Unified Treatment of Rotation, Surface Vibration and Pairing Vibration of Atomic Nucleus*)

Akitsu IKEDA**)

Department of Physics, Osaka University, Toyonaka, Osaka

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The generator coordinate method is formulated so as to be suitable for description of the nuclei in the transition region. To take into account the pairing correlation, the BCS wave function constructed with various single particle states in a deformed potential well is used as an intrinsic wave function, from which eigenstates of angular momentum are projected out. Using the quadrupole deformation parameters, $\beta$ and $\gamma$, and the gap parameters, $\Delta_n$ and $\Delta_p$, as generator coordinates, the trial wave function is generated by superposing the projected wave functions with various values of the generator coordinates. Such a choice of the generator coordinates enables us to treat the rotation, the surface vibration and the pairing vibration in a unified way. The generator function, which serves as a weight function, is obtained by solving an integral equation derived through the variational principle. It is shown that our projected wave functions have the same symmetry properties as those of wave functions given by the Bohr model.

§ 1. Introduction

It is well known that in the nuclei far from the doubly closed shell the low-lying excited states have collective nature. We restrict ourselves to the doubly even nuclei in this article.

In the spherical region, the energy spectra and the electromagnetic properties of the low-lying excited states are at least qualitatively well described as the quadrupole harmonic vibration of the nuclear surface in the framework of the Bohr model. The microscopic treatment based on linearization approximation of the equation of motion has been proved to be successful to the same extent. There are, however, observed many deviations from the predicted harmonic vibration such as unequidistant level spacings of one- and two-phonon states, splitting of the two-phonon triplet states, missing of one or two members of the triplet, cross-over transitions from the two-phonon state to the ground state and so on. All of these experimental results show that there are strong anharmonic effects in the so-called spherical vibrational nuclei. Therefore it is desirable to develop a more rigorous theory which is not based on the harmonic vibration. More direct evidence for anharmonic effects would be the quadrupole moment of the first excited $2^+$ state ($Q_{2^+}$). Unfortunately the experimental evidence is

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** Present address: Department of Physics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo.
not conclusive in regard to this quantity. There have been several reports\(^7\) that \(Q_2\) is very large compared with the single particle estimate in the typical vibrational nuclei such as Cd isotopes. On the other hand, it has recently been reported by Simpson et al.\(^8\) that \(Q_2\) of Cd\(^{114}\) is very small.

The low-lying excited states of the deformed nuclei constitute several rotational bands whose energy spectra seem to obey the \(I(I+1)\) rule.\(^{1,2,4}\) These spectra are understood in the Bohr model as the rotation of a deformed nucleus as a whole (rotor model). A microscopic method to evaluate the moment of inertia is the cranking model.\(^9\) In order to reproduce the order of magnitude of experimentally obtained moments of inertia it is essential\(^{10}\) to use the wave functions in which the pairing correlation is taken into account. Deviations of the energy spectra from the \(I(I+1)\) rule are always observed in actual nuclei. Many works have been performed to explain the deviation from various points of view, which may be classified into three groups: (1) the rotation-vibration coupling including the centrifugal stretching,\(^{11,12,15}\) (2) the Coriolis' anti-pairing effect\(^{14,16}\) (a kind of rotation-pairing vibration coupling) and (3) higher order effects of the cranking model.\(^{17}\) However, if all of these effects are taken into account simultaneously,\(^{17}\) theoretical estimate of the deviation gives too large a correction to explain the experimental results. There are two defects in the usual treatments: (1) semi-classical nature of the cranking model and (2) the special assumption about the nuclear shape.

From the above-mentioned facts we can see that there are many difficulties in the conventional treatments in which all nuclei are divided into two categories, i.e. spherical or deformed, and the approximation of harmonic vibration either around a spherical shape or a deformed shape.

Besides spherical and deformed nuclear regions of the periodic table there is a region of onset or disappearance of nuclear deformation, which is called the transition region. Properties of the low-lying excited states of nuclei in this region are very much different from the predictions of either the phonon model or the rotor model. The experimental evidences suggest a strong vibration-rotation coupling. Therefore it is desirable especially in this region that a more rigorous method should be developed to treat the rotation and the vibration in a unified way.

There are experimental evidences that nuclear shapes change with not only mass number but also excitation energy. For example, at higher excitation energy rotational spectra\(^{18}\) were observed in the doubly closed shell nucleus O\(^{16}\). This fact shows that even the doubly closed shell nucleus can be deformed, although the deformation may not be always of quadrupole type. This possibility of variation of the nuclear shape with excitation energy may be common in all nuclei. So there arises an interesting problem of how the nuclear shape varies with increasing excitation energy. It is therefore necessary to develop a theoretical method in which the nuclear shape is allowed to change with excitation.
energy.

Baranger and Kumar\(^{19-21}\) made many calculations to investigate collective motions of the nuclei in the transition region within the framework of the Bohr model. The "pairing-plus-quadrupole mode"\(^{22}\) was employed. Rather good agreements with experimental results were obtained. However, the method used has some defects: (1) It cannot be generalized to more realistic nuclear forces than the pairing-plus-quadrupole one. (2) Therefore multipole components of the nuclear force other than monopole and quadrupole ones are neglected. (3) Exchange characters of the nuclear force are not considered at all.

In studying the collective modes of the medium-weight and heavy nuclei, the generator coordinate method (GCM)\(^{25-29}\) seems most appropriate. This method is based on the variational principle and is superior to the macroscopic treatment of the collective motion (Bohr model) because redundant variables inherent to the Bohr model are integrated out and interactions between nucleons are considered rigorously. Trial wave functions in the GCM are eigenstates of angular momentum in contrast to the usual selfconsistent field method.\(^{29}\) Equilibrium deformations and fluctuations around an equilibrium deformation in excited states can be described naturally. This method has been already applied to the nuclei in the transition region by Onishi and Yoshida\(^{27}\) with restriction to axially symmetric deformation and to light nuclei by Une.\(^{28}\) A purpose of the present paper is to generalize the formalism of reference \(^{27}\) to axially non-symmetric deformation.

Since the GCM is based on the variational principle, it is very important to choose a trial wave function which simulates the actual wave function as far as possible. In choosing the generator coordinates we should make full use of the successful results obtained by many authors in the course of various attempts.

It has been established that the low-lying excited states of the heavy nuclei far from the doubly closed shell are closely connected with quadrupole deformation of the nuclear density as can be seen from the structure of the Bohr model.\(^{25,27}\) Since the nuclear force is of short-range nature, we can reasonably expect that a nucleon feels a field of the same shape as the nuclear density. For the sake of simplicity, we employ the anisotropic harmonic oscillator potential well with various quadrupole deformations\(^{30}\) without restriction to axial-symmetry as the construction potential. The parameters \(\beta\) and \(\gamma\) which characterize deformation are used as generator coordinates.

After Bohr, Mottelson and Pines\(^{31}\) pointed out the existence of an energy gap in the intrinsic excitation spectrum of nuclei, it was proposed by Belyaev\(^{32}\) to apply the BCS wave function for a description of nuclei. This wave function has attained many successes particularly in calculations of various matrix elements of transition\(^{35-38}\) and moments of inertia.\(^{39}\) Though the BCS wave function, which is specified by a value of the energy gap, can explain many properties of nuclei, there remain residual interactions which are not taken into account.
in it. A more favourable wave function can be obtained by superposing BCS
wave function with various energy gaps. In this way fluctuations around an
equilibrium energy gap can be considered. So we employ the energy gap $A$
as another generator coordinate.

