On the Short Range Interaction between Pion and Nucleon

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Asymptotic behaviors of the so-called force functions, dispersion integrals over left-hand cuts of partial-wave amplitudes, in pion-nucleon scattering are investigated. By virtue of the exponential decrease of fixed \( \cos \theta \) amplitudes for \( s \to +\infty \) with \( |\cos \theta|<1 \), the left-hand cut contributions to these amplitudes, from which we obtain the force function by doing the partial-wave projection, can be evaluated in terms of the right-hand cut integrals. The empirical formulae for high-energy scattering are used for evaluating these right-hand cut integrals. The two cases are considered, in which Krisch's formula and Orear's one are used for the large-angle scattering, respectively. For the case of Krisch's formula, it is found that the asymptotic behavior of \( \pi N \) force function for any partial wave is dominated by the contribution from a diffraction-scattering term and this contribution gives rise to an attractive short-range force, and that for the \( P_{11} \) state, this attractive force seems to be somewhat weaker than that required in the previous \( N/D \) bootstrap calculation. For the case of Orear's formula, no definite conclusion is obtained because of the possibility that large angle regions give rise to large contributions.

§ 1. Introduction

Many important investigations and suggestions as to the bootstrap hypothesis of hadrons have so far been done by making \( N/D \) calculations of various partial-wave amplitudes. It was, however, hard to draw any unambiguous conclusion, because of the lack of our knowledge on short range interactions among hadrons.

In this theory, the whole nature of dynamical force acting between the underlying two hadrons is condensed into the so-called force function \( B(s) \) which is the integral over the left-hand cuts of the partial-wave amplitudes. Today, most physicists may recognize that not only contributions from nearby singularities but also those from distant ones are equally important even for the calculations of low energy amplitudes. The short range interaction contributes only to the distant singularities, and one knows little about both of them.

How can one go further with by-passing the difficulty relating to the short range interactions? Fortunately, the total contribution from the distant singularities to the partial-wave amplitude or to the force function for low energy can be approximately described in terms of only the average, not local, properties of the distant singularities, because of the large distance between these singularities and the low energy physical region. In some of the \( N/D \) calculations,\(^{1,2}\)
all the distant singularities were substituted by a few fictional poles which were expected for the low energy region to give the contribution approximately equal to that of the actual singularities. The positions of the poles, however, can presumably be determined by a definite criterion which is independent of the local structure of actual singularities. This last fact is important because we have little information about the distant singularities. The residues of the poles are retained as the parameters of the theory and one can finally determine them by requiring the crossing symmetry for the low-energy partial-wave amplitudes.

Detailed calculations along the line of thought described above recently been done for \( \pi N \) scattering, and some success has been obtained.\(^5\) It is found in these calculations that the effective short range force required by the crossing symmetry in the sense mentioned above is very strong and attractive so that the nucleon pole as a zero of the \( D \)-function appears very close to the empirical mass of nucleon. The calculated \( \pi N \) coupling constant is also close to the empirical one. This result is certainly satisfactory. Nevertheless, we still have the following question: Can the actual distant singularities produce so strong attractive short range force as required by the crossing symmetry? To give an answer to this question we shall, in this paper, try to evaluate the asymptotic behavior of the force function of \( \pi N \) scattering. This asymptotic behavior is sensitively affected by the distant left-hand singularities. Of course, we have no new method to evaluate the distant left-hand cut directly. Instead, we shall utilize dispersion relations as the sum rules with which the integral over the left-hand cut, i.e. the force function, is described in terms of the physical amplitudes.\(^*\)

In \( \S \) 2, we shall give the preliminary discussion for our schema of calculations. First, we discuss the full amplitudes rather than partial wave amplitudes, and define the generalized force function which is a function of the squared energy \( s \) and \( \cos \theta \). This discussion might clarify the situation we shall encounter. The force function for the partial wave amplitude is the partial wave projection of this function. The details of the calculations for \( \pi N \) scattering are given in \( \S \) 3. In \( \S \) 4, we shall compare our result with that obtained in the \( N/D \) calculations.\(^9\)

\section*{\( \S \) 2. The preliminary discussion}

For brevity, we discuss in this section the scattering of two neutral spinless equal-mass particles. Consider the invariant amplitude \( F(s, \cos \theta) \) of this scattering. The generalized force function \( B(s, \cos \theta) \) is introduced by writing \( F(s, \cos \theta) \) as follows for \(|\cos \theta|<1\):

\(^*\) The similar approach is found in recent papers.\(^4\) In these papers, only the low energy behavior of the force function is investigated and the asymptotic behavior of this function is not derived but simply assumed. We saw these papers when we were almost finishing our work.
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\begin{equation}
F(s, \cos \theta) = B(s, \cos \theta) + \frac{1}{\pi} \int_{R}^{a} \frac{\text{Im} F(s', \cos \theta)}{s' - s} ds',
\tag{2.1}
\end{equation}

where \( R \) denotes the threshold of the right-hand cut.

When \( F(s, \cos \theta) \) satisfies an unsubtracted dispersion relation for fixed \( \cos \theta \), we have

\begin{equation}
B(s, \cos \theta) = \frac{1}{\pi} \int_{L}^{a} \frac{\text{Im} F(s', \cos \theta)}{s' - s} ds',
\tag{2.2}
\end{equation}

where \( L \) denotes threshold of the left-hand cut. However, we do not assume in this paper that \( F(s, \cos \theta) \) should satisfy the dispersion relation. Recent investigations of Regge pole models seem to suggest that \( F(s, \cos \theta) \) has an essential singularity at \( s = \infty \).\(^5\) It is dangerous to assume a priori the dispersion relation for \( F(s, \cos \theta) \) with a finite number of subtractions. Instead, we only assume Eq. (2.1) to hold. We have the experimental result that the fixed \( \cos \theta \) amplitude decreases exponentially for \( s \rightarrow +\infty \) if \(-1 < \cos \theta < 1\). Therefore the integral in the right side of Eq. (2.1) converges. For \( \cos \theta = \pm 1 \), the integral may diverge but \( F(s, \cos \theta) \) is finite there. The points \( \cos \theta = \pm 1 \) are the singular points of \( B(s, \cos \theta) \).

Let us consider the partial wave amplitude

\begin{equation}
F_i(s) = \frac{1}{2} \int_{-1}^{1} F(s, \cos \theta) P_i(\cos \theta) d \cos \theta.
\tag{2.3}
\end{equation}

Taking the partial wave projection on both sides of Eq. (2.1) we obtain

\begin{equation}
F_i(s) = B_i(s) + \frac{1}{\pi} \int_{R}^{a} \frac{\text{Im} F_i(s')}{s' - s} ds',
\tag{2.4}
\end{equation}

and

\begin{equation}
B_i(s) = \frac{1}{2} \int_{-1}^{1} B(s, \cos \theta) P_i(\cos \theta) d \cos \theta.
\tag{2.5}
\end{equation}

In obtaining the second term on the right-hand side of Eq. (2.4), we first expand \( \text{Im} F(s, \cos \theta) \) into Legendre series, the order of integration over the right-hand cut and the summation about angular-momentum states can be exchanged as long as the integral on the right-hand side of (2.4) converges for all partial waves; thus, we obtain Eq. (2.4). Under the above condition of the integral in (2.4), \( B_i(s) \) must be finite although \( B(s, \cos \theta) \) has singularities at \( \cos \theta = \pm 1 \) as mentioned above. If \( F_i(s) \) satisfies the unsubtracted dispersion relation, it is found that \( B_i(s) \) is the conventional force function, i.e.

\begin{equation}
B_i(s) = \frac{1}{\pi} \int_{-\infty}^{L} \frac{\text{Im} F_i(s')}{s' - s} ds'.
\tag{2.6}
\end{equation}

In the bootstrap calculations, it was so far assumed that \( F_i(s) \) satisfies the unsubtracted dispersion relation. However, a doubt has recently arisen; if
$F(s, \cos \theta)$ has an essential singularity at $s=\infty$, $F_a(s)$ might also has the similar singularity. For generality, we then assume in this paper only the convergence of the integral over the right-hand cut, that is, the integral in (2·4). The lack of the relation such as (2·6) gives rise to no new difficulty in solving the $N/D$ problem. Actually we have there no use for Eq. (2·6) as long as the values of $B_a(s)$ are given for all the physical $s$.

Our aim in this paper is to calculate the asymptotic form of the force function $B_a(s)$ for $s \to +\infty$. For this purpose, we first calculate $B(s, \cos \theta)$. Let us start from the fundamental equation (2·1). For the values of $s$ and $\cos \theta$ in the physical region except for $\cos \theta = \pm 1$, Eq. (2·1) can be written as

$$B(s, \cos \theta) = \text{Re} F(s, \cos \theta) - \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im} F(s', \cos \theta) \, ds'}{s' - s}, \quad (2·7)$$

where $P$ means the Cauchy principal value of the integral. We consider this equation as a sum rule with which one can in principle calculate $B(s, \cos \theta)$ by inserting empirical formulae of $F(s, \cos \theta)$ into the right-hand side.

