calculated data input. Variation of $\sigma(v)$ with $v$. First four layers input. Layers 5 and 6 determined.}
\end{figure}

$\sigma(\tau_n) = 1 \text{ ms} \quad \nu_2 = 2.500 \pm 0.008 \text{ km s}^{-1} \quad \sigma(\rho) = 1 \text{ m} \quad \nu_6 = 3.000 \pm 0.006 \text{ km s}^{-1}$
Table 3(c)

Model 3P. Layer 6, \( np = 11, D = 5 \text{ km} \). Digitized data values. (First layer only input.) Partial derivatives multiplied by standard deviations. 

(See Appendix II)

<table>
<thead>
<tr>
<th>( x_{n,1} ) (km)</th>
<th>( \frac{\partial \phi_N}{\partial \phi_{N-1}} )</th>
<th>( \frac{\partial \phi_N}{\partial \phi_{N-1}} )</th>
<th>( \frac{\partial \phi_N}{\partial \phi_{N-1}} )</th>
<th>( \frac{\partial \phi_N}{\partial \phi_{N-1}} )</th>
<th>( \frac{\partial \phi_N}{\partial \phi_{N-1}} )</th>
<th>( \frac{\partial \phi_N}{\partial \phi_{N-1}} )</th>
<th>( \frac{\partial \phi_N}{\partial \phi_{N-1}} )</th>
<th>( \frac{\partial \phi_N}{\partial \phi_{N-1}} )</th>
<th>( \frac{\partial \phi_N}{\partial \phi_{N-1}} )</th>
<th>( \frac{\partial \phi_N}{\partial \phi_{N-1}} )</th>
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</tbody>
</table>

\[
\frac{\sigma^2(M)}{\langle \Delta^2 \rangle} = 9 \times 10^{-5}
\]

\[
\frac{\sigma^2(t_n)}{\langle \Delta^2 \rangle} = 7.299 \times 10^{-4}
\]

\[
\frac{\sigma^2(t_{N-1})}{\langle \Delta^2 \rangle} = 1.242 \times 10^{-4}
\]

NB. In \( ds(c) \) we have collected the terms \( d\tau_N, d(SD), s(ZD); d(XD) \).
Table 3(d)

Model 3P. Layer 6, $np = 10$, $D = 9$ km. Digitized data values. (First four layers input.) Partial derivatives multiplied by standard deviations. (See Appendix II)

<table>
<thead>
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<th>$X_{N,t}$ (km)</th>
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$$\frac{\sigma^2(s(M))}{\langle \Delta t^2 \rangle} = 1.68 \times 10^{-3}$$

$$\frac{\sigma^2(t_c) c^2}{\langle \Delta t^2 \rangle} = 1.683 \times 10^{-4}$$

$$\frac{\sigma^2(t_{C-1}) c^2}{\langle \Delta t^2 \rangle} = 7.128 \times 10^{-5}$$

NB. In $d\alpha(c)$ we have collected the terms $d\alpha_n$, $d(\alpha SD)$, $d(\alpha ZD)$, $d(\alpha XD)$. 

Determination of seismic interval velocities
Fig. 12. Model 3P. Full error calculation, using calculated data input. Variation of \( \sigma(v) \) with \( v \). First four layers input. Layers 5 & 6 determined.

\[
\begin{align*}
\sigma(\tau_N) & = 1 \text{ ms} \\
\sigma(\rho) & = 1 \text{ m} \\
\end{align*}
\]

(a) \( v_5 = 2.500 \pm 0.009 \text{ km s}^{-1} \) \hspace{1cm} (b) \( v_5 = 2.500 \pm 0.013 \text{ km s}^{-1} \)

\( v_5 = 3.000 \pm 0.005 \text{ km s}^{-1} \) \hspace{1cm} \( v_6 = 3.000 \pm 0.017 \text{ km s}^{-1} \)