Intrinsic wave functions prepared in the above-mentioned way are not
eigenstates of angular momentum. Using the projection operator introduced by
Peierls and Yoccoz, eigenstates of angular momentum are projected out from
the intrinsic wave functions. This kind of angular momentum projection is just
a quantum mechanical treatment of nuclear rotations.

Fluctuations around an equilibrium deformation may be interpreted as the
so-called surface vibrations including $\beta$- and $\gamma$-vibration and fluctuations around
an equilibrium energy gap is the pairing vibration. Our method enables us to
treat the nuclear rotation, the surface vibrations and the pairing vibration in a
unified way, and mutual couplings are taken into account automatically.

The Hamiltonian used consists of the single particle energy and the interaction
between nucleons with all the exchange characters. The interaction can be
of any non-singular radial dependence.

In § 2 our formalism is developed. The method of the numerical calculation
is described in the Appendix. Some remarks are given in § 3.

§ 2. Description of the formalism

We describe the formalism in the following order: (1) the construction
potential and the single particle states in it, (2) the construction of the intrinsic
wave functions, (3) the construction of the projected wave functions and their
symmetry properties, (4) the variational principle and trial wave functions and
(5) calculation of the kernels and the matrix elements.

2-1 Single-particle states in quadrupole field

As mentioned in § 1, we use the following construction potential to obtain
single particle wave functions,

$$H_{c} = -\frac{\hbar^2}{2M} \rho^{2} + \frac{M}{2} (\omega_{x}^{2} x^{2} + \omega_{y}^{2} y^{2} + \omega_{z}^{2} z^{2}) + C \bar{T} \cdot \bar{s} + D \bar{I}^{2}$$

$$= H_{sph} + H_{def},$$

where $M$ is the nucleon mass and $H_{sph}$ and $H_{def}$ stand for a spherical field and
a deformed field respectively. After the transformation of the variables

$$x = \sqrt{\frac{M}{\hbar}} \omega_{b}(\beta, \gamma) \cdot x', \cdots,$$

they are given by

$$H_{sph} = \frac{\hbar \omega_{b}(\beta, \gamma)}{2} [ - \rho^{2} + r^{2} ] + C \bar{T} \cdot \bar{s} + D \bar{I}^{2},$$
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\[ H_{\text{rot}} = \frac{-\hbar\omega_0(\beta, \gamma)}{2} \left[ \frac{\omega_x^2 - \omega_y^2(\beta, \gamma)}{\omega_y^2(\beta, \gamma)} x^2 + \frac{\omega_y^2 - \omega_z^2(\beta, \gamma)}{\omega_z^2(\beta, \gamma)} y^2 + \frac{\omega_z^2 - \omega_x^2(\beta, \gamma)}{\omega_x^2(\beta, \gamma)} z^2 \right] , \]  

(4)

where \( \omega_x(\beta, \gamma) \) will be defined later. Usually \( H_{\text{rot}} \) is expressed as follows:

\[ H_{\text{rot}} = -\hbar\omega_0(\beta, \gamma) r^2 \sum_{m} \alpha_m Y_{2m}(\theta, \phi) \]

or

\[ H_{\text{rot}} = -\hbar\omega_0(\beta, \gamma) \sqrt{\frac{5}{4\pi}} \beta \left[ \cos \left( r - \frac{2\pi}{3} \right) x^2 + \cos \left( r + \frac{2\pi}{3} \right) y^2 + \cos \gamma \cdot z^2 \right] , \]

(6)

where \( \alpha's \) and \( (\beta, \gamma) \) are connected by

\[ \alpha_z = \alpha_{-z} = \frac{\beta}{\sqrt{2}} \sin \gamma , \quad \alpha_1 = \alpha_{-1} = 0 , \quad \alpha_0 = \beta \cos \gamma . \]

(7)

The relation between \( (\omega_x, \omega_y, \omega_z) \) and \( (\beta, \gamma) \) is

\[ \omega_x^2 = \omega_y^2(\beta, \gamma) \left[ 1 - 2 \sqrt{\frac{5}{4\pi}} \beta \cos \left( r - \frac{2\pi}{3} \right) \right] , \]

\[ \omega_y^2 = \omega_z^2(\beta, \gamma) \left[ 1 - 2 \sqrt{\frac{5}{4\pi}} \beta \cos \left( r + \frac{2\pi}{3} \right) \right] , \]

\[ \omega_z^2 = \omega_x^2(\beta, \gamma) \left[ 1 - 2 \sqrt{\frac{5}{4\pi}} \beta \cos \gamma \right] . \]

(8)

From the volume conservation, \( \omega_0(\beta, \gamma) \) is determined by

\[ \omega_0(\beta, \gamma) = \omega_0(\beta = 0) \left[ 1 - 2 \sqrt{\frac{5}{4\pi}} \beta \cos \left( r - \frac{2\pi}{3} \right) \right] \]

\[ \times \left[ 1 - 2 \sqrt{\frac{5}{4\pi}} \beta \cos \left( r + \frac{2\pi}{3} \right) \right]^{-\nu/4} \]

(9)

A reasonable choice of the value of \( \hbar\omega_0(\beta = 0) \) is

\[ \hbar\omega_0(\beta = 0) = 41 \cdot A^{-\nu/4} \text{ MeV} , \]

where \( A \) is the mass number of the nucleus under consideration.

We expand the eigenfunctions of \( H_e \) in terms of the ones of \( H_{\text{sph}} \) which are denoted as \( \phi_{\alpha} \), \( \alpha \) being an abbreviation of an abbreviation of a set of quantum numbers \( (j_a j_a j_a) \). Since \( H_{\text{rot}} \) causes \( \Delta m = 2 \) mixing, the energy matrix is divided into two parts, one of which belongs to \( m = j, j - 2, \ldots, -j + 1 \), and the other to \( m = j - 1, j - 3, \ldots, -j \). Introducing a quantum number \( q \), we assign \( q = 1/2 \) to the wave functions belonging to the former group and \( q = -1/2 \) to the latter one. Denoting the expansion coefficients by \( a_{\alpha q}(\beta, \gamma) \), the eigenfunctions \( \phi_{\alpha q}(\beta, \gamma) \) of \( H_e \) are given as

\[ \phi_{\alpha q}(\beta, \gamma) = \sum_{\alpha} a_{\alpha q}(\beta, \gamma) \phi_{\alpha} , \]

(10)
where the index \( \sigma \) stands for a set of quantum numbers of the eigenfunction of \( H_e \) other than \( q \). We write down the secular equations obeyed by \( a \)'s,

\[
\sum_{m_a > 0} a_{q \rightarrow a, m_a} \langle a | m_a | H_e | b, m_b \rangle = E_{a \rightarrow b} a_{q \rightarrow a, m_a},
\]

\[
\sum_{m_a > 0} a_{q \rightarrow a, m_a} \langle a - m_a | H_e | b - m_b \rangle = E_{a - \rightarrow b} a_{q \rightarrow a, m_a}.
\]

Here \( m_a > 0 \) implies that the summation is made over \( m_a = -(j-1) \) or \(-j, \cdots, -3/2, 1/2, 5/2, \cdots \), and \( a \) stands for \((n, l, \mathbf{j})\).