Because of the exponential decreasing of $F(s, \cos \theta)$ for $s \to +\infty$ and $-1 < \cos \theta < 1$, the denominator in the integrand of the right-hand side of (2·7) can be expanded by using the formal identity,

$$\frac{1}{s' - s} = \frac{1}{s} - \frac{s'}{s^2} - \cdots - \frac{s'^{n-1}}{s^n} + \left(\frac{s'}{s}\right)^n \frac{1}{s' - s},$$

and we find that this integral generally gives the term $O(s^{-1})$ for $s \to +\infty$, so that the first term on the right-hand side of (2·7), which is decreasing exponentially for $s \to +\infty$, gives only a negligibly small contribution. Thus for large $s$, we have

$$B(s, \cos \theta) = \sum_{m=1}^{n} b_m(\cos \theta) s^{-m} + R_n(s, \cos \theta), \quad (2·8)$$

where

$$b_m(\cos \theta) = \frac{1}{\pi} \int_{-\infty}^{\infty} s'^{m-1} \text{Im} F(s', \cos \theta) \, ds' \quad (2·9)$$

and

$$R_n(s, \cos \theta) = -\frac{P}{\pi} \int_{s'}^{\infty} \left(\frac{s'}{s}\right)^n \frac{\text{Im} F(s', \cos \theta) \, ds'}{s' - s}. \quad (2·10)$$

Since $R_n(s, \cos \theta) \sim O(s^{-n})$ for $s \to +\infty$, the series in (2·8) gives the asymptotic expansion of $B(s, \cos \theta)$. When we take the value of $\cos \theta$ more and more closely to the unity, the exponential decreasing of $\text{Im} F$ may become slower and slower. Therefore, the coefficient $b_m$ becomes larger and larger and diverges when $\cos \theta \to +1$. This corresponds to the fact that no asymptotic expansion exists for $\cos \theta = +1$, where $F$ no longer shows the exponential decrease. Near the backward the similar situation occurs. To calculate $B_a(s)$, we take the partial
wave projection \((2\cdot5)\). It might be expected, at first sight, from the above arguments that the dominant contributions to the partial wave \(B_t(s)\) might come from the scattering near the forward and backward directions. However, we shall see in the next section that there exist large cancellations among the contributions from the scattering near the forward direction. One cannot \emph{a priori} assume the dominance of the contributions from the near forward and near backward regions. It is, at any rate, important to investigate the behavior of \(B(s, \cos \theta)\) for \(\cos \theta \rightarrow 1\).

The asymptotic expansion can be used only in the region \(s > s_c\), the critical value \(s_c\) may depend on \(\cos \theta\). In the next section, we shall see that this region is just given by \(t < -t_c\). Here \(t\) is the Mandelstam momentum-transfer variable and \(t_c\) is a certain positive constant. In order to find what happens near the forward, one therefore needs to calculate the principal-value integral in \((2\cdot7)\) explicitly. After this integration, we change the variables from \((s, \cos \theta)\) to \((s, t)\)\(^{39}\) and calculate the asymptotic behavior for \(t\) fixed, \(t > -t_c\). For this case, the first term on the right-hand side of \((2\cdot7)\) cannot \emph{a priori} be assumed to be negligible. The real-imaginary ratio of \(F\) is known only for \(t = 0\). We have to make a certain assumption for the real-imaginary ratio for \(t \neq 0\). We shall find in the next section that this first term gives, for \(\pi N\) scattering, only nonleading contribution to the asymptotic form of \(B_t(s)\). After our investigations of the near forward region, it will be found that the dominant terms of \(B_t(s)\) for large \(s\) can be calculated again by using the asymptotic expansion \((2\cdot8)\).

\section*{§ 3. Calculations for the pion-nucleon scattering}

\textit{a. Kinematics}

The pion-nucleon scattering amplitude with definite isospin \(I\) can be expressed in terms of two invariant amplitudes, \(A^I\) and \(B^I\) as\(^{37}\)

\[ T^I = -A^I + iF^I \cdot \frac{q_1 + q_2}{2} \tag{3.1} \]

where \(q_1\) and \(q_2\) are the initial and final momenta of the pion, respectively, and \(A^I\) and \(B^I\) are functions of the three Mandelstam variables, \(s = (p_1 + q_1)^2\), \(t = (p_2 - p_1)^2\) and \(u = (p_1 - q_2)^2\) (see Fig. 1); in terms of the center-of-mass scattering angle \(\theta\) in the \(s\) channel, \(t\) is given by

\[ t = -2q_s^2(1 - \cos \theta), \tag{3.2} \]

where \(q_s\) is the center-of-mass momentum,\(^{**}\)

\[ 4q_s^2 = (1 - \frac{(M - \mu)^2}{s})(s - (M + \mu)^2). \tag{3.3} \]

\(^{39}\) For the \(t\) fixed, one never changes the variable in the integrand. Such an integral has no connection with \(B(s, \cos \theta)\), because this integral is a part of the \(t\)-fixed dispersion relation but not of the \(\cos \theta\) fixed dispersion relation.

\(^{**}\) The masses of nucleon and pion are denoted by \(M\) and \(\mu\), respectively.
The partial-wave amplitude

\[ f_{i \pm}^I(s) = \exp(i \delta_{i \pm}^I(s)) \frac{\sin \beta_{i \pm}^I(s)}{q_s}, \quad (3.4) \]

which corresponds to the state with total isospin \( I \), orbital angular momentum \( l \)
and total angular momentum \( l \pm \frac{1}{2} \), is given by the formula\(^7\),\(^8\)

\[ f_{i \pm}^I(s) = \frac{1}{2} \int_{-1}^{+1} d \cos \theta [P_1(\cos \theta)f_{i \pm}^I(s, \cos \theta) + P_{i \pm 1}(\cos \theta)f_{i}^I(s, \cos \theta)], \quad (3.5) \]

where

\[ f_{i}^I = \frac{E(s) + M}{8\pi \sqrt{s}} [A^I + (\sqrt{s} - M)B^I], \]

\[ f_{i}^I = \frac{E(s) - M}{8\pi \sqrt{s}} [-A^I + (\sqrt{s} + M)B^I] \quad (3.6) \]

and

\[ E(s) = (s + M^2 - \mu^2)/2\sqrt{s} . \]

Let us introduce the following amplitudes which correspond to the helicity nonflip and flip amplitudes\(^9\) except for a certain simple factor;

\[ h_{i}^I = \frac{f_{i}^I + f_{-i}^I}{2} = \frac{\sqrt{s}}{16\pi s} [2MA^I + (s - M^2 - \mu^2)B^I], \]

\[ h_{-i}^I = \frac{f_{i}^I - f_{-i}^I}{2} = \frac{1}{16\pi s} [(s + M^2 - \mu^2)A^I + M(s - M^2 + \mu^2)B^I]. \quad (3.7) \]

The helicity ± states form the two channels for the definite total angular momentum \( J \); one can write four helicity partial-wave amplitudes in the matrix form,

\[ h^{IJ} = \begin{pmatrix} h_{i \pm}^{IJ} & h_{-i \pm}^{IJ} \\ h_{i \pm}^{IJ} & h_{-i \pm}^{IJ} \end{pmatrix} , \quad (3.8) \]

where the subscripts ± represent the final and initial helicities. The invariance under parity and time reversal transformations reduces the number of independent amplitudes to the following two amplitudes, in the new notation,

\[ h_{i \pm}^{IJ} = h_{i \pm}^{IJ} = h_{-i \pm}^{IJ} , \]

\[ h_{i \pm}^{IJ} = h_{i \pm}^{IJ} = h_{-i \pm}^{IJ} , \quad (3.9) \]

and we have

\[ h_{1 \pm}^{IJ} = \frac{1}{2} \int_{-1}^{+1} h_{i \pm}^{IJ}(P_{J \mp 1/2}(z) + P_{J \pm 1/2}(z))dz , \]

\[ h_{2 \pm}^{IJ} = \frac{1}{2} \int_{-1}^{+1} h_{i \pm}^{IJ}(P_{J \mp 1/2}(z) - P_{J \pm 1/2}(z))dz . \quad (3.10) \]
The transformation to the parity-definite states is generated by the unitary matrix
\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \]
and applying this to the matrix (3·8), one finds
\[ f_{\pm}^{l} = h_{1}^{lJ} + h_{2}^{lJ} \quad \text{for} \quad l = J - \frac{1}{2} \]
and
\[ f_{\mp}^{l} = h_{1}^{lJ} - h_{2}^{lJ} \quad \text{for} \quad l = J + \frac{1}{2}. \]

b. Elimination of kinematical cut

From (3·5) and (3·6), one sees that the partial-wave amplitude \( f_{\pm}^{l}(s) \) has a kinematical cut which comes from the factor \( \sqrt{s} \). To eliminate this kinematical cut, Frautschi and Walecka\(^5\) carried out the \( N/D \) formalism in the \( W \)-plane, where \( W^2 = s \). However, our scheme of calculation has deep connection with kinematics on Mandelstam’s \( s-t-u \) plane, it is difficult to carry out our scheme in the \( W \)-plane; we take other method which is equivalent to that of Frautschi and Walecka, and which is able to carry on in the \( s \)-plane.