We have carried out trials on calculated models to investigate the effects of some of these errors, such as missing out a layer (Fig. 15 and Table 4) and erroneous upper velocities (Fig. 16). Obviously, the layer immediately after the erroneous layer is strongly affected, but it seems that the error is averaged over two layers, and the correct velocity is again found in the succeeding layers. This appears to be similar to the result of using Dix's equation (Dix 1955), where rms velocities for the layers \( N - 1 \) and \( N \) are used to calculate \( v_N \).
4. Comparison of the method with Dix's method

We have therefore used the calculated data from model 1, without dip, as a test of Dix's method, to compare it with the one described here. We have only considered layers 4, 5, 6 assuming the upper layers 1, 2, 3 to be known (Table 1). If we compare the results of Table 4 (ii) with the results of Tables 4(i), the corresponding geometric ray path method solutions, we find that there is a systematic error in the former. When we further consider the effect of missing out layer 4, Tables 4(d) and (e), we find that there is little change in the velocity of layer 6, but that this is also the case with the geometric ray path method solution, Table 4d(i) and 4e(i). This result proves that: the exact ray path solution is useful in the case of long profiles where, as implied by Dix (1955), Brown (1969) and Shah & Levin (1973), the approximation \( x_0 \) is no longer valid; for deep multi-layered systems where the rejection of the terms \( a_4 x^6 + a_5 x^8 + \ldots \) leads to a systematic error (Cressman 1968); and above all in the case of dip. Errors due to dip have been studied for common reflection point configurations by the authors already cited, as well as by Levin (1971), but the error in using the Dix equation in a fixed end point (sonobuoy) profile has not been treated.

5. Use of the variance

It can be seen that the use of the minimum variance \( \sigma^2(\delta t_N) \) to define the velocity \( v_N \) has been justified experimentally. Fig. 14 shows the curves for the velocities which would be obtained using the parameters:

\[
\langle \delta t_{N,j} \rangle, \quad \langle |\delta t_{N,j}| \rangle \approx \langle (\delta t_{N,j})^2 \rangle, \quad \text{and} \quad \sigma^2(\delta t_{N,j}) = [\langle (\delta t_{N,j})^2 \rangle - \langle (\delta t_{N,j}) \rangle^2]
\]

in the case where an erroneous \( \tau_{N,4} \) from the normal incidence reflection profile has been used to calculate the models. The last parameter when minimized traces a curve parallel to the data points 1, 2, 3. The variance can be further minimized using the correction

\[
\tau_{N, v(new)} = \tau_{N, v(old)} - \Delta T_N, \quad 0
\]

\[
= \tau_{N, v(old)} - \langle \delta t_{N,j} \rangle \mid \sigma(t) \text{ minimum}
\]

so that the final curve passes through the data points, reducing \( \sigma(t_N) \).
USE OF ΔT₀
MODEL 1

FIG. 14. Time-distance curves generated by the minima of the functions <Δt>, |<Δt>|, |<Δt>|², σ(Δt), where the model starting times τ₁ and τ₂ are in error by ⟨Δt⟩ = 100 ms. These times are then corrected to provide the close fit to the data points 1, 2, 3, shown by the thin line σ(Δt). Correct velocity is 5.4 km s⁻¹.

Before: 

v(Δt) = 4.3 km s⁻¹ (dots)  
ν(Δt) = 5.3 km s⁻¹ (dashes)  
ν(Δt²) = 5.3 km s⁻¹ (dashes)  
ν(σ(Δt)) = 5.82 km s⁻¹ (thick line)

After: 

ν(σ(Δt)) = 5.36 km s⁻¹ (thin line)
Determination of seismic interval velocities

Fig. 15. Study of effect of missing out a layer. Model 1, using calculated data. Plot of $\sigma(t)$ against $v$ for the case where an upper layer, 4, has been missed out completely. $N =$ no. of data points. $D =$ length of profile (km).