Noticing that \( \langle a | H_e | b \rangle \) is real and \( \langle a - m_a | H_e | b - m_b \rangle = (-) \langle a - m_a | H_e | b | m_b \rangle \), we can see that \( a_{q \rightarrow a, m_a} \) and \( (-) \langle a - m_a, a_{q - \rightarrow a, m_a} \rangle \) obey the same secular equation and \( a \)'s can be real quantities. Then we get

\[
E_{a \rightarrow b} = E_{a - \rightarrow b} = E_a,
\]

and the following phase convention is used,

\[
a_{q \rightarrow a, m_a} = (-1)^{j_a - m_a} a_{q - \rightarrow a, m_a}.
\]

Under the above phase convention the time reversed state \( \psi_{\tilde{q} \rightarrow q} \) of \( \psi_{q \rightarrow q} \) is connected to \( \psi_{q - \rightarrow q} \) as follows,

\[
\psi_{\tilde{q} \rightarrow q} = (-1)^{j_a - q} \psi_{q - \rightarrow q}.
\]

### 2-2 Intrinsic wave functions

Since the pairing correlation plays an important role in low-lying excited states of heavy nuclei as mentioned in § 1, we employ wave functions taken account of the pairing correlation as intrinsic ones. These can be obtained by applying the BCS procedure to the following generating Hamiltonian,

\[
H_p = \sum_{\beta, \gamma} (E_\beta - \lambda) C_{\beta \rightarrow q} \beta \gamma C_{\gamma \rightarrow q} \beta \gamma - \frac{g}{4} \sum_{\alpha, \beta, \alpha, \beta} C_{\alpha \rightarrow q} \beta \gamma C_{\alpha \rightarrow q} \beta \gamma C_{\beta \rightarrow q} \beta \gamma C_{\gamma \rightarrow q} \beta \gamma,
\]

where \( C_{\alpha \rightarrow q} \beta \gamma \) is the creation operator of a nucleon in a state labelled by quantum numbers \( \alpha \) and \( q \) which assign an orbit in the deformed field specified by \( \beta \) and \( \gamma \), and \( \lambda \) is the chemical potential which ensures the expectation value of the nucleon number operator to be the neutron or proton number. Here it should be noted that the Hamiltonian (15) is only a generating one, and \( g \) is the strength parameter of the pairing interaction used to get wave functions in which the pairing correlation is taken into account. Therefore \( g \) may be interpreted as a variable to generate wave functions with various energy gaps.

According to the usual BCS theory, we perform the Bogoliubov-Valatin transformation,

\[
\alpha_{q \rightarrow q} = u_{q \rightarrow q} C_{q \rightarrow q} + v_{q \rightarrow q} C_{q \rightarrow q}^+,
\]

\[
C_{q \rightarrow q}^+ = u_{q \rightarrow q} \alpha_{q \rightarrow q} + v_{q \rightarrow q} \alpha_{q \rightarrow q}^+.
\]
where $\alpha_{\sigma q}$ is the creation operator of a quasi-particle. The coefficients of the transformation, $u_{\sigma q}$ and $v_{\sigma q}$, are given in the form

$$|u_{\sigma q}|^2 = \frac{1}{2} \left( 1 + \frac{E_q - \lambda}{\sqrt{(E_q - \lambda)^2 + \Delta^2}} \right),$$  

$$|v_{\sigma q}|^2 = \frac{1}{2} \left( 1 - \frac{E_q - \lambda}{\sqrt{(E_q - \lambda)^2 + \Delta^2}} \right),$$  

(18)

where $\Delta$ is the energy gap. The quantities $\Delta$ and $\lambda$ are determined by solving the BCS equations:

$$\frac{4}{g} = \sum_{\sigma q} \frac{1}{\sqrt{(E_q - \lambda)^2 + \Delta^2}},$$  

$$2N = \sum_{\sigma q} \left[ 1 - \frac{E_q - \lambda}{\sqrt{(E_q - \lambda)^2 + \Delta^2}} \right],$$  

(19)  

(20)

where $N$ is the neutron or proton number. Under the time reversal operation on Eq. (16), we obtain

$$\alpha_{\sigma q} = u_{\sigma q}^* C_{\sigma q} + v_{\sigma q}^* C_{\sigma q}.$$  

(21)

On the other hand, from the definition we have

$$\alpha_{\sigma q} = u_{\sigma q} C_{\sigma q} + v_{\sigma q} C_{\sigma q}.$$  

(22)

From Eqs. (21), (22), (18) and (12) it is clear that we can use the following phase convention,

$$u_{\sigma q} = u_{\sigma q} = u_{\sigma} = \pm \frac{1}{2} \left[ 1 + \frac{E_q - \lambda}{\sqrt{(E_q - \lambda)^2 + \Delta^2}} \right],$$  

$$v_{\sigma q} = v_{\sigma q} = v_{\sigma} = \pm \frac{1}{2} \left[ 1 - \frac{E_q - \lambda}{\sqrt{(E_q - \lambda)^2 + \Delta^2}} \right].$$  

(23)

The BCS wave function is given in the form

$$\phi_0(\beta, \gamma, \Delta) = \prod_{\sigma} (u_{\sigma} (\beta, \gamma, \Delta) + v_{\sigma} (\beta, \gamma, \Delta) C_{\sigma_{1/2}} (\beta, \gamma) C_{\sigma_{1/2}}^+(\beta, \gamma) |0\rangle,$$  

(24)

where $|0\rangle$ stands for the nucleon vacuum. This wave function is the quasi-particle vacuum; $\alpha \phi_0 (\beta, \gamma, \Delta) = 0$. Thus the intrinsic wave functions are obtained.

The BCS wave function (24) can be written without any approximation as follows:

$$\phi_0(\beta, \gamma, \Delta) = \prod_{\sigma} u_{\sigma} (\beta, \gamma, \Delta) \exp \left[ \sum_{\sigma} v_{\sigma} (\beta, \gamma, \Delta) C_{\sigma_{1/2}} (\beta, \gamma) C_{\sigma_{1/2}}^+(\beta, \gamma) \right] |0\rangle,$$  

(25)

because when the exponential on the right-hand side is expanded in a power series the terms with more than two $C^+$ operators with the same $\sigma$ vanish iden-
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tically. (All $u_\alpha$ are assumed to be not equal to zero; if some of them are zero, Eq. (25) should be modified.) The expression (25) is of a form convenient for the later development. The nucleon operators in the exponent expressed on the deformed well basis are now transformed into the spherical basis. Any spherical basis may be used, but it is convenient to choose it so that the single particle states with various values of deformation parameters may be expressed as simply as possible. We make an approximation to neglect the dependence of $\hbar \omega_\nu (\beta, \gamma)$ on deformation parameters (Eqs. (9) and (10)). In this case, as a spherical basis we can use the single particle states of $H_{\text{sph}}$ in Eq. (3) with replacement of $\omega_\nu (\beta, \gamma)$ by $\omega_\nu (\beta = 0)$. The intrinsic wave function (25) may be rewritten in terms of the nucleon operator defined in the spherical potential well and the results are given by

$$
\phi_0 (\beta, \gamma, \Delta) = \left[ \prod_\alpha u_\alpha (\beta, \gamma, \Delta) \right] \exp \left[ \frac{i}{\hbar} \sum_{\alpha, \beta} f_{\alpha \beta} (\beta, \gamma, \Delta) C_\alpha^+ C_\beta^+ \right] |0\rangle ,
$$

(26)

where $C_\alpha^+$ is the creation operator of a nucleon in an eigenstate of $H_{\text{sph}}$ specified by quantum number $\alpha = (n_{\text{a}a}, j_\alpha m_\alpha)$. The relation between $C_\alpha^+$ and $C_\nu (\beta, \gamma)$ is just the same as Eq. (10), that is,

$$
C_\nu (\beta, \gamma) = \sum_{\alpha} a_{\nu,\alpha} (\beta, \gamma) C_\alpha^+ .
$$

(27)

Here, $f_{\alpha \beta} (\beta, \gamma, \Delta)$ are given as

$$
f_{\alpha \beta} (\beta, \gamma, \Delta) = \sum_{\nu} (-1)^{j_\alpha - j_\beta} a_{\nu, \alpha} (\beta, \gamma) a_{\nu - \beta, \beta} (\beta, \gamma) \frac{u_\nu (\beta, \gamma, \Delta)}{u_\alpha (\beta, \gamma, \Delta)} ,
$$