We now introduce new amplitudes, i.e. \( H_{1}^{lJ} = h_{1}^{lJ}/\sqrt{s} \) and \( H_{2}^{lJ} = h_{2}^{lJ} \). It is seen from (3·7) and (3·10) that \( H_{1}^{lJ} \)'s have no kinematical cut, and the matrix (3·8) is written as
\[ h^{lJ} = \begin{pmatrix} \sqrt{s}H_{1} & H_{2} \\ H_{1} & \sqrt{s}H_{1}^{*} \end{pmatrix}, \]
where we have suppressed the superscripts of the matrix elements. Let us apply the two-channel \( ND^{-1} \) method to these helicity amplitudes;
\[ h^{lJ} = N \cdot D^{-1}, \]
where
\[ N = \begin{pmatrix} \sqrt{s}N_{1} & N_{2} \\ N_{1} & \sqrt{s}N_{1}^{*} \end{pmatrix}, \]
and
\[ D = \begin{pmatrix} D_{1} & \sqrt{s}D_{2} \\ \sqrt{s}D_{2}^{*} & D_{1} \end{pmatrix}. \]

\(^{5}\) A function \( \sqrt{s}h_{1}^{lJ} \) can also be used for eliminating the kinematical cut. If unsubtracted dispersion relation is assumed for this function, we encounter the integral
\[ \frac{1}{(\pi \sqrt{s})^{3}} \int_{(m \pm \rho)^2}^{\infty} ds' \sqrt{s'} \operatorname{Im} h_{1}(s'-s), \]
as corresponding to the last term of Eq. (3·23). The diffraction model, discussed in §3c, will give the result that \( \operatorname{Im} h_{1} \sim C\sqrt{s} \) for \( s \to +\infty \). (See Eqs. (3·26) and (3·36).) Thus, the above integral diverges. Therefore, we must assume a once subtracted dispersion relation for this function which gives the same equations as that described in the text for the choice, \( H_{1}^{lJ} = h_{1}^{lJ}/\sqrt{s} \).
The functions \( N_t \) and \( D_t \) have no kinematical cut, and one sees that the product \( \mathbf{N} \cdot \mathbf{D}^{-1} \) leads to the \( \sqrt{s} \) dependence of matrix elements as just seen on the right side of (3.13). The force functions are introduced in the matrix form,

\[
\mathbf{B} = \begin{pmatrix} \sqrt{s} B_1, & B_2 \\ B_3, & \sqrt{s} B_1 \end{pmatrix},
\]

(3.17)

and we have, corresponding to Eq. (2.4),

\[
H_t = B_t + \frac{1}{\pi} \int_0^\infty \text{Im} H_t ds'.
\]

(3.18)

The matrices (3.13) \( \sim \) (3.17) can be diagonalized by means of the unitary transformation (3.11), and we obtain the expressions for \( f_{iz} \):

\[
\begin{pmatrix} f_{iz}^+, & 0 \\ 0, & f_{(i+1)-} \end{pmatrix} = \begin{pmatrix} \sqrt{s} N_t + N_b, & 0 \\ 0, & \sqrt{s} N_1 - N_2 \end{pmatrix} \begin{pmatrix} D_1 + \sqrt{s} D_b, & 0 \\ 0, & D_1 - \sqrt{s} D_b \end{pmatrix}^{-1}.
\]

(3.19)

The \( D \) function for \( f_{iz}^+ \), i.e. \( D_1 + \sqrt{s} D_b \), can be continued to the other branch of \( \sqrt{s} \) and this second sheet \( D \) function is equal to the \( D \) function of \( f_{(i+1)-} \), this property, MacDowell symmetry, \(^{10}\) is that just realized by Frautschi and Walecka\(^9\) in their \( W \)-plane method. Our method is thus equivalent to that of Frautschi and Walecka.

In correspondence to Eq. (2.1), we introduce two generalized force functions \( B_i^t(s, \cos \theta) \) and \( B_{iz}^t(s, \cos \theta) \) by

\[
\begin{align*}
\sqrt{s} h_i^t(s, \cos \theta) &= B_i^t(s, \cos \theta) + \frac{1}{\pi} \int_{(M+)^2} \text{Im} h_i^t(s', \cos \theta) ds' \frac{s'}{s'-s}, \\
\sqrt{s} h_{iz}^t(s, \cos \theta) &= B_{iz}^t(s, \cos \theta) + \frac{1}{\pi} \int_{(M+)^2} \text{Im} h_{iz}^t(s', \cos \theta) ds' \frac{s'}{s'-s}.
\end{align*}
\]

(3.20)

(3.21)

The \( B_i^t \)'s in (3.17) or (3.18) are obtained by performing the partial-wave projections similar to (3.10) on these \( B_i(s, \cos \theta) \). The force functions, \( B_i^t \) and \( B_{iz}^t \), for the definite-parity partial-wave are found from (3.19) to be \( (\sqrt{s} B_1 + B_t) \) and \( (\sqrt{s} B_1 - B_t) \), respectively, and then we get finally

\[
B_{iz}^t(s) = \frac{1}{2} \int_{-1}^{+1} dz \left\{ [\sqrt{s} B_{iz}^t(s, z) + B_{iz}^t(s, z)] P_t(z) \\
+ [\sqrt{s} B_{iz}^t(s, z) - B_{iz}^t(s, z)] P_{iz+1}(z) \right\}.
\]

(3.22)

c. Contribution from diffraction scattering

We start from the equations which are obtained by writing Eqs. (3.20) and (3.21) in the form corresponding to Eq. (2.7),

\[
\sqrt{s} B_t(s, z) = \text{Re} h_t(s, z) - \frac{\sqrt{s}}{\pi} \int_0^\infty \frac{\text{Im} h_t(s', z) ds'}{\sqrt{s'}(s'-s)},
\]

(3.23)
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\[ B_i(s, z) = \text{Re} \ h_i(s, z) - \frac{1}{\pi} P \int_{s_0}^{s} \frac{\text{Im} \ h_i(s', z) ds'}{s' - s}, \]  

(3.24)

where \( s > s_0 = (M + \mu)^2 \), \( z = \cos \theta \) and the isospin indices are suppressed. These equations will be used as the sum rules for calculating \( B_i(s, \cos \theta) \), as discussed in § 2. We first try to evaluate the integral terms on the right-hand sides of these equations.

To give \( \text{Im} \ h_i \) explicitly, we shall, in this section, assume a simple model in which only the diffraction terms are retained. The \( \pi^*P \) elastic differential cross section near the forward, \( P_{lab} \geq 1 \text{ BeV}/c \), is described by a simple formula

\[
\frac{d\sigma}{dt} = \frac{d\sigma}{dt} \bigg|_{t=0} \exp(i\pi t) \cdot \frac{C}{s},
\]

(3.25)

Since the near forward amplitude is almost imaginary, the above formula and the optical theorem suggest

\[
\text{Im} \ h_i = \frac{q_s}{8\pi} \sigma_{tot}(s) e^{i\pi t}, \quad (i=1, 2)
\]

(3.26)

where \( \sigma_{tot}(s) \) is the total cross section and \( A = B/2 \); we have used the fact that the diffraction scattering contributes only to the spin-nonflip amplitude \( f_1 \) so that \( h_1 \) and \( h_2 \) get the same contribution [see Eq. (3.7)].