(a) $v_1 = 2.300 \pm 0.001 \text{ km s}^{-1}$  \hspace{1cm} $\sigma(t) = 0.34 \text{ ms}$; \hspace{1cm} $\sigma(v) = 1.03 \text{ m s}^{-1}$

$V_6 = 2.990 \pm 0.006 \text{ km s}^{-1}$  \hspace{1cm} $\sigma(t) = 0.24 \text{ ms}$; \hspace{1cm} $\sigma(v) = 3.20 \text{ m s}^{-1}$

(b) $v_7 = 2.300 \pm 0.004 \text{ km s}^{-1}$  \hspace{1cm} $\sigma(t) = 1.53 \text{ ms}$; \hspace{1cm} $\sigma(v) = 3.89 \text{ m s}^{-1}$

$V_8 = 3.000 \pm 0.014 \text{ km s}^{-1}$  \hspace{1cm} $\sigma(t) = 0.41 \text{ ms}$; \hspace{1cm} $\sigma(v) = 3.64 \text{ m s}^{-1}$

$\sigma(t) = 3.64 \text{ ms}$; \hspace{1cm} $\sigma(v) = 14.1 \text{ m s}^{-1}$

See NB in caption of Fig. 8.
Fig. 16. Study of effect of an erroneous velocity in a previous layer. Model 3P,
using calculated data. Plot of $\sigma(t)$ against $\nu$ for the case where:

(a) $\nu_4$ is 1.9 km s$^{-1}$ instead of 2.1 km s$^{-1}$.

\[ \begin{align*}
\nu_5 &= 2.680 \pm 0.013 \text{ km s}^{-1} & \sigma(t) &= 1.78 \text{ ms}; \\
\nu_6 &= 3.020 \pm 0.021 \text{ km s}^{-1} & \sigma(\nu) &= 12.96 \text{ m s}^{-1}
\end{align*} \]

(b) $\nu_4$ is 2.3 km s$^{-1}$ instead of 2.1 km s$^{-1}$.

\[ \begin{align*}
\nu_5 &= 2.160 \pm 0.004 \text{ km s}^{-1} & \sigma(t) &= 0.56 \text{ ms}; \\
\nu_6 &= 2.990 \pm 0.008 \text{ km s}^{-1} & \sigma(\nu) &= 4.30 \text{ m s}^{-1}
\end{align*} \]

See caption of Fig. 8 for NB.
Comparison of (i) new method with (ii) Dix's method, using calculated data values for model 1. \( N = \) no. of data points. \( D = \) distance in km.

(a) \( N = 5; \ D = 5 \). (b) \( N = 5; \ D = 9 \). (c) \( N = 9; \ D = 9 \). (d) \( N = 5; \ D = 5 \) (miss out layer 4). (e) \( N = 5; \ D = 9 \) (miss out layer 4).

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The use of the two time functions \( \sigma^2(\delta t_N) \) and \( \langle \delta t_{N,i} \rangle \) thus establishes a powerful method as can be seen in Figs 10 and 14, where it has been used with digitized data. Theoretically, to find \( v_N = f(t_N, x_N) \) we should try to minimize \( \sigma^2(v_N) \), but this is a function of several terms (equation (II.6)).
The fact that we minimize $\sigma^2(\delta t_N)$ thus has three implications. The first is that we consider only errors in $t_N$, and not in $x_N$, which are therefore contained in $\sigma^2(t_N)$ in the determination of the velocity $v_N$. The second implication is that we neglect the errors in the preceding layers, except in the calculation of the uncertainty of the velocity. This last approximation is valid since the variance $\sigma^2(t_N)$ is larger, except for calculated data, or varies with velocity more rapidly than the sum of the rest of the quantities in the error equation (equation (II.8)). This can be seen in Fig. 13, where we plot the variation with velocity of $\sigma(v_N)$ and of $[\sigma^2(v_N) - \sigma^2(t_N)(v/\Delta t_N)^2]^{1/2}$ separately. This latter contains the variances and covariances of all the other terms. It is noticeable that this is held relatively constant by the covariances, thus showing the importance of these quantities (Brandt, 1973, p. 33).

Finally, we minimize the variance, $\sigma^2(\delta t)$ of the time differences rather than the more usually employed variance, $\sigma^2(\delta t)$, since we admit that there is a possible error in the $\tau_{N-1}$ value of the comparison model, and, as already demonstrated, this can be corrected by the use of a curve, parallel to the data points, given by the minimum of $\sigma^2(\delta t)$ (Fig. 14).