(28)

and have the following properties,

$$
f_{\alpha \beta} (\beta, \gamma, \Delta) = - f_{\beta \alpha} (\beta, \gamma, \Delta) ,
$$

(29a)

$$
f_{\alpha \beta} (\beta, \gamma = 0, \Delta) = 0 \quad \text{for} \quad m_\alpha + m_\beta \neq 0 ,
$$

(29b)

$$
f_{\alpha \beta} (\beta, \gamma, \Delta) = 0 \quad \text{for} \quad m_\alpha + m_\beta = \text{odd}
$$

(29c)

and

$$
f_{a - m_\alpha, b - m_\beta} (\beta, \gamma, \Delta) = (-1)^{j_\alpha - m_\alpha + j_\beta - m_\beta} f_{a m_\alpha, b m_\beta} (\beta, \gamma, \Delta) ,
$$

(30)

where subscripts $a$ and $b$ stand for $(n_{\text{a}a}, j_\alpha)$ and $(n_{\text{b}b}, j_\beta)$ respectively.

2-3 Projected wave functions and their symmetry properties

The intrinsic wave functions given by Eq. (26) is not an eigenfunction of angular momentum. Therefore it is necessary to project out the wave function with definite angular momentum $I$ and its $z$-component $M$. This projection can be performed by using the projection operator

$$
P_{IMK} = \frac{2I + 1}{8\pi^2} \int_0^{2\pi} d\varphi \int_0^{2\pi} d\chi \int_0^\pi \sin \vartheta d\vartheta D_{IMK}^I (\varphi, \vartheta, \chi) R_{\varphi \chi} .
$$

(31)

In Eq. (31) the rotation operator $R_{\varphi \chi}$ is defined by
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\[ R_{\text{ph}} = \exp(-i\varphi J_z) \exp(-i\theta J_y) \exp(-i\gamma J_x), \]  

(32)

where \( J_z, J_y \) are the \( z \)- and the \( y \)-component of the angular momentum operator respectively and \( D_{MK}^J(\varphi, \theta, \gamma) \) is the hermitian conjugate of the ordinary representation of rotational group. The projected wave function is given by

\[ \Phi_{IMK}(\beta, \gamma, \delta) = P_{IMK} \phi_0(\beta, \gamma, \delta), \]  

\[ = \frac{2I+1}{8\pi^2} \int_0^{2\pi} d\varphi \int_0^{2\pi} d\chi \int_0^\pi \sin \theta d\theta \hat{D}_{MK}^I(\varphi, \theta, \gamma) \phi_0(\beta, \gamma, \delta; \theta, \chi), \]  

(33)

where

\[ \phi_0(\beta, \gamma, \delta; \theta, \chi) = \prod \psi_c(\beta, \gamma, \delta) \exp\left[ \frac{1}{2} \sum_{a, \beta} f_{a \beta}(\beta, \gamma, \delta; \theta) C_a^+ C_{\beta}^+ \right] |0\rangle, \]  

(34)

and \( \Omega \) stands for Eulerian angles \((\varphi, \theta, \gamma)\). The quantities \( f_{a \beta}(\beta, \gamma, \delta; \theta) \) are defined by

\[ f_{a \beta}(\beta, \gamma, \delta; \theta) = \exp(-i(m_a + m_\beta) \varphi) \sum_{j, m_a \beta} \exp(-i(m_a' + m_\beta') \gamma) \times (j a m_a j m_\beta | J a m_a + m_\beta) \times d_{\nu a \mu \beta, m_a' m_\beta}^\nu(\beta, \gamma, \delta). \]  

(35)

The previous notation \( f_{a \beta}(\beta, \gamma, \delta) \) can then be regarded as the abbreviation of \( f_{a \beta}(\beta, \gamma, \delta; 000) \). In Eq. (35) \( d_{\nu a \mu \beta, m_a' m_\beta}^\nu(\beta, \gamma, \delta) \) is defined by removing the \( \varphi \) and \( \gamma \) dependent factors from \( \hat{D}_{MK}^I(\varphi, \theta, \gamma) \). The projected wave function (33) constitutes a trial wave function for variational calculation.

At this stage, it may be useful to investigate the symmetry properties of the projected wave functions (33).

(1) The intrinsic wave function (26) has the following property,

\[ \phi_0(\beta, -\gamma, \delta) = R_{\text{ph}} \phi_0(\beta, \gamma, \delta). \]  

(36)

Proof: The explicit expressions of both sides of Eq. (36) are given by

\[ \phi_0(\beta, -\gamma, \delta) = \prod \psi_c(\beta, -\gamma, \delta) \exp\left[ \frac{1}{2} \sum_{a, \beta} f_{a \beta}(\beta, -\gamma, \delta; \theta) C_a^+ C_{\beta}^+ \right] |0\rangle, \]  

(37a)

where

\[ f_{a \beta}(\beta, -\gamma, \delta) = \sum_{\nu q} (-1)^{1/2} a_{q \nu, a}(\beta, -\gamma) a_{q - \nu, \beta}(\beta, -\gamma) \psi_c(\beta, -\gamma, \delta) u_{\nu q}(\beta, -\gamma, \delta), \]  

(37b)

and

\[ R_{\text{ph}} \phi_0(\beta, \gamma, \delta) = \prod \psi_c(\beta, \gamma, \delta) \exp\left[ \frac{1}{2} \sum_{a, \beta} f_{a \beta}(\beta, \gamma, \delta; 00 \frac{\pi}{2}) C_a^+ C_{\beta}^+ \right] |0\rangle, \]  

(38a)

where

\[ f_{a \beta}(\beta, \gamma, \delta; 00 \frac{\pi}{2}) = \sum_{\nu q} (-1)^{1/2} a_{q \nu, a}(\beta, \gamma) a_{q - \nu, \beta}(\beta, \gamma) \psi_c(\beta, \gamma, \delta) u_{\nu q}(\beta, \gamma, \delta) \times \exp\left[ -i(\pi/2)(m_a + m_\beta) \right]. \]  

(38b)
In order to prove Eq. (36), we must know the relation between \( a_{\alpha\beta}(\beta, -\gamma) \) and \( a_{\alpha\beta}(\beta, \gamma) \). The secular equations for these quantities are

\[
\sum_{\alpha} a_{\alpha\beta}(\beta, \gamma) \langle \alpha | H_\varepsilon(\beta, \gamma) | \beta \rangle = E_\varepsilon(\beta, \gamma) a_{\alpha\beta}(\beta, \gamma), \tag{39a}
\]

\[
\sum_{\alpha} a_{\alpha\beta}(\beta, -\gamma) \langle \alpha | H_\varepsilon(\beta, -\gamma) | \beta \rangle = E_\varepsilon(\beta, -\gamma) a_{\alpha\beta}(\beta, -\gamma), \tag{39b}
\]

where the explicit form of \( H_\varepsilon(\beta, \gamma) \) is given in Eq. (1). If we use the relation

\[
H_\varepsilon(\beta, -\gamma) = R_{2\pi/3} R_\varepsilon(\beta, -\gamma) R_{2\pi/3}, \tag{40}
\]

where \( R^{-1} \) is the inverse of \( R \), the matrix element appearing on the left-hand side of Eq. (39) becomes

\[
\langle \alpha | H_\varepsilon(\beta, -\gamma) | \beta \rangle = \exp[i(\pi/2)(m_\alpha - m_\beta)] \langle \alpha | H_\varepsilon(\beta, \gamma) | \beta \rangle. \tag{41}
\]