For \( \sigma_{tot}(s) \), we have the empirical formula for \( I=1/2 \), 11)

\[
\sigma_{tot}(s) = \sigma_\infty + \frac{C}{\sqrt{s}}
\]

(3.27)

with

\[
\sigma_\infty \approx 58.0 \text{ BeV}^{-1} \quad \text{and} \quad C \approx 70.2 \text{ BeV}^{-1}.
\]

(3.28)

The diffraction scattering may correspond to the first term on the right-hand side of (3.27). The term \( C/\sqrt{s} \) seems to correspond to what should be represented by Regge pole terms in \( t \) channel. Therefore this second term is neglected in this section and will be considered later as a correction. We further simplify the expression (3.26) by using the high-energy approximations; \( q_s \approx \frac{1}{4} (s - s_0) \) and \( t \approx -\xi(s - s_0) \) with \( \xi = \frac{1}{2} (1 - z) \), thus

\[
\text{Im} \ h_i = \frac{1}{16\pi} \sigma_\infty(s - s_0)^{3/2} \exp(-A\xi(s - s_0)).
\]

(3.29)

Now we substitute (3.29) into (3.23) and (3.24), i.e.

\[
[B_i(s, z)]_{\text{diff}} = -\frac{\sigma_\infty}{16\pi} I(s, z),
\]

(3.30)

*) The unit \( \hbar = c = 1 \) is used.
\[ [B_i(s, z)]_{s=0} = -\frac{\sigma_{i}}{16\pi} I_1(s, z), \]

where

\[ I_n(s, z) = \frac{P}{\pi} \int_{s_0}^{s} \frac{(s'-s_0)^{\mu}}{s'^{(n-1)/2}} \exp\left(-A\xi(s'-s_0)\right) ds'. \quad (3.31) \]

The principal value integrals (3.31) can be easily carried out [the details will be given in Appendix 1];

\[ I_1 = \sqrt{\frac{v}{\pi}} \left[ \frac{1}{\sqrt{x}} - 2e^{-x} \int_0^{\sqrt{x}} \exp(u^2) du \right], \quad (3.32) \]

\[ I_2 \approx \frac{1}{v - s_0} \left[ \frac{2(\pi s_0 v)^{1/2}}{v - s_0} e^{-x} \int_0^{\sqrt{x}} \exp(u^2) du - \frac{v}{v - s_0} e^{-x} Ei(x) \right] - \frac{2s_0}{v - s_0} \left[ \sqrt{\pi} \exp(-s_0 A^2) \int_0^{\sqrt{s_0 A^2}} \exp(u^2) du - \frac{1}{2} \exp(-s_0 A^2) Ei(s_0 A^2) \right], \quad (3.33) \]

where \( Ei(x) \) is the exponential integral, \( x = A\xi(s-s_0) = At \) and \( v = s - s_0 \).

The above result for \( I_1 \) is exact, while for \( I_2 \) we have replaced the factor \( 1/\sqrt{s'} \) by \( 1/(\sqrt{s'} - s_0 + \sqrt{s_0}) \).

For large \( x \), the quantity inside the brackets in (3.32) has the asymptotic expansion

\[ e^{-x} \int_0^{\sqrt{x}} \exp(u^2) du \sim \frac{1}{2} \left[ x^{-1/2} + \left( \frac{1}{2} \right) x^{-3/2} + \left( \frac{3}{4} \right) x^{-5/2} + \ldots \right]. \quad (3.34) \]

If we fix \( x(z=\pm 1) \), this expansion gives the asymptotic expansion of \( I_1 \) which is a series of inverse power in \( s \). The similar expansion is also obtained for \( I_2 \). It is noticed that the last terms on the right-hand side of (3.32) has already the definite \( s \)-dependence as \( v^{-1} \). Collecting these expansions, we finally find the asymptotic expansions for \( B_i(s, z) \), These expansions are nothing but the asymptotic expansion discussed in § 2. In Fig. 2, we plot values of the exact form of \( (v/\pi)^{1/2} I_1(s, z) \) and of first few terms of the expansion obtained by making use of (3.34). The latter is a good approximation of the former only for \( x \) larger than a certain critical value \( x_0; x_0 \sim \sqrt{\gamma} \), thus one sees that the expansion can be used only for \( |t| > x_0/A \), which is already mentioned in § 2.

The exact function changes its sign near \( t=0 \) and diverges at \( t=0 \) as \( \sim (-t)^{-1/2} \).

Let us consider the partial wave projection; in terms of \( x, z = 1 - 2x/Av \); and

\[ \frac{1}{2} \int_{-1}^{1} dx P_i(z) B_i(s, z) = \frac{1}{Av} \int_0^{Av} dx P_i(1 - \frac{2x}{Av}) B_i. \quad (3.35) \]

\(^{\text{e}^x Ei(x) = P \int_0^{\infty} \frac{e^{-x}}{t} dt.\)
Defining

\[ I_n^{(q)}(s) = \frac{1}{2} \int_{-1}^{+1} dz P_n(z) I_n(s, z), \]  

we obtain, for large \( s \),

\[ I_1^{(q)}(s) \sim \frac{1}{\sqrt{\pi A^{3/2}}} \frac{1}{s}, \]  

\[ I_1^{(s)}(s) \sim I_1^{(q)} + \frac{2}{\sqrt{\pi A^{3/2}}} \frac{1}{s}, \]  

\[ I_1^{(s)}(s) \sim -\frac{1}{\pi A} \frac{\ln(As)}{s} + \frac{\delta_1}{s}, \]  

\[ I_1^{(s)}(s) \sim I_1^{(s)} + \frac{\delta_1}{s}, \]  

where terms \( O(s^{-2}) \) are neglected. Details of calculations in obtaining the above result and expressions for \( \delta_1 \)'s are given in Appendix 1. There occurs an interesting cancellation in doing the partial wave projection; for \( I_1^{(s)} \), we find that \( \int_0^\infty dx (\sqrt{1/x - 2e^{-x^2}} \exp(u^2) du) = 0 \), this means that the contribution from the region, \( 0 < x < x_0 \), and that from the region \( x > x_0 \) are cancelled with each other [see Fig. 2], the former contribution corresponds to repulsive force and the latter to attractive force; the s-wave projection is the integral over the region \( 0 \leq x \leq Av \), thus the cancellation becomes incomplete and the remaining contribution corresponds to repulsive force. Utilizing the cancellation, one can write that \( I_1^{(s)} = - (1/Av) \int_0^\infty dx I_1(s, x) \); the asymptotic expansion, \( I_1(s, x) \sim -\sqrt{\pi/\pi} \times (1/2x^{3/2}) + \cdots \), can now be used and one is able to make sure of the right-hand side of \( (3 \cdot 37) \). The similar situation occurs for \( I_1^{(s)} \). But the cancellation is more incomplete, \( e^{-xE_1(x)} \) diverges at \( x = 0 \) as \( \ln x \) and behaves as \( x^{-1} \) for large \( x \), thus \( \int_0^\infty dx e^{-xE_1(x)} = \ln \infty \) and therefore we have the logarithmic term \( \ln(As)/s \) as seen in \( (3 \cdot 39) \). This contribution has a negative sign, and thus corresponds to the attractive force, and is obtained again by using the asymptotic expansion for the integrand. The other two terms on the right side of \( (3 \cdot 33) \) give rise to the term \( O(s^{-1}) \).

Using the numerical value for \( B \) in \( (3 \cdot 25) \), i.e.

\[ B = 2A \sim 8 \text{BeV}^{-1}, \]  

\[ (3 \cdot 42) \]  

\[ \text{Fig. 2. The function } (\pi/\nu)^{3/2} I_1; \text{ The solid line shows exact values and the dashed line shows the contributions from the first few terms of the asymptotic expansion.} \]
one obtains\(^{(*)}\)

\[
I_{i}^{(0)} \sim 0.071 \frac{1}{s}, \quad I_{i}^{(1)} \sim 0.212 \frac{1}{s},
\]

\[
I_{s}^{(0)} \sim 0.175 \frac{1}{s} - 0.080 \frac{\ln 4s}{s},
\]

\[
I_{s}^{(1)} \sim 0.237 \frac{1}{s} - 0.080 \frac{\ln 4s}{s}.
\]

(3.43a), (3.43b)

Our most interesting channel is \(P_{11}\) state, i.e. \(I = J = \frac{1}{2}\) \(P\)-wave state. From Eqs. (3.22), (3.30) and (3.36), \(B_{2}^{-}(s)\), the force function for \(P_{11}\) state, is given in the form

\[
B_{2}^{-}(s) = \sqrt{s} \frac{1}{16\pi} \left[ -\sigma_{\text{e}} (I_{s}^{(0)} + I_{i}^{(1)}) \right] + \frac{1}{16\pi} \left[ \sigma_{\text{e}} (I_{s}^{(0)} - I_{i}^{(1)}) \right].
\]

(3.45)

It should be noticed that the right-hand side of (3.45) is independent of the isospin index, because the model used here, namely the diffraction model, should be common for all isospin states. In (3.45), one sees that the contribution from the helicity non-flip amplitude gets the factor \(\sqrt{s}\), which enhances, at high energy, the contribution from this amplitude; inserting the relations (3.43) and (3.44) into (3.45) and using the values of (3.28), one finally finds

\[
B_{2}^{-}(s) \sim 0.18 \frac{\ln 4s}{\sqrt{s}} - 0.47 \frac{1}{\sqrt{s}}. \quad \text{(BeV}^{-1})
\]