It should be pointed out here that the difference between a model velocity and the velocity at the minimum of $\sigma(t)$ is not a linear function of the variances $\sigma^2(t_N)$ or $\sigma^2(v_N)$ (Figs 10 and 13). Thus a straightforward least squares scheme (Sattlegger 1966) can be dangerous, as we have found out when trying to use Lagrangian interpolation to find the minimum of $\sigma^2(t_N)$ (Patriat & Limond, in preparation).

6. Conclusion

We have developed a new method of obtaining interval seismic velocities from the time–distance data of long variable angle reflection profiles. The details of this method are described more fully in another paper (Patriat & Limond, in preparation).

Even though the method needs the velocities, depths and dips of the upper $N-1$ layers to calculate the velocity in the $N$th layer, we can define the uncertainty with which the velocity is determined, taking account of the random errors in the layers 1 to $N$ and the propagation of the errors from the upper $N-1$ layers into the $N$th layer. This method is stable since only the data points $t_N$ and $x_N$ for the layer in question, are used. Thus even if a systematic error is encountered, this has been shown to be averaged over two layers, and the following layer is only very slightly affected (Table 4, Figs 15 and 16). This is similar to Dix’s results (Dix 1955).

Contrary to Dix’s method, however, we use no approximations in the Snell’s Law geometric ray path formulae, so that as demonstrated, the method gives more accurate results (Table 4). Furthermore, we can treat uniformly dipping layers using analytical expressions even in the case where we do not have a common reflection point; a most important aspect when considering disposable sonobuoy results.

Present work is directed towards the use of the method in cases where there are velocity gradients in the layers and where the interfaces do not dip uniformly (Patriat & Limond, in preparation). One could also consider the three-dimensional problem as posed by data obtained from more detailed surveys. Further possibilities are being explored to employ the exact equations in a velocity spectral study (Taner & Koehler 1969).

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The method of velocity determination was originally developed by one of us (Ph.P) while engaged in research at the Institut Français du Pétrole. We thank the Geology Division of IFP for permitting us to continue work on this subject.

We thank M. Chataux for the figures, and also P. Bois, B. Damotte, M. Lavergne and C. Willm of the Institut Français du Pétrole; C. Brewitt-Taylor, M. Fowler and D. Wright of the University of Cambridge for useful discussions. J. M. Limond provided linguistic technical and typing skills, for which we are grateful.

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Appendix I

Many of these symbols are explained in the text where they occur and several are explained in the diagrams; for these reasons only a short description of each is given here. Please note the explanation of symbols used for variance and co-variance in the miscellaneous section.

Mathematical symbols

Miscellaneous

\( N \) Layer index; no. of data points (on graphs).

\( i \) Data point index.

\( p \) Seismic ray parameter (in some obvious cases this is not used, e.g. \( a_{N-1} = a_{N-1,p} \)).

Ray index.
Theoretical interpolated ray index.

\[ \sigma^2(f) = \langle (f - \mu_f)^2 \rangle = \text{variance of } f. \]

\[ \sigma(f, g) = \langle fg \rangle - \langle f \rangle \langle g \rangle; \text{ co-variance of } f \text{ and } g. \]

\[ \langle f \rangle = \text{mean of } f. \]

\[ np \text{ No. of data points.} \]

\[ C_f \text{ Co-variance matrix of vector } f. \]

\[ T^T \text{ Transpose matrix of } T. \]