If we insert Eq. (41) into Eq. (39b), it is seen that the two secular equations are identical. The quantities \( a_{\alpha\beta}(\beta, \gamma) \) and \( a_{\alpha\beta}(\beta, -\gamma) \exp[i(\pi/2)m_\alpha] \) are equal to each other except an arbitrary common phase. Since it is convenient to make both of them real, we use the following phase convention

\[
a_{\alpha\beta}(\beta, -\gamma) \exp[i(\pi/2)(m_\alpha - q)] = a_{\alpha\beta}(\beta, \gamma). \tag{42}
\]

It is also clear that the energy eigenvalues of the secular equations (39a) and (39b) are equal to each other,

\[
E_\varepsilon(\beta, \gamma) = E_\varepsilon(\beta, -\gamma). \tag{43}
\]

From Eqs. (19), (20), (23) and (43), we obtain

\[
\begin{bmatrix}
\nu_\varepsilon(\beta, -\gamma, \Delta)
\end{bmatrix} = \begin{bmatrix}
\nu_\varepsilon(\beta, \gamma, \Delta)
\end{bmatrix},
\]

\[
\begin{bmatrix}
u_\varepsilon(\beta, -\gamma, \Delta)
u_\varepsilon(\beta, \gamma, \Delta)\end{bmatrix},
\]

which is a short form of two equations. By making use of Eqs. (42) and (44) it can be easily proved that

\[
\prod_\varepsilon \nu_\varepsilon(\beta, -\gamma, \Delta) = \prod_\varepsilon \nu_\varepsilon(\beta, -\gamma, \Delta),
\]

\[
f_{\alpha\beta}(\beta, -\gamma, \Delta) = f_{\alpha\beta}(\beta, \gamma, \Delta; 00 \pi/2). \tag{45}
\]

Thus Eq. (36) is proved.

Under the projection operator on both sides of Eq (36), we obtain

\[
P_{IM\varepsilon K}(\beta, -\gamma, \Delta) = (-i)^K P_{IM\varepsilon K}(\beta, \gamma, \Delta), \tag{47a}
\]

where the expression of the right-hand side is obtained as follows:

\[
P_{IM\varepsilon K}R_{00/n\varepsilon K}(\beta, \gamma, \Delta) = \int d\Omega D_{K\varepsilon K}(\Omega) R_\varepsilon R_{00/n\varepsilon K}(\beta, \gamma, \Delta)
\]

\[
= \int d\Omega \sum_K D_{K\varepsilon K}((00 \pi/2)) D_{K\varepsilon K}(\Omega) R_\varepsilon R_{00/n\varepsilon K}(\beta, \gamma, \Delta)
\]
Unified Treatment of Rotation, Surface Vibration and Pairing Vibration

From Eq. (47) it is concluded that there do not exist odd-K components in our projected wave functions. The symmetry property considered here corresponds to the $R_{3}$-symmetry of the Bohr model. Then we can prove the following relation:

$$\phi_{0}(\beta, \gamma, \Delta) = R_{0x0y} \phi_{0}(\beta, \gamma, \Delta).$$

Proof: We write down the right-hand side of Eq. (48) as

$$R_{0x0y} \phi_{0}(\beta, \gamma, \Delta) = \prod u_{\epsilon}(\beta, \gamma, \Delta) \exp \left[ \frac{1}{2} \sum_{m_{a}m_{b}} f_{a\beta}(\beta, \gamma, \Delta; \varphi_{a} \varphi_{b}) 0 \pi \right] C_{a}^{+} C_{b}^{+} |0\rangle.$$

Then, we proceed as follows:

$$f_{a\beta}(\beta, \gamma, \Delta; \varphi_{a} \varphi_{b}) = \sum_{m_{a}m_{b}} f_{a_{m_{a}}m_{b}}(\beta, \gamma, \Delta) \frac{\pi_{a_{m_{a}}}(0 \pi \pi)}{\pi_{m_{b}}(0 \pi \pi)}$$

$$= \sum_{m_{a}m_{b}} (-1)^{m_{a}m_{b}} \exp \left[ i \pi (m_{a} + m_{b}) \right] (-1)^{m_{a}m_{b}}$$

$$\times \sum_{m_{a}m_{b}} \chi_{a_{m_{a}}}(\beta, \gamma) \chi_{m_{b}}(\beta, \gamma) \frac{\varphi_{a}^{+}(\beta, \gamma, \Delta)}{u_{\epsilon}(\beta, \gamma, \Delta)}$$

$$= f_{a\beta}(\beta, \gamma, \Delta),$$

where Eq. (13) and the relation, $m_{a} + m_{b} =$ even for non-vanishing $f_{a\beta}(\beta, \gamma, \Delta)$, were used. Thus Eq. (48) is proved directly.

This equation presents another useful relation:

$$P_{1MxK} \phi_{0}(\beta, \gamma, \Delta) = (-1)^{t} P_{1MxK} \phi_{0}(\beta, \gamma, \Delta).$$

In deriving the above equation, the same technique as used in Eq. (47a) can be used. It is then easy to see that the projected wave function with odd spin has not $K=0$ component. Since there is no $K=odd$ component as has already been shown, it follows that there does not exist a state with $I=1$ among the projected wave functions. In the case of axially-symmetric deformation, i.e. $\gamma=0$, the spin $I$ must be even, because there exists only the $K=0$ component in the projected wave function when $\gamma=0$, as can be seen from Eqs. (29b), (33) and (35). In reference 27) this was not given a complete proof, but only concluded by assuming the invariance of the original function $\phi_{0}(\beta, \gamma, \Delta)$ with respect to $180^\circ$ rotation about an axis perpendicular to the symmetry axis. The symmetry property (50) corresponds to $K_{3}$-symmetry of the Bohr model. Finally the symmetry property, which corresponds to the $R_{3}$-symmetry of the Bohr model, is proved. We start by proving the following relation:

$$\phi_{0}(\beta, \gamma + \frac{2\pi}{3}, \Delta) = R_{0x/3} \phi_{0}(\beta, \gamma, \Delta).$$

Proof: We give here the expression of the right-hand side of Eq. (51),
\[ R_{e\tau/n\beta/\phi_0}(\beta, \gamma, \mathcal{A}) = \prod_{\epsilon} u_{\epsilon}(\beta, \gamma, \mathcal{A}) \exp \left[ \frac{1}{2} \sum_{\alpha, \beta} f_{\alpha \beta}(\beta, \gamma, \mathcal{A}; 0, \frac{\pi}{2}, \frac{\pi}{2}) C_\alpha^+ C_\beta^+ \right] |0\rangle , \]  

(52a)

where

\[ f_{\alpha \beta}(\beta, \gamma, \mathcal{A}; 0, \frac{\pi}{2}, \frac{\pi}{2}) = \sum_{m_{\alpha'}, m_{\beta'}} f_{\alpha_{\beta_{\gamma_{\mathcal{A}'}}}}(\beta, \gamma, \mathcal{A}) \mathcal{D}_{m_{\alpha}m_{\alpha'}}^{l_{\beta}} \left( \frac{0, \frac{\pi}{2}}{2} \right) \mathcal{D}_{m_{\beta}m_{\beta'}}^{l_{\gamma}} \left( \frac{0, \frac{\pi}{2}}{2} \right) . \]  

(52b)

Since the deformed potential well \( H_c(\beta, \gamma + \frac{2\pi}{3}) \) is connected with \( H_c(\beta, \gamma) \) by

\[ H_c(\beta, \gamma) = R_{e\tau/n\beta/\phi_0}^{-1} H_c(\beta, \gamma + \frac{2\pi}{3}) R_{e\tau/n\beta/\phi_0}, \]  