(3.46)

d. Contributions from large-angle scattering

The formula (3.25) must be modified when \(|t|\) becomes large. For high-energy large-angle proton-proton scattering, we have an accumulation of data sufficient for obtaining certain empirical formulae. There are two types of the empirical formulae; one of them is Krisch's formula,\(^{15}\)

\[
\frac{d\sigma}{dt} = \frac{\sigma_{\text{e}}^{2}}{16\pi} \left[ \sum_{i=1}^{3} K_{i} \exp(-a_{i}\beta^{2} p_{i}^{2}) \right],
\]

(3.47)

and the other Orear's formula,\(^{14}\)

\[
\frac{d\sigma}{d\Omega} = \frac{L}{s} \exp(-b p_{0}),
\]

(3.48)

where \(p_{i} = q_{i} \sin \theta \) and \(\beta\) is the velocity of the incident particles in the center-of-mass system. For pion-nucleon scattering, the accumulation of data is not yet sufficient. We then assume that Krisch's or Orear's formula may still be valid for \(\pi N\) scattering. These formulae exhibit the forward-backward symmetry of proton-proton scattering. Therefore, for \(\pi N\) scattering, we expect these formulae

\(^{(*)}\) \(s\) is measured in \(\text{BeV}^{2}\).
to be applicable for $\theta<90^\circ$. Krisch made attempt to fit $\pi N$ data with his formula (3.47) and gave the numerical values for the parameters,$^{19}$

$$\frac{\sigma_{\pi N}}{4\sqrt{\pi}} K_1 = 8.02 \text{ BeV}^{-1}, \quad a_1 = 3.86 \text{ BeV}^{-2},$$

$$\frac{\sigma_{\pi N}}{4\sqrt{\pi}} K_2 = 1.64 \times 10^{-1} \text{ BeV}^{-1}, \quad a_2 = 1.06 \text{ BeV}^{-2}, \quad (3.49)$$

$$\frac{\sigma_{\pi N}}{4\sqrt{\pi}} K_3 = 1.31 \times 10^{-3} \text{ BeV}^{-1} \text{ and } a_3 = 0.47 \text{ BeV}^{-3}.$$  

For the Orear formula, we shall use the same values as those given for proton-proton scattering, that is, the dimensionless constant $L=600 \text{ BeV}^2 \text{ mb}=1500$ and $b^{-1}=158 \text{ MeV}$. These two expressions for the differential cross-section have quite different asymptotic behaviors for fixed $\cos \theta$. At present, one knows that both asymptotic behaviors are consistent with the requirement arising from the analytic properties of the scattering amplitudes,$^{10}$ and does not know which is preferable for the fit to experimental data. However, the Veneziano model,$^5$ which has recently been proposed for exhibiting the duality and the super-converging amplitudes, suggests that Krisch's form is more preferable. In this section, we shall calculate both cases separately.

(The case for Krisch's formula.)

There are many ambiguities in determining the amplitudes; we have two independent amplitudes $h_1$ and $h_2$, we have no data to determine the phase of amplitudes. We then simply assume that $\text{Im } h_1$ has a following form,$^*${\footnote{\text{Im } h_1^{(t)} \text{ represents the diffraction term.}}}

$$\text{Im } h_1 = \sum_{i=1}^{3} \text{Im } h_1^{(t)} = \sum_{i=1}^{3} \frac{\sigma_{\pi N}}{8\pi} K_i q_i \exp (-a_i \beta^2 p_i^2)$$

$$= \sum_{i=1}^{3} \frac{1}{16\pi} (s-s_0)^{1/2} \exp [-\bar{a}_i (s-s_0) (1-z^2)], \quad \bar{a}_i = a_i/4, \quad (3.50)$$

where we have taken $\beta=1$. The right-hand side of (3.50) might give the possibly largest value for Im $h_1$ which is obtained under the situation that the cross section described by (3.47). Inserting (3.50) into the right-hand side of (3.23), we may find the possibly largest contribution from large angle scattering;

$$\sqrt{s} B_1^{(t)} (s, z) = -\left(\frac{\sigma_{\pi N}}{16\pi} \right) \frac{1}{\pi} \int_{s_0}^{\infty} ds' (s'-s_0)^{1/2} \exp \left[-\bar{a}_i (1-z^2) (s'-s_0) \right]$$

$$= -\left(\frac{\sigma_{\pi N}}{16\pi} \right) I_1^{(t)} (s, z), \quad (3.51)$$

where $I_1^{(t)} (s, z)$ is the function obtained by the exchange $A_1^\pm \rightarrow \bar{a}_1 (1-z^2)$ on the
right-hand side of (3·33). We further obtain the partial wave projection [see Appendix 2]

\[ \frac{1}{2} \int_0^1 dz I_s^{(0)}(s, z) \sim -\frac{1}{4\pi} \frac{\ln(4\bar{a}s)}{\bar{a}s} . \]  

(3·52)

Here, it should be noted that the formula (3·47) can be used only for \( \theta < 90^\circ \).

Using the numerical values (3·49), one finally get

\[ \frac{1}{2} \sqrt{s} \sum_{i=1}^3 B_i^{(0)}(s, z) \sim +\left(0.093 \frac{\ln 3.8s}{\sqrt{s}} - 0.1\right) + \left(0.007 \frac{\ln s}{\sqrt{s}} - 0.02\right) + \left(0.001 \frac{\ln 0.5s}{\sqrt{s}} - 0.004\right) . \]  

(3·53)

The first term on the right-hand side is the contribution from diffraction scattering and should be compared with the contribution from \( I_s^{(0)} \) to the first term on the right-hand side of (3·46), that is, \( \sim (0.09 \ln 4s/\sqrt{s} - 0.2/\sqrt{s}) \). The second and third terms represent the contributions from large angle scattering, and we see that these are small compared with the contribution from diffraction scattering. The signs of the second and third parentheses have no any meaning because we cannot determine the relative signs among the three terms in (3·50).

(The case for Orear's formula.)

In this case, it is assumed that

\[ \left[ \text{Im } h_i \right]_{\text{Orear}} = \frac{\{L(s-s_0)\}^{1/2}}{4s} \exp\{-(\bar{b}(s-s_0))^{1/2} \sin \theta \} . \]  

(3·54)

and

\[ \bar{b} = b/4 \approx 1.58 \text{ BeV}^{-1} . \]

It should be noted that the diffraction-scattering term has been separated out from (3·54). Carrying out the integration (3·23), with (3·54), and the partial wave projection similar to the left-hand side of (3·53), one gets [see the Appendix for details of the calculations]

\[ \frac{1}{2} \sqrt{s} \int_0^1 dz [B_i(s, z)]_{\text{Orear}} \sim 0.42 - \frac{1}{\sqrt{s}} . \]  

(3·55)

This contribution is seen as more converging than the contribution from diffraction scattering, which appears as \( \sim \ln s/\sqrt{s} \). However, the large coefficient on the right-hand side of (3·55) makes the former to become comparable to the latter at \( s \sim 100 \text{ BeV}^2 \). The large coefficient is due to the large value of \( L \). When we calculate the integral (3·23), we have extrapolated the formula (3·48) or (3·54) into the low-energy region. There is especially in the case of the Orear formula a strong possibility that this extrapolation gives rise to a dangerous error. The factor \( L/s \) in front of the exponential in (3·48) has been normalized
at high energy, \( s = 10 \sim 20 \text{BeV}^2 \), to give the correct order of the cross section, so that the simple extrapolation in the original form may give too large low-energy amplitude. For this reason, it should be considered that we have over-estimated in obtaining the result (3.55).

e. Other contributions

In the previous calculations, we have extrapolated the high-energy formulae, in their original forms, into the low-energy region so that the effects of the low-energy resonances could not be taken into account. Now we try to get certain improvement.