Time

\[ t_{N, p} \text{ Minimum travel time on variable angle reflection record of wave reflected from interface } N \text{ at angle defined by ray parameter } p. \]

\[ t_{N, i} \text{ Minimum travel time at data distance } x_{N, i} \text{ on variable angle reflection record for wave reflected from interface } N. \]

\[ t_{N, x} \text{ Minimum travel time on variable angle reflection record of wave reflected from interface } N \text{ at angle defined by ray parameter } j. \]

\[ \tau_N \text{ Minimum travel time on normal incidence reflection record of wave reflected from interface } N. \]

\[ SD \text{ Downward single way travel time up to interface } N-1 \text{ for ray perpendicular to interface } N. \]

\[ \delta t_{N, j} = t_{N,i} - t_{N,j}; \text{ difference between observed and calculated travel times for layer } N \text{ at } x_{N,i}. \]

\[ \Delta t_{N, j} = t_{N,j} - t_{N-1,j}; \text{ travel time within layer } N \text{ of ray } j. \]

\[ T_{N, 0} \text{ Two-way travel time at } x_{N, j} = 0. \]

\[ \Delta T_{N, 0} \text{ The correction to be applied to times } t_{N,j} \text{ to bring the parallel theoretical curve closer to the observed data. This quantity is equal to } \langle \delta t \rangle \text{ when } \sigma(\delta t) \text{ is at a minimum.} \]

Angle

\[ \sigma_{N-1, p} \text{ Angle between ray } p \text{ and vertical in layer } N. \]

\[ \theta_N \text{ Dip of interface } N \text{ measured along the direction of seismic profile.} \]

\[ \beta_{N, p} \text{ Angle between vertical and ray } p \text{ reflected at interface } N. \]

\[ \delta_N \text{ True dip of layer } N. \]

\[ \phi_N \text{ Apparent dip of layer } N \text{ along the direction of the profile measured in terms of time difference } (\tau_{N,a} - \tau_{N,b}). \]

\[ \gamma_{N, N-2} \text{ Angle between ray perpendicular to interface } N \text{ and vertical, in layer } N-1. \]

Angles are measured positive anticlockwise.

Velocity

\[ v_{N} \text{ Seismic interval velocity within layer } N. \]

\[ \frac{(v_a)_N}{2} \text{ Root mean square velocity (Dix 1955).} \]

\[ v(M)_N \text{ Reduced velocity used in Uncertainty calculation (see Appendix II, equation } II.7). \text{ cf. Use of } ds(M)_N. \]

Length

\[ x_{N, p} \text{ Theoretical horizontal distance from source to receiver for ray parameter } p. \]

\[ x_{N, i} \text{ Observed horizontal distance from source to receiver for layer } N. \]

\[ h_N \text{ Thickness of layer } N. \]

\[ P_N \text{ Depth of layer } N \text{ (from origin to intersection of interface } N \text{ with vertical axis).} \]
Determination of seismic interval velocities

$Z_{N,j}$ Depth at which ray $j$ intersects interface $N$.

$X_{N,j}$ Horizontal distance at which ray $j$ intersects interface $N$.

$s_{N,j}$ Total path travelled by ray $j$ in layer $N$.

$\rho$ Horizontal distance between measurements of $\tau_{N,a}$ and $\tau_{N,b}$ on normal incidence reflection profile.

$X_D$ Horizontal distance of intersection with interface $N-1$ of ray perpendicular to interface $N$.

$Z_D$ Depth of intersection with interface $N-1$ of ray perpendicular to interface $N$.

$R_{N,p}$ Distance travelled by downgoing ray $p$ in layer $N$.

$ds(M)_N$ Derivative of a reduced distance used in the uncertainty calculation.

Appendix II

Calculation of uncertainties

The velocity of a layer is determined from the average of several ray path trajectories, so that assuming the upper $N-1$ layers to be known,

$$v_N = \langle v_{N,j} \rangle = \langle s_{N,j} / (t_{N,j} - t_{N-1,j}) \rangle$$

The relevant equations to determine the quantities $s_{N,j}$, $t_{N,j}$, $t_{N-1,j}$ have already been derived (equations (2.6)-(2.18)).