(53)

the secular equation for \( a_{\epsilon q, a}(\beta, \gamma + \frac{2\pi}{3}) \) can be written as

\[ \sum_{\alpha} \left[ \sum_{m_{\alpha'}} a_{\epsilon q, a_{\gamma_{\mathcal{A}'}}}(\beta, \gamma + \frac{2\pi}{3}) \mathcal{D}_{m_{\alpha}m_{\alpha'}}^{l_{\beta}} \left( \frac{0, \frac{\pi}{2}}{2} \right) \langle m_{\alpha} H_c(\beta, \gamma) | m_{\beta}' \rangle \right] = E_\epsilon \left( \beta, \gamma + \frac{2\pi}{3} \right) \left[ \sum_{m_{\alpha'}} a_{\epsilon q, b_{\gamma_{\mathcal{A}'}}}(\beta, \gamma + \frac{2\pi}{3}) \mathcal{D}_{m_{\alpha}m_{\alpha'}}^{l_{\gamma}} \left( \frac{0, \frac{\pi}{2}}{2} \right) \right] . \]  

(54)

This is identical with Eq. (39a). Therefore the energy eigenvalues of Eq. (54) are equal to those of Eq. (39a),

\[ E_\epsilon \left( \beta, \gamma + \frac{2\pi}{3} \right) = E_\epsilon (\beta, \gamma) , \]  

(55)

and, moreover, we obtain

\[ a_{\epsilon q, a}(\beta, \gamma + \frac{2\pi}{3}) = \sum_{m_{\alpha'}} a_{\epsilon q, a_{\gamma_{\mathcal{A}'}}}(\beta, \gamma) \mathcal{D}_{m_{\alpha}m_{\alpha'}}^{l_{\beta}} \left( 0, \frac{\pi}{2} \right) \exp \left[ i (\pi/2) (q + 2m_{\alpha'}) \right] , \]  

(56)

where the phase factor, \( \exp[i(\pi/2)q] \), was inserted to guarantee that both of \( a_{\epsilon q, a}(\beta, \gamma) \) and \( a_{\epsilon q, a}(\beta, \gamma + \frac{2\pi}{3}) \) be real. From Eqs. (19), (20), (23) and (55), we obtain

\[ \begin{pmatrix} v_\epsilon(\beta, \gamma + \frac{2\pi}{3}, \mathcal{A}) \\ u_\epsilon(\beta, \gamma + \frac{2\pi}{3}, \mathcal{A}) \end{pmatrix} = \begin{pmatrix} v_\epsilon(\beta, \gamma, \mathcal{A}) \\ u_\epsilon(\beta, \gamma, \mathcal{A}) \end{pmatrix} . \]  

(57)

Substituting Eqs. (56) and (57) into Eqs. (52a) and (52b), one can obtain Eq. (51). Q.E.D.

Using Eq. (51), one can obtain an important property of the projected wave function:

\[ P_{IMK} \phi_0(\beta, \gamma + \frac{2\pi}{3}, \mathcal{A}) = \sum_{K'} \mathcal{D}_{KK'}^{l_{\mathcal{A}'}} \left( \frac{0, \frac{\pi}{2}}{2} \right) P_{IKM} \phi_0(\beta, \gamma, \mathcal{A}). \]  

(58)

In the above consideration concerning the symmetry properties, only the
quadrupole nature of the deformation was used. Therefore the results obtained here are valid for any single particle potential with quadrupole deformation irrespective of its radial shape other than $H_c$.

2-4 Variational principle and trial wave functions

According to the generator coordinate method, the wave function which describes the collective motion is expressed as superposition of the projected wave functions $\phi_{IKM}(\beta, \gamma, \Delta)$ with various values of the deformation parameters and the energy gap parameters. If the generator function is denoted by $f_{K}^{I,n}(\beta, \gamma, \Delta)$, then the wave function is given by

$$\Psi_{IM}^{n} = \sum_{K} \int_{0}^{\alpha_{m}} d\beta \int_{0}^{\alpha_{m}} d\Delta \int_{0}^{(2\pi)} d\gamma \phi_{IKM}(\beta, \gamma, \Delta) f_{K}^{I,n}(\beta, \gamma, \Delta),$$

where $\Delta_{m}$ and $\beta_{m}$ are taken to be sufficiently large. The integral domain of the variable $\gamma$ is reduced from $[0, 2\pi]$ to $[0, \frac{\pi}{2}]$ by using Eqs. (47) and (58). The quantum number $n$ specifies the mode of the collective motion. The generator function $f_{K}^{I,n}(\beta, \gamma, \Delta)$ is determined by making the following expression stationary with respect to the variation of the generator function:

$$E_{n} = \langle \Psi_{IM}^{n} | H | \Psi_{IM}^{n} \rangle,$$

where $H$ represents the Hamiltonian of the system of nucleons under consideration. This condition leads to the integral equation

$$\sum_{K'} \int d\beta' d\gamma' d\Delta' \left[ H_{KK'}^{\beta \gamma \Delta} (\beta' \gamma' \Delta') - E_{n} N_{KK'}^{\beta \gamma \Delta} (\beta' \gamma' \Delta') \right] f_{K}^{I,n}(\beta' \gamma' \Delta') = 0.$$

The projected energy kernel $H_{KK'}^{\beta \gamma \Delta} (\beta' \gamma' \Delta')$ and overlap kernel $N_{KK'}^{\beta \gamma \Delta} (\beta' \gamma' \Delta')$ are defined by

$$\begin{bmatrix}
H_{KK'}^{\beta \gamma \Delta} (\beta' \gamma' \Delta') \\
N_{KK'}^{\beta \gamma \Delta} (\beta' \gamma' \Delta')
\end{bmatrix} = \langle \phi_{IKM}^{\beta, \gamma, \Delta} | H \phi_{IKM}^{\beta', \gamma', \Delta'} \rangle.
$$

When solving the integral equation (61), the symmetry properties (47), (50) and (58) provide sufficient boundary conditions in the variable $\gamma$.

2-5 Calculation of kernels and matrix elements

This subsection is devoted mainly to the evaluation of the two kernels defined in Eq. (62). To avoid complexity, we use the following abbreviated notation in the subsequent discussions:

$$\delta = (\beta, \gamma, \Delta), \quad \int d\beta \int d\gamma \int d\Delta = \int d\delta, \quad \Omega = (\varphi, \theta, \chi).$$

If Eqs. (33) and (34) are substituted into Eq. (62), then the following expression is obtained by using the invariance of the Hamiltonian under rotation and the
orthonormality of the $D$-functions:

$$\begin{bmatrix} N_{kk'}(\delta, \delta') \\ H_{kk'}(\delta, \delta') \end{bmatrix} = \int d\Omega D_{kk'}^{*}(\delta) \begin{bmatrix} n(\delta, \delta'; \Omega) \\ h(\delta, \delta'; \Omega) \end{bmatrix}. \quad (63)$$

where

$$\begin{bmatrix} n(\delta, \delta'; \Omega) \\ h(\delta, \delta'; \Omega) \end{bmatrix} = N(\delta) N(\delta') \langle 0 | \exp[\frac{1}{\hbar} \sum_{\alpha \beta} f_{\alpha \beta}(\delta) C_{\alpha} C_{\beta}] \begin{bmatrix} 1 \\ H \end{bmatrix} \times \exp[\frac{1}{\hbar} \sum_{\alpha \beta} f_{\alpha \beta}(\delta'; \Omega) C_{\alpha} C_{\beta}^{+}] | 0 \rangle. \quad (64)$$