Let us go back to the formula (3.27). The term \( C/\sqrt{s} \), which has been neglected in §3c, seems to correspond to what should be represented by Regge pole terms in the \( t \) channel.\(^{13} \) In Fig. 3, the right-hand side of (3.27) is compared with the experimental values. It is found that (3.27) gives approximately a local average of the experimental curve in the low-energy region. This is the so-called duality\(^{17} \) discovered recently in the Regge pole theory. If we take full Regge representations, such local averaging may realize in higher accuracy and be expected to hold even for \( t \neq 0 \). Regge representations are not, however, used in this paper, because this representation introduces \( \ln s \) term into the exponent of the expression similar to (3.25), and this makes the integrations of (3.23) and (3.24) very much difficult. Furthermore, the present Regge-pole fits of the amplitudes are not yet unambiguous. Instead, let us assume the simple form

\[
\text{Im } h \sim \frac{q_0}{8\pi} \left( \frac{C}{\sqrt{s}} \right) e^{At}, \tag{3.56}
\]

where we do not take \( A \) to be the same value with that for the diffraction peak; we wish Eq. (3.56) to be a certain approximation of the effective Regge term. The \( A \) may be larger than \( \alpha \) (the slope of the Regge trajectory) because this exponential term describes also a certain \( t \) dependence of the Regge residue; we
then expect that $1 \text{BeV}^{-2} \leq A \leq 4 \text{BeV}^{-2}$.

For very large $s$, only the average, but not the local, properties of low-energy $\text{Im} h_t$ can determine its contributions to the integrals in (3.23) and (3.24). The difference between the actual low-energy amplitude and the locally averaging amplitude may consist of super-convergent amplitudes, and it may give only smaller terms for large $s$.

Inserting (3.56) into (3.23) and (3.24), and using (3.22), we obtain the following contribution to $B_{1-}$:

$$- \frac{C}{16\pi} \left[ \sqrt{s} (I_s^{(0)} + I_s^{(1)}) - (I_s^{(0)} - I_s^{(1)}) \right],$$

which gives for large $s$ [see Appendix 1]

$$\frac{(+0.37)}{\sqrt{s}} \text{ with } A = 4 \text{ BeV}^{-2}$$

and

$$\frac{(+0.77)}{\sqrt{s}} \text{ with } A = 1 \text{ BeV}^{-2}.$$  

Besides the above calculation which utilizes the duality, the correction to the force function from the low-energy integrals can roughly be obtained by taking into account only resonance peaks. For $B_{1-}$, the only remarkable peak is due to $P_{11}(1440 \text{MeV})$ resonance and this gives the contribution about $\frac{+0.3}{\sqrt{s}}$ which is consistent with (3.58).

Another contributions to be discussed are the first terms on the right-hand sides of Eqs. (3.23) and (3.24). Here we need to know the real part of amplitudes. At present, the real-imaginary ratio is known only for $t=0$; in Fig. 4, the experimental values\textsuperscript{39} of this ratio $|R|$ are shown and we see that $|R|$ decreases as $\sim s^{-n}$ with $n \approx \frac{1}{2}$. Although we have no data for $t \neq 0$, the $R$ may be insensitive to $t$ for the near forward scattering. This is consistent with the pure diffraction model of very high-energy scattering, where we can hardly expect to have a sudden change of $R$, because of the homogeneity of the extended target particle.

In order to find the order of the contributions, we shall simply assume that

$$\text{Re } h_t = R \text{ Im } h_t \propto s^{3/2} e^{At},$$

where $R$ is considered to be independent of $t$, and the last expression is obtained from (3.29).

Performing the partial wave projection (3.35), we get the contribution to the partial wave
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For the short range interaction to be 

\[ B \text{ function to be } \text{const } \times R s^{-1/2}; \text{ the substitution of the relation, } |R| \sim s^{-n} (n \geq \frac{1}{2}) , \text{ shows that this contribution gives only a non-leading term at most } O(s^{-1}). \]

Contrary to the simple assumption made above, if we get non-vanishing real amplitudes in the limit \( s = \infty \), which could happen possibly in large \( |\ell| \), the contribution is still given as \( CR s^{-1/2} \) but here \( R = 0 \) at \( s = + \infty \). This term is still smaller than the leading term of (3·46) [Remember that the \( \text{Re } h_t \) term in (3·23) has no \( \sqrt{s} \) factor]. The coefficient \( C \) is expected to be very small, since the contribution may come from the large \( |\ell| \) region. In conclusion, one can say that the contributions from \( \text{Re } h_t \) terms gives only non-leading terms which are negligible for large \( s \).

§ 4. Summary and conclusions

The fixed \( \cos \theta \) dispersion relations have been used as sum rules for calculating the asymptotic form of the force function of partial wave amplitudes. From the calculations in § 3, one finds the several interesting results:

(1) The diffraction scattering amplitude gives rise to the term which represents the existence of the attractive short range force.\(^\text{a}\) For the force function of the \( P_{11} \) state, one sees from (3·46) that there is also the term having the minus sign (repulsive force) and for high energy, the force function is dominated by the attractive-force term.

(2) The contributions from the low-energy resonance amplitude is shown to be attractive force. For the \( P_{11} \) state, this contribution almost cancels the above mentioned (repulsive-force) term of the diffraction scattering contribution. [See Eqs. (3·46) and (3·58).]

(3) For the contributions from large angle scattering, we have investigated two cases, for which Krisch’s and Orear’s formulae are assumed, respectively. In the case of Krisch’s formula, the contribution from large angle scattering is small compared with that from the diffraction scattering. [See (3·53).] While in the case of the Orear formula, we have reached no definite conclusion; it is not impossible that the large angle scattering gives the contribution comparable to that from diffraction scattering.

(4) The terms \( \text{Re } h_t \) in (3·23) and (3·24) are negligible.

The Veneziano model leads to the asymptotic behavior, for fixed \( \cos \theta \), which is consistent with Krisch’s formula but not Orear’s. The super-convergent relations, the duality, and the possibility of rising trajectories are exhibited in terms of the Veneziano amplitudes. This model taught us how one can incorporate both Regge asymptotic behaviors in \( s \)- and \( t \)-channels in a consistent manner with the super-convergent relations. In this situation, one can predict unique asympt-

\(^{a}\) The reason for the short range is due to the fact that the asymptotic behavior of the term has the factor, \( \ln s \), since if the main contributions come from the parts of the left-hand cut existing in the finite \( s \)-domain, one gets the asymptotic behavior to be at most \( \sim 1/\sqrt{s} \).
totic behavior for fixed $\cos \theta$ and the result is consistent with Krisch's formula. This fact is very encouraging; we then take Krisch's formula as the most plausible one and we can, in this case, ignore the contributions from large angle scattering; the force function for large $s$ is therefore dominated by the first term of the diffraction scattering contribution (3.46) which shows an attractive force, since the second term is cancelled by the low energy correction (3.58), as mentioned already. Thus, we have an attractive force for the short range interaction between a pion and a nucleon in the $P_{11}$ state.

In order to see how strong this attractive force is, we compare the asymptotic form of our force function with that obtained in the N/D bootstrap calculation in reference 2). In that paper, the $N$ and $D$ functions were calculated but $B$ (force function) was not. Now, let us calculate the $B$ function by using the parametrization stated there: the $N$ function consists of the two poles at $s=0$ and $-50M^2$ ($M=$the nucleon mass), and the Cauchy integral term over nearby left-hand cut. We approximate this integral term by a pole at $s=0.7M^2$, then we can calculate the $D$ function analytically and get the values of $D$ at $s=-50M^2$, 0 and $0.7M^2$. Now $B$-function is easily calculated since the residue of $B$-pole is simply written as (the residue of $N$-pole) $\times$ (the value of $D^{-1}$ at $s$ considered). For large $s$, the $B$ function is dominated by the distant-pole term which is expressed as $47M/(s+50M^2)$, In Fig. 5, the values of this pole term are given. The curve for our diffraction scattering contribution $0.18(\ln 4s/\sqrt{s})$, which is the dominant term in our force function, is also shown. Our force function gives about $30\sim50\%$ of the magnitude of the values obtained in the N/D calculation, for $10\text{BeV}^2<s<100\text{BeV}^2$. Although the former gives larger values than the latter for $s>800\text{BeV}^2$, it seems that the strength of our force function at high energy is somewhat weak, and a certain help of inelastic effects should be needed for producing the nucleon pole dynamically.

From the above argument, one knows that our attractive short range force has a certain correlation with the mechanism of diffraction scattering. This is very interesting, because the diffraction scattering is common in all pseudo-scalar meson-baryon scattering processes and contributes to all partial-waves in nearly the same situation. The former fact means that the attractive short-range force can exist between pair of any pseudo-scalar meson and any baryon. The latter predicts that we might still have an attractive short-range force for higher angular momentum states, of course it is
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weakened by a stronger centrifugal repulsion. This situation is seen in the difference between the right-hand sides of (3·44a) and (3·44b). It is very interesting to compare these predictions with the empirical facts that the present phase-shift analysis shows that there exist bound states and/or resonances in most states, and that we have resonances in all pseudo-scalar meson-baryon scatterings except for \( K^+ - N \) scattering.