The uncertainty of the velocity can be found in the following way (Wolberg 1967)

The differential of the velocity, $v_N$ (equation (3.1))

$$dv_{N,j} = \left( \frac{\partial v_{N,j}}{\partial s_{N,j}} \right) ds_{N,j} + \left( \frac{\partial v_{N,j}}{\partial t_{N,j}} \right) dt_{N,j} + \left( \frac{\partial v_{N,j}}{\partial t_{N-1,j}} \right) dt_{N-1,j}$$

$$= \frac{ds_{N,j}}{\Delta t_{N,j}} - v_N \frac{dt_{N,j}}{\Delta t_{N,j}} + v_N \frac{dt_{N-1,j}}{\Delta t_{N,j}}$$

where

$$\Delta t_{N,j} = t_{N,j} - t_{N-1,j} \quad dt_{N,j} = t_{N+1,j} - t_{N,j}$$

The uncertainty in the velocity, for small differences $dt_N$, $dt_{N-1}$, $ds_N$ is given by

$$\sigma^2(v_N) = \langle (dv_N)^2 \rangle$$

$$= \left\langle \left( \frac{ds_N}{\Delta t_N} \right)^2 \right\rangle + v_N^2 \left[ \left\langle \left( \frac{dt_N}{\Delta t_N} \right)^2 \right\rangle + \left\langle \left( \frac{dt_{N-1}}{\Delta t_N} \right)^2 \right\rangle \right] - 2v_N \left( \frac{ds_N dt_N}{\Delta t_N^2} \right)$$

$$+ \text{other co-variance terms}$$

The error terms have been separated.

$\sigma^2(t_N) = \langle (dt_N)^2 \rangle$ represents the scatter in the time data for layer $N$.

$\sigma^2(t_{N-1}) = \langle (dt_{N-1})^2 \rangle$ represents the scatter in the time data for layer $N-1$, or the top boundary of layer $N$.

$\sigma^2(s_N) = \langle (ds_N)^2 \rangle$ represents the uncertainty in the ray path within the layer $N$. This necessarily contains the uncertainties from the upper $N-1$ layers.

The co-variance terms express the interdependence of these various uncertainties. Unfortunately $s_N$ is itself an unknown, a function of other variables, and more particularly of the velocity $v_N$. 
From geometry, for layers with constantly dipping interfaces,

\[
\begin{align*}
    s_N &= 2a_N \cos (\theta_N) \cos (\theta_N - \theta_{N-1}) \\
    &= \frac{\cos (2\theta_N - \theta_{N-1} - \alpha_{N-1})}{\cos (\theta_N - \theta_{N-1} - \alpha_{N-1})}
\end{align*}
\]

for \( P_N, \theta_N, \alpha_{N-1} \) depend on the velocity of the layer \( v_N \) so that we must reduce \( s_N \) to a function of variables whose uncertainties are known or can be estimated.

Finally, \( s_N = fn(v_{N-1}, \theta_{N-1}, \tau_N, SD, XD, ZD, \gamma_{N-2}, \alpha_{N-2}, Z_{N-1}, X_{N-1}, v_N) \)

and

\[
    ds_N = \left( \frac{\partial s_N}{\partial v_{N-1}} \right) dv_{N-1} + \ldots + \left( \frac{\partial s_N}{\partial X_{N-1}} \right) dX_{N-1} + \left( \frac{\partial s_N}{\partial v_N} \right) dv_N
\]

we introduce a reduced distance and velocity \( s(M)_N \) and \( v(M)_N \), where

\[
    ds(M)_N = ds_N - \left( \frac{\partial s_N}{\partial v_{N-1}} \right) dv_{N-1}
\]

and

\[
    \langle (dv(M)_N)^2 \rangle = \left\langle \left( \frac{ds(M)_N}{\Delta t_N} \right)^2 \right\rangle + v_N^2 \left[ \left\langle \left( \frac{dt_N}{\Delta t_N} \right)^2 \right\rangle + \left\langle \left( \frac{dt_{N-1}}{\Delta t_N} \right)^2 \right\rangle \right] \\
    + 2v_N \left[ \left\langle \left( \frac{ds(M)_N}{\Delta t_N} \right) \times \left( \frac{dt_{N-1}}{\Delta t_N} \right) \right\rangle - v_N \left\langle \frac{dt_{N-1} dt_N}{\Delta t_N^2} \right\rangle \right] - \left\langle \frac{ds(M)_N dt_{N-1}}{\Delta t_N^2} \right\rangle
\]

is independent of the velocity derivative \( dv_N \).