In Eq. (64) $N(\delta)$ is the normalization constant given by

$$N(\delta) = \prod_{\sigma} u_{\delta}(\delta).$$

It is now necessary to give an explicit form of the Hamiltonian. Assuming that the effective interaction is a two-body one which has no hard core, it is represented as

$$H = \sum_{\alpha, \beta} \epsilon_{\alpha \beta} C_{\alpha} C_{\beta} + \frac{1}{4} \sum_{\alpha \beta \gamma} V_{\alpha \beta \gamma \delta} C_{\alpha}^{+} C_{\beta}^{+} C_{\gamma} C_{\delta}. \quad (65)$$

The single particle energy is expressed by $\epsilon_{\alpha \beta}$, which consists of the kinetic energy and the single particle potential which may contain the spin-orbit force if necessary. The matrix element of the effective interaction is given by $V_{\alpha \beta \gamma \delta}$, which satisfies the following antisymmetry relation:

$$V_{\alpha \beta \gamma \delta} = - V_{\beta \alpha \gamma \delta} = - V_{\alpha \beta \delta \gamma} = V_{\beta \alpha \delta \gamma}. \quad (66)$$

The matrix element, $n(\beta \gamma \alpha \beta' \gamma' \alpha'; \delta \delta'; \Omega)$ is evaluated as follows:

$$n(\delta, \delta'; \Omega) = N(\delta) N(\delta') \exp[\frac{1}{\hbar} \text{Tr} \{ \ln (1 + M(\delta, \delta'; \Omega)) \}], \quad (67)$$

where

$$M_{\alpha \beta}(\delta, \delta'; \Omega) = - \sum_{\gamma} f_{\alpha \beta}(\delta'; \Omega) f_{\gamma \beta}(\delta). \quad (68)$$

According to the method given in reference 39), the other matrix element $h(\delta, \delta'; \Omega)$ is given by

$$h(\delta, \delta'; \Omega) / n(\delta, \delta'; \Omega) = \sum_{\alpha \beta} \epsilon_{\alpha \beta} \rho_{\alpha \beta}(\delta, \delta'; \Omega) + \sum_{\alpha \beta \gamma \delta} V_{\alpha \beta \gamma \delta} \left[ \frac{1}{2} \rho_{\alpha \gamma}(\delta, \delta'; \Omega) \rho_{\beta \delta}(\delta, \delta'; \Omega) + \frac{1}{\hbar} \kappa_{\alpha \beta}(\delta, \delta'; \Omega) \kappa_{\gamma \delta}(\delta, \delta'; \Omega) \right]. \quad (69)$$

where the generalized density and pair matrices are defined by

$$\rho_{\alpha \beta}(\delta, \delta'; \Omega) = \left[ \frac{M(\delta, \delta'; \Omega)}{1 + M(\delta, \delta'; \Omega)} \right]_{\alpha \beta}, \quad (70a)$$

$$\kappa_{\alpha \beta}(\delta, \delta'; \Omega) = \sum_{\gamma} f_{\alpha \beta}(\delta) \left[ \frac{1}{1 + M(\delta, \delta'; \Omega)} \right]_{\gamma \beta}, \quad (70b)$$
The energy kernel, $h(\delta, \delta'; \Omega)$, can be brought into a form convenient for the numerical computations using the technique developed by Baranger. The matrix element of the effective interaction is written in two ways; one is convenient to treat the pairing type interaction and the other is useful for describing the particle-hole interaction

$$V_{ab\beta\delta} = -\frac{1}{2} \sum_{J\Omega} G(abcd; J)(ja_{m_a}jb_{m_\beta}|JM)(ja_{m_\tau}jb_{m_\tau}|JM), \quad (71a)$$

$$V_{ab\beta\delta} = -\frac{1}{2} \sum_{J\Omega} F(acdb; J)s_\tau(ja_{m_a}ja_{m_\tau}-m_\tau|JM)s_\beta(ja_{m_\beta}jb_{m_\beta}-m_\beta|JM), \quad (71b)$$

where $s_\tau = (-1)^{J_\tau-M_\tau}$. Next the density- and the pair matrix are expanded as follows:

$$\rho_{ab,J\Omega}(\delta, \delta'; \Omega) = \sum_{m_a} (-1)^{J_\alpha-m_\beta}(ja_{m_a}j_b-m_\beta|JM)\rho_{ab}(\delta, \delta'; \Omega), \quad (72)$$

$$\kappa_{ab,J\Omega}(\delta, \delta'; \Omega) = \sum_{m_a} (ja_{m_a}j_b|JM)\kappa_{ab}(\delta, \delta'; \Omega). \quad (73)$$

Then, the energy kernel is expressed as

$$h(\delta, \delta'; \Omega)/n(\delta, \delta'; \Omega) = \sum_{a\beta} \kappa_{a\beta}\rho_{ab,\alpha}(\delta, \delta'; \Omega)$$

$$+ \sum_{J\Omega} \left[-\frac{1}{2} F(abcd; J)(-1)^{J_\tau-J_\sigma+m_\tau} \rho_{ab,J\Omega}(\delta, \delta'; \Omega)\rho_{cd,J-M}(\delta, \delta'; \Omega) \right.$$

$$\left.+ \frac{1}{2} G(abcd; J)\kappa_{ab,J\Omega}(\delta, \delta'; \Omega)\kappa_{cd,J\Omega}(\delta, \delta'; \Omega) \right]. \quad (74)$$

The process of the numerical calculation is as follows: After the single particle states in the deformed field (1) are obtained, the BCS equations are solved. Then, the intrinsic wave functions are constructed. The energy kernel and the overlap kernel are calculated as a function of $\beta, \gamma, \Delta$ and the Eulerian angles. From these kernels the projected overlap and energy kernels are calculated by Eq. (63). When the integral equation (61) is solved, the eigenfunction and the corresponding energy eigenvalue $E_n^I$ are obtained.

Since the intrinsic wave functions have various values of $K$, we are forced to perform integration over all of the Eulerian angles. This procedure of evaluating the projected kernels requires a large amount of computation time. A method to reduce the computation time is developed by making use of some symmetry properties of the density- and the pair matrix, and is explained in the Appendix.

Once the wave function is obtained we can calculate the various nuclear quantities such as the magnetic moment, the quadrupole moment and the transition probability of electric and magnetic radiations. To calculate these quantities we need the matrix element of the general one-body tensor operator $O^k_q$, whose expression is given by
\[ \langle \mathcal{Y}^n_{I'M'|J} | O^h_q | \mathcal{Y}^n_{IM} \rangle = \frac{8\pi^2}{2I' + 1} \sum_{\kappa',\kappa} (I M q | I'M') \sum_{\mu,\nu} (I \mu \nu | I' K' ) \]
\[ \times \int \mathcal{D} f_{J'K'}(\Omega) \, d\Omega \int f_{K'J}(\delta') f_{J'K}(\delta) \, d\delta \, d\delta' \langle a | O^h_q | b \rangle \rho_{ab,\kappa}(\delta', \delta; \Omega) n(\delta', \delta; \Omega), \]

where the reduced matrix element is given by

\[ \langle a | O^h_q | b \rangle = \sqrt{2j_a + 1} \sum_{\alpha,\beta} (j_m \rho_{q \alpha} | j_{a \beta} m_a \rangle \langle j_{a \beta} m_a | O^h_q | j_m \rho_{q \alpha} \rangle. \]

The method explained in the Appendix can be used in the evaluation of the matrix element (75).

§ 3. Concluding remarks

We have developed a microscopic theory to describe the rotation, the surface vibration and the pairing vibration of the medium-weight and heavy nuclei in a unified way. The rotation is described quantum mechanically by the angular momentum projection in contrast to the semi-classical method, i.e. the cranking model.