Appendix 1

The functions \( I_n(s, z) \) and \( I_n^{(i)}(s) \)

We first carry out integrations

\[
I_n(s, z) = \frac{P}{\pi} \int_0^\infty ds' \frac{\sqrt{s' - s_0}}{s'(n-1/2)} \exp\left(-A\tilde{z}(s' - s_0) \right) \tag{A·1}
\]

for \( n=1, 2 \) and 3, where \( \tilde{z} = (1-z)/2 \).

The following formulae are very useful:

\[
P \int_0^\infty dy \frac{\exp\left(-A\tilde{z}y^2\right)}{y^2} = -e^{-\tilde{z}} Ei(x), \tag{A·2}
\]

\[
\int_0^\infty dy \exp\left(-A\tilde{z}y^2\right) \frac{1}{y + \sqrt{v}} = e^{-\tilde{z}} \left[ \sqrt{\pi} \int_0^{\sqrt{v}} du \exp(u^2) - \frac{1}{2} Ei(x) \right], \tag{A·3}
\]

where \( x = A\tilde{z}v, \ v > 0 \) and \( Ei(x) \) is the exponential integral which is defined by

\[
Ei(x) = \int_0^\infty \frac{e^{-u}}{u} du. \tag{A·4}
\]

From (A·2) and (A·3) we find that

\[
P \int_0^\infty dy \frac{\exp\left(-A\tilde{z}y^2\right)}{y - \sqrt{v}} = -e^{-\tilde{z}} \left[ \sqrt{\pi} \int_0^{\sqrt{v}} du \exp(u^2) + \frac{1}{2} Ei(x) \right]. \tag{A·5}
\]

Hereafter, we suppress the \( P \)-symbol. Thus, using (A·3) and (A·5) we obtain

\[
I_1(s, z) = \frac{2}{\pi} \int_0^\infty dy \frac{y^2 \exp\left(-A\tilde{z}y^2\right)}{\sqrt{y^2 - v} - s_0} \quad (y^2 = s' - s_0, \ v = s - s_0)
\]

\[
= \frac{2}{\pi} \int_0^\infty dy \exp\left(-A\tilde{z}y^2\right) + \frac{2v}{\pi} \int_0^\infty dy \exp\left(-A\tilde{z}y^2\right) \frac{1}{y^2 - v}
\]

\[
= \sqrt{\frac{v}{\pi}} \left[ \frac{1}{\sqrt{x}} - 2e^{-\tilde{z}} \int_0^{\sqrt{v}} du \exp(u^2) \right], \tag{A·6}
\]

\[
I_2(s, z) = \frac{2}{\pi} \int_0^\infty dy \frac{y^2 \exp\left(-A\tilde{z}y^2\right)}{\sqrt{y^2 + s_0} (y^2 - v)}
\]

\[
= \frac{2}{\pi} \int_0^\infty dy \frac{y^2 \exp\left(-A\tilde{z}y^2\right)}{(y + \sqrt{s_0}) (y^2 - v)}
\]

\[
= \sqrt{\frac{v}{\pi}} \left[ \frac{1}{\sqrt{x}} - 2e^{-\tilde{z}} \int_0^{\sqrt{v}} du \exp(u^2) \right],
\]
\begin{equation}
\frac{2\sqrt{vs_0}}{\sqrt{\pi} (v - s_0)} e^{-x} \int_0^{vs_0} du \exp (u^2) - \frac{v}{\pi (v - s_0)} e^{-x} Ei(x)
\end{equation}

\begin{equation}
- \frac{2s_0}{\pi (v - s_0)} \left[ \sqrt{\pi} \exp (-A \xi s_0) \int_0^{YAt_s} du \exp (u^2) \right.
\end{equation}

\begin{equation}
\left. - \frac{1}{2} \exp (-A \xi s_0) Ei(A \xi s_0) \right]\quad (A \cdot 7)
\end{equation}

[where the approximation \((y^2 + s_0)^{3/2} = y + \sqrt{s_0}\) has been used], and

\begin{equation}
I_s(s, z) = \frac{2}{\pi} \int_0^\infty dy \frac{y^2 \exp (-A \xi y^2)}{(y^2 + s_0)} \left( y^2 - v \right)
\end{equation}

\begin{equation}
= \frac{-2\sqrt{v}}{\sqrt{\pi} (v + s_0)} e^{-x} \int_0^{YAt_s} du \exp (u^2)
\end{equation}

\begin{equation}
+ \frac{\sqrt{s_0}}{v + s_0} \exp (A \xi s_0) \left( 1 - \frac{2}{\sqrt{\pi}} \int_0^{YAt_s} du \exp (-u^2) \right). \quad (A \cdot 8)
\end{equation}

Next, we shall calculate the partial wave projections

\begin{equation}
I_n^{(0)}(s) = \frac{1}{2} \int_{-1}^1 dz P_l(z) I_n(s, z)
\end{equation}

\begin{equation}
= \frac{1}{Av} \int_0^{A\theta} dx P_l \left( 1 - \frac{2x}{Av} \right) I_n.
\end{equation}

(A \cdot 9)

Thus,

\begin{equation}
I_1^{(0)}(s) = \frac{1}{Av} \int_0^{A\theta} dx I_1
\end{equation}

\begin{equation}
= \frac{1}{Av} \sqrt{\frac{v}{\pi}} \left[ 2 \sqrt{Av} + 2 \int_0^{YAt_s} dy \left[ \frac{d}{dy} \exp (-y^2) \right] \int_0^y du \exp (u^2) \right] (y^2 = x)
\end{equation}

\begin{equation}
= \frac{2}{\sqrt{\pi} A^3 v} e^{-A\theta} \int_0^{YAt_s} du \exp (u^2).
\end{equation}

Using the asymptotic expansion \((3 \cdot 34)\), we find for \(s \to +\infty\)

\begin{equation}
I_1^{(0)}(s) \sim \frac{1}{\sqrt{\pi} A^3} \frac{1}{s} + O\left( \frac{1}{s^2} \right).
\end{equation}

(A \cdot 10)

Similarly we obtain

\begin{equation}
I_1^{(1)}(s) = I_1^{(0)}(s) - \frac{1}{2} \left( \frac{2}{Av} \right)^2 \int_0^{A\theta} dx x I_1
\end{equation}

\begin{equation}
= I_1^{(0)}(s) + \frac{4}{\sqrt{\pi} A^3} \left[ \frac{1}{v} - \frac{1 + Av}{\sqrt{Av}} e^{-A\theta} \int_0^{YAt_s} du \exp (u^2) \right]
\end{equation}

\begin{equation}
\sim \frac{3}{\sqrt{\pi} A^3} \frac{1}{s} + O\left( \frac{1}{s^2} \right).
\end{equation}

(A \cdot 11)
Making use of the relation

\[ \text{Ei}(x) = \gamma + \ln x + \int_0^x dy e^{-y} y \]  

(A.12)

one obtains

\[ \int_0^L e^{-y} \text{Ei}(x) = \lim_{\epsilon \to 0} \left\{ -e^{-\epsilon} \text{Ei}(x) \right\}^L_0 + \int_0^L \frac{dy}{y} \]

\[ = -e^{-L} \text{Ei}(L) + \gamma + \ln L, \]  

(A.13)

where \( \gamma \) is Euler's constant. Thus we have

\[
I^{(0)}_2(s) = \frac{1}{Av} \int_A^s \frac{dx I_2}{I_2}
\]

\[= \frac{1}{\pi A} \frac{1}{v-s_0} \left[ e^{-Av} \text{Ei}(Av) - \gamma - \ln(Av) \right] \]

\[= \frac{2}{\sqrt{\pi A} v} \left( \frac{1}{v-s_0} \right) \int_0^{Av} du \exp(u^2) \right] - \sqrt{Av} \right) \]

\[+ \frac{2}{\sqrt{\pi A} v} \frac{1}{v-s_0} \left( \int_0^{\sqrt{Av}} du \exp(u^2) \right) \]

\[- \frac{1}{\sqrt{\pi A} v} \frac{1}{v-s_0} \left[ e^{-As_0} \text{Ei}(As_0) - \gamma - \ln(As_0) \right] \]

\[\sim - \frac{1}{\pi A} \frac{\ln(As)}{s} + \frac{\mathcal{D}_1}{s} + O\left(\frac{\ln s}{s^2}\right), \]  

(A.14)

where

\[\mathcal{D}_1 = \frac{1}{\pi A} \left[ 2 \sqrt{\pi} e^{-As_0} \int_0^{\sqrt{As_0}} du \exp(u^2) + \ln(As_0) - e^{-As_0} \text{Ei}(As_0) \right] \]

and we have also used the asymptotic expansion for \( e^{-y} \text{Ei}(x) \):

\[ e^{-y} \text{Ei}(x) \sim -\frac{1}{x} + \frac{1}{x^2} + \frac{3}{x^3} + \cdots. \]  