Each of the co-variance terms:

\[
    \langle ds(M)_N dt_{N-1} \rangle
\]

\[
    = \left( \frac{\partial s_N}{\partial v_{N-1}} \right) dv_{N-1} dt_{N-1} + \ldots + \left( \frac{\partial s_N}{\partial X_{N-1}} \right) dX_{N-1} dt_{N-1} \text{ at } N-1
\]

\[
    = \left( \frac{\partial s_N}{\partial v_{N-1}} \right) \sigma(v_{N-1}, t_{N-1}) + \ldots + \left( \frac{\partial s_N}{\partial X_{N-1}} \right) \sigma(X_{N-1}, t_{N-1})
\]

and likewise,

\[
    \langle ds(M)_N dt_N \rangle = \left( \frac{\partial s_N}{\partial v_{N-1}} \right) \sigma(v_{N-1}, t_N) + \ldots + \left( \frac{\partial s_N}{\partial X_{N-1}} \right) \sigma(X_{N-1}, t_N)
\]

Formulae II.8, 9, 10 can be expressed more simply in terms of the co-variance matrix (e.g. Brandt 1973),

\[
    \langle dv(M)_N \rangle^2 = C_M = TC_f \text{ } T^T
\]

where the co-variance matrix

\[
    C_f = f \text{ } f^T
\]
Determination of seismic interval velocities

expresses the uncertainty of all the variables in (II.2) and (II.6), i.e.

\[ f = (dv_{N-1}, d\theta_{N-1}, dx_{N-2}, d\gamma_{N,N-2}, dZ_{N-1}, dX_{N-1}, d(SD), d(XD), \]

\[ d(ZD), d\tau_{N}, dt_{N}, dt_{N-1} ) \]  \hspace{1cm} (II.15)

For the calculations used in the text, we have used the variances of all these variables for the determination of the error (i.e. the diagonal of the covariance matrix \( C_f \)) but we have considered the co-variances of \( t_N \) and \( t_{N-1} \) (see II.9, 10) judging that the co-variances of the other terms have been taken into account in the calculation of \( \langle dv(M)_{N-1}^2 \rangle \) from which many of them are determined.

The influence of these uncertainties on the velocity error, \( \langle dv(M)^2 \rangle \), are contained in the propagation matrix:

\[ T = \begin{pmatrix}
\frac{\partial v_{N,1}}{\partial v_{N-1}} & \frac{\partial v_{N,1}}{\partial \theta_{N}} & \frac{\partial v_{N,1}}{\partial \alpha_{N-2}} & \frac{\partial v_{N,1}}{\partial \gamma_{N,N-2}} & \frac{\partial v_{N,1}}{\partial Z_{N,1}} & \frac{\partial v_{N,1}}{\partial X_{N,1}} \\
\frac{\partial v_{N,1}}{\partial (SD)} & \frac{\partial v_{N,1}}{\partial (XD)} & \frac{\partial v_{N,1}}{\partial (ZD)} & \frac{\partial v_{N,1}}{\partial \tau_{N}} & \frac{\partial v_{N,1}}{\partial t_{N}} & \frac{\partial v_{N,1}}{\partial t_{N-1}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial v_{N,1}}{\partial \theta_{N}} & \frac{\partial v_{N,1}}{\partial \alpha_{N-2}} & \frac{\partial v_{N,1}}{\partial \gamma_{N,N-2}} & \frac{\partial v_{N,1}}{\partial Z_{N,1}} & \frac{\partial v_{N,1}}{\partial X_{N,1}} & \frac{\partial v_{N,1}}{\partial (SD)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial v_{N,1}}{\partial \tau_{N}} & \frac{\partial v_{N,1}}{\partial t_{N}} & \frac{\partial v_{N,1}}{\partial t_{N-1}} & \frac{\partial v_{N,1}}{\partial \theta_{N}} & \frac{\partial v_{N,1}}{\partial \alpha_{N-2}} & \frac{\partial v_{N,1}}{\partial \gamma_{N,N-2}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial v_{N,1}}{\partial t_{N-1}} & \frac{\partial v_{N,1}}{\partial t_{N-1}} & \frac{\partial v_{N,1}}{\partial t_{N-1}} & \frac{\partial v_{N,1}}{\partial t_{N-1}} & \frac{\partial v_{N,1}}{\partial t_{N-1}} & \frac{\partial v_{N,1}}{\partial t_{N-1}} \\
\end{pmatrix} \]

where \( n_p = \text{no. of data points}. \)

Finally the uncertainty in the velocity

\[ \sigma^2(v_N) = \langle (dv_N)^2 \rangle \]

\[ \approx \langle (dv(M)_N)^2 \rangle + \left( \frac{\partial \delta_{N}^2}{\partial v_N} \right) \left( \frac{\partial v(M)_N}{\partial \Delta t_N^2} \right) \]  \hspace{1cm} (II.17)

The uncertainties in \( \theta_N, \alpha_{N-1}, P_N \) can now be estimated using the equations (2.6)–(2.15). From

\[ d\theta_N = \left( \frac{\partial \theta_N}{\partial \theta_{N-1}} \right) d\theta_{N-1} + \left( \frac{\partial \theta_N}{\partial v_N} \right) dv_N + \left( \frac{\partial \theta_N}{\partial \alpha_{N-1}} \right) d\alpha_{N-1} + \left( \frac{\partial \theta_N}{\partial \gamma_{N,N-2}} \right) d\gamma_{N,N-2} \]

\[ \sigma^2(\theta_N) = \left( 1 - \frac{v_N}{v_{N-1}} \right)^2 \sigma^2(\theta_{N-1}) + \tan^2(\theta_N - \theta_{N-1}) \]

\[ \times \left[ \frac{\sigma^2(v_N)}{v_N^2} + \frac{\sigma^2(v_{N-1})}{v_{N-1}^2} + \frac{\sigma^2(\gamma_{N,N-2})}{\tan^2(\gamma_{N,N-2} - \theta_{N-1})} \right] \]  \hspace{1cm} (II.19)
Similarly,
\[ \sigma^2(\alpha_{N-1}) = \left(1 - \frac{v_N}{v_{N-1}}\right)^2 \sigma^2(\theta_{N-1}) + \tan^2(\alpha_{N-1} - \theta_{N-1}) \]
\[ \times \left[ \frac{\sigma^2(v_N)}{v_N^2} + \frac{\sigma^2(v_{N-1})}{v_{N-1}^2} + \frac{\sigma^2(\alpha_{N-2})}{\tan^2(\alpha_{N-2} - \theta_{N-1})} \right] \]  \hspace{1cm} (II.20)

The depth to layer \(N\) is: (Fig. 5)
\[ P_N = XD \tan(\theta_N) + D_N/\cos(\theta_N) + ZD \]  \hspace{1cm} (II.21)
where
\[ D_N = (\tau_N/2 - SD) v_N \]  \hspace{1cm} (II.22)
and thus,
\[ \sigma^2(P_N) = \tan^2(\theta_N) \sigma^2(XD) + \left(\frac{v_N}{2 \cos(\theta_N)}\right)^2 \sigma^2(\tau_N) + \sigma^2(ZD) + \left(\frac{1}{\cos(\theta_N)}\right)^2 \sigma^2(SD) \]
\[ + \left(\frac{D_N}{v_N \cos(\theta_N)}\right)^2 \sigma^2(v_N) + \left(\frac{X D + D \sin(\theta_N)}{\cos^2(\theta_N)}\right)^2 \sigma^2(\theta_N) \]  \hspace{1cm} (II.23)

Likewise, it is possible to calculate the uncertainties in \(Z_N, X_N, \gamma_N, \alpha_{N-1}, d(SD), d(ZD)\) in readiness for the calculation of the velocity uncertainty in the next layer, \(\sigma^2(v_{N+1})\).