(A.15)

In the similar way

\[ I^{(0)}_2(s) = I^{(0)}_0 - \frac{1}{2} \left( \frac{2}{Av} \right)^2 \int_0^A dx x I_2 \]

\[\sim I^{(0)}_0 + \frac{\mathcal{D}_2}{s}, \]  

(A.16)

where

\[\mathcal{D}_2 = \frac{2}{\pi A} \frac{4}{3} \sqrt{\frac{s_0}{s}} - \frac{2}{\pi A s_0} \left[ A s_0 - (1 + As_0) e^{-As_0} \text{Ei}(As_0) + \gamma + \ln(As_0) \right] \]
Further we obtain
\begin{equation}
I_a^{(0)}(s) = \frac{1}{A \sqrt{s_0}} \left[ e^{A s_0} \left( 1 - \frac{2}{\sqrt{\pi}} \int_0^{Y \Delta s_0} du \exp(-u^2) \right) - 1 \right] \frac{1}{s} + O\left( \frac{1}{s^2} \right),
\end{equation}
\begin{equation}
I_a^{(1)}(s) \sim I_a^{(0)} + \left\{ \frac{4}{\sqrt{\pi} A \Delta s_0} \left[ 1 - e^{A s_0} (1 - A s_0) \left( 1 - \frac{2}{\sqrt{\pi}} \int_0^{Y \Delta s_0} du \exp(-u^2) \right) \right] \right\}.
\end{equation}

\textbf{Appendix 2}

\textit{Calculations for large angle scattering}

For Krisch's formula, we encounter the integral $I_a^{(i)}(s, z)$:
\begin{equation}
I_a^{(i)}(s, z) = \frac{1}{\pi} \int_0^\infty ds' \sqrt{\frac{s'}{s_0}} \exp\left[ -\alpha_i (1 - z^2) (s' - s) \right] \int_0^{Y \Delta s_0} du \exp(u^2).
\end{equation}

The result of this integration is just given by (A·7) with replacing $A \xi$ by $\bar{a}_i (1 - z^2)$. The partial wave projection becomes
\begin{equation}
\frac{1}{2} \int_0^\infty dz I_a^{(i)}(s, z)
= \frac{\sqrt{s_0}}{\sqrt{\pi} (v - s_0)} \int_0^1 dz \left[ \sqrt{v} \exp[-\bar{a}_i v (1 - z^2)] \int_0^{Y \Delta s_0} du \exp(u^2) \right.
- \sqrt{s_0} \exp[-\bar{a}_i s_0 (1 - z^2)] \int_0^{Y \Delta s_0} du \exp(u^2) \bigg]
- \frac{v}{2\pi (v - s_0)} \int_0^1 dz \exp[-\bar{a}_i v (1 - z^2)] Ei(\bar{a}_i v (1 - z^2))
+ \frac{s_0}{2\pi (v - s_0)} \int_0^1 dz \exp[-\bar{a}_i s_0 (1 - z^2)] Ei(\bar{a}_i s_0 (1 - z^2)).
\end{equation}

For the first integral, the following is obtained:
\begin{equation}
J = \int_0^1 dz \exp[-aw (1 - z^2)] \int_0^{Y \Delta s_0} du \exp(u^2)
= \int_0^1 dy \frac{\sqrt{y}}{\sqrt{1 - y^2}} \left[ \exp(-aw y^2) \int_0^{Y \Delta s_0} du \exp(u^2) \right], \quad (a > 0, w > 0)
\end{equation}
using the formula
\begin{equation}
\int_0^a dt \exp(-at^2) \sin(2xt) = \frac{1}{\sqrt{\pi a}} \exp(-x^2/a) \int_0^{a/\sqrt{\pi a}} du \exp(u^2),
\end{equation}
and interchanging the order of integrations we get
$J = \frac{\pi}{2\sqrt{aw}} \int_0^\infty dt \exp(-t^2/aw)J_i(2t)$

$= \frac{\sqrt{\pi}^3}{4} \exp(-aw/2)I_{1a}(aw/2)$

$= \frac{\pi}{4} \sqrt{\frac{1}{aw}} (1-e^{-aw})$, \hspace{1cm} (A·23)

For the second integral,

$\int_0^1 dz \exp[-\bar{a}\xi(1-z^2)]E_i(\bar{a}\xi(1-z^2))$

$= \frac{1}{\sqrt{\bar{a}\xi}} \int_0^{\sqrt{\bar{a}\xi}} dy \frac{y}{\sqrt{\bar{a}\xi-y^2}} \exp(-y^2)E_i(y^2)$

$= \frac{1}{\sqrt{\bar{a}\xi}} \int_0^{\sqrt{\bar{a}\xi}} dy \frac{y}{\sqrt{\bar{a}\xi-y^2}} \left(1+\frac{1}{2}y^2+\cdots\right)$

$\sim \frac{1}{\sqrt{\bar{a}\xi}} \ln(4\bar{a}\xi) + \frac{\gamma}{2\sqrt{\bar{a}\xi}}$, \hspace{1cm} (v\to\infty) \hspace{1cm} (A·24)

where $\lambda$ should be chosen to be sufficiently large but $\lambda\ll v$ and we have also used the asymptotic expansion (A·15). For the third integral, because of the small numerical value of $\bar{a}s_0(=1\sim0.2)$, it can be approximated, using (A·12), as

$\int_0^1 dz \exp[-\bar{a}s_0(1-z^2)]E_i(\bar{a}s_0(1-z^2))$

$= \frac{1}{\sqrt{\bar{a}s_0}} \int_0^{\sqrt{\bar{a}s_0}} dy \frac{y}{\sqrt{\bar{a}s_0-y^2}} \exp(-y^2)E_i(y^2)$

$\sim \frac{1}{\sqrt{\bar{a}s_0}} \int_0^{\sqrt{\bar{a}s_0}} dy \frac{y}{\sqrt{\bar{a}s_0-y^2}} \left(\gamma + 2\ln y\right)$

$= \gamma - \frac{2}{\sqrt{\bar{a}s_0}} + \ln(4\bar{a}s_0)$. \hspace{1cm} (A·25)

As a consequence we obtain for $s\to\infty$

$\frac{1}{2} \int_0^1 dz I_i(s,z) \sim -\frac{1}{4\pi\bar{a}_t} \ln(4\bar{a}_t) + \frac{\bar{D}_s}{s}$, \hspace{1cm} (A·26)

where $\bar{D}_s$ is approximated by

$\bar{D}_s = \frac{1}{2} \left[ -\frac{\gamma}{2\pi\bar{a}_t} + \frac{\sqrt{\pi}s_0}{2\bar{a}_t} \exp(-\bar{a}s_0) - \frac{s_0}{\pi} \left(\gamma - \frac{2\ln(4\bar{a}s_0)}{\sqrt{\bar{a}s_0}}\right)\right]$. \hspace{1cm} (A·27)

For Orear's formula, the main contribution to the force function comes again from the helicity non-flip amplitude.
\[ \left[ \sqrt{s} B_2(s, z) \right]_{\text{Born}} = -\frac{\sqrt{Ls}}{4\pi} \int_0^\infty ds' \frac{\sqrt{s' - s_0} \exp\left[ -\frac{\sqrt{s' - s_0}}{s' - s_0} \sqrt{1 - z^2} \right]}{\sqrt{s' (s' - s)}} \] 

\[ \approx -\frac{\sqrt{Ls}}{4\pi} \left\{ -\frac{\sqrt{v}}{(\sqrt{v} + \sqrt{s_0}) (v + s_0)} \exp(-d\sqrt{v}) Ei(d\sqrt{v}) \right. 
\left. - \frac{\sqrt{v}}{(\sqrt{v} - \sqrt{s_0}) (v + s_0)} \exp(d\sqrt{v}) Ei(-d\sqrt{v}) \right. 
\left. + \frac{1}{v - s_0} \exp(d\sqrt{s_0}) Ei(-d\sqrt{s_0}) \right. 
\left. + \frac{1}{v + s_0} \left[ ci(d\sqrt{s_0}) \sin(d\sqrt{s_0}) - si(d\sqrt{s_0}) \cos(d\sqrt{s_0}) \right] \right. 
\left. + \frac{1}{v + s_0} \left[ ci(d\sqrt{s_0}) \cos(d\sqrt{s_0}) + si(d\sqrt{s_0}) \sin(d\sqrt{s_0}) \right] \right\} \] 

where \( d = \sqrt[4]{1 - z^2} \) and \( Ei(-x) \), \( ci(x) \) and \( si(x) \) are the exponential, cosine and sine integrals respectively. The partial wave projections of the first two terms are carried out in the same approximation as (A·24), and for other three terms, we calculate numerically.

References

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