The treatment of the surface vibration is completely general: We have included the variable \( r \) and the harmonic approximation is not assumed at all. From the mathematical considerations on the symmetries of the intrinsic wave functions, we have obtained the selection rules for possible values of the nuclear spin and the boundary conditions which should be imposed on the generator function about the variable \( r \). The basis of these considerations lies in the fact that there are 24 different sets of the principal axes of an ellipsoid which are equivalent to each other as the reference frame in which to describe the intrinsic wave functions.

Quite a new method for the pairing vibration is proposed. This method has been applied to the schematic system of \( n \)-particles moving in two non-degenerate \( j \)-shells and interacting via the pairing force, which is amenable to exact calculation. The calculated excitation energy of the first excited state (the pairing vibration) agreed well with the exact one.

Some features of the collective motion explained with the Bohr model can be understood in the method developed here: If the generator function \( f_{J}^I(\beta, r, \Delta) \) is localized around \( \beta = \beta_0 (\beta_0 \text{ is large}) \), then the overlap kernel has sharp peaks at \( \theta = 0 \) and \( \pi \). In this case the \( D \)-function can be expanded around \( \theta = 0 \) and \( \pi \) up to the second order in \( \theta \). Then we would obtain the \( I(I+1) \)-rule for the energy spectra of the rotational band. Moreover, the \( K \)-mixing would be small because \( d_{J}^{K}(\theta) \) has not large value around \( \theta = 0 \) and \( \pi \), for \( K \neq K' \), and the concept of the \( \beta \)- and the \( r \)-vibration would become good. However, these features appear only as a limiting case of large deformation. It should be noted that the mutual couplings between the collective modes are taken into account.
very naturally in the present method.

The numerical calculation for actual nuclei is now in progress.

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Appendix

The method of reducing the integral domain of the Eulerian angles

Operating a factor depending on \( \varphi \) among the three factors in \( R_{\sigma \chi} \) onto the bra vector in Eq. (32), the matrix \( M \) defined in (68) can be rewritten as

\[
M_{\alpha \beta}(\delta, \delta'; \varphi \vartheta \chi) = -\sum_{\beta'} F_{\alpha \beta}(\delta'; \vartheta \chi) G_{\xi \beta}(\delta; \varphi),
\]  
(A·1)

where

\[
F_{\alpha \beta}(\delta; \vartheta \chi) = f_{\alpha \beta}(\delta; 0 \vartheta \chi),
\]  
(A·2)

\[
G_{\xi \beta}(\delta; \varphi) = f_{\xi \beta}(\delta; - \varphi 0 0).
\]  
(A·3)

Defining a matrix \( M^{(1)} \) by

\[
M_{\alpha \beta}^{(1)}(\delta, \delta'; \varphi \vartheta \chi) = -\sum_{\beta'} F_{\alpha \beta}^{*}(\delta'; \vartheta \chi) G_{\xi \beta}(\delta; \varphi),
\]  
(A·4)

another kind of the density and pair matrices connected with \( M^{(1)} \) are introduced:

\[
\rho_{\alpha \beta}^{(1)}(\delta, \delta'; \varphi, \vartheta, \chi) = \left[ \frac{M^{(1)}(\delta, \delta'; \varphi \vartheta \chi)}{1 + M^{(1)}(\delta, \delta'; \varphi \vartheta \chi)} \right]_{\alpha \beta},
\]  
(A·5)

\[
\xi_{\alpha \beta}^{(1)}(\delta, \delta'; \varphi \vartheta \chi) = \left[ \frac{1}{1 + M^{(1)}(\delta, \delta'; \varphi \vartheta \chi)} F^{*}(\delta'; \varphi \chi) \right]_{\alpha \beta},
\]  
(A·6)

\[
\xi_{\alpha \beta}^{(1)}(\delta, \delta'; \varphi \vartheta \chi) = \left[ G(\delta; \varphi) \frac{M^{(1)}(\delta, \delta'; \varphi \vartheta \chi)}{1 + M^{(1)}(\delta, \delta'; \varphi \vartheta \chi)} \right]_{\alpha \beta}.
\]  
(A·7)

It can be easily seen from the definitions of \( f_{\alpha \beta} \), (28), that there exist the following relations

\[
G_{\alpha \beta}(\delta; 2\pi - \varphi) = G_{\alpha \beta}(\delta; \pi - \varphi) = G_{\alpha \beta}(\delta; \varphi),
\]  
(A·8)

\[
F_{\alpha \beta}(\delta; \vartheta, 2\pi - \chi) = -F_{\alpha \beta}(\delta; \vartheta, \pi - \chi) = -F_{\alpha \beta}(\delta; \vartheta, \chi),
\]  
(A·9)

\[
F_{\alpha \beta}(\delta; \pi - \vartheta, \chi) = (-1)^{m_{a} + m_{b}} F_{\alpha \beta}^{*}(\delta; \vartheta, \chi).
\]  
(A·10)

Using Eqs. (A·8) \sim (A·10), one obtains

\[
M_{\alpha \beta}(\delta, \delta'; 2\pi - \varphi, \vartheta, \chi) = M_{\alpha \beta}(\delta, \delta'; \pi - \varphi, \vartheta, \chi) = M_{\alpha \beta}^{(3)}(\delta, \delta'; \varphi \chi),
\]  
(A·11)

\[
M_{\alpha \beta}(\delta, \delta'; \varphi, \vartheta, 2\pi - \chi) = M_{\alpha \beta}(\delta, \delta'; \varphi, \vartheta, \pi - \chi) = M_{\alpha \beta}^{(3)}(\delta, \delta'; \varphi \chi).\n\]  
(A·12)
If one put Eqs. (A·8)∼(A·12) into Eqs. (70a)∼(70c) and (A·5)∼(A·7), one obtains
\[
\rho_{ab} (\delta, \delta'; 2\pi - \varphi, \vartheta, \chi) = \rho_{ab} (\delta, \delta'; \pi - \varphi, \vartheta, \chi) = \rho^{(1)}_{ab} (\delta, \delta'; \varphi, \vartheta, \chi), \quad (A·13)
\]
\[
\rho_{ab} (\delta, \delta'; \varphi, \vartheta, 2\pi - \chi) = \rho_{ab} (\delta, \delta'; \varphi, \vartheta, - \chi) = \rho^{(1)}_{ab} (\delta, \delta'; \varphi, \vartheta, \chi), \quad (A·14)
\]
\[
\kappa_{ab} (\delta, \delta'; 2\pi - \varphi, \vartheta, \chi) = \kappa_{ab} (\delta, \delta'; - \varphi, \vartheta, \chi) = \kappa^{(1)}_{ab} (\delta, \delta'; \varphi, \vartheta, \chi), \quad (A·15)
\]
\[
\kappa_{ab} (\delta, \delta'; \varphi, \vartheta, 2\pi - \chi) = \kappa_{ab} (\delta, \delta'; \varphi, \vartheta, - \chi) = \kappa^{(1)}_{ab} (\delta, \delta'; \varphi, \vartheta, \chi), \quad (A·16)
\]
\[
\kappa_{ab} (\delta, \delta'; 2\pi - \varphi, \vartheta, \chi) = \kappa_{ab} (\delta, \delta'; - \varphi, \vartheta, \chi) = \kappa^{(1)}_{ab} (\delta, \delta'; \varphi, \vartheta, \chi), \quad (A·17)
\]
\[
\kappa_{ab} (\delta, \delta'; \varphi, \vartheta, 2\pi - \chi) = \kappa_{ab} (\delta, \delta'; \varphi, \vartheta, - \chi) = \kappa^{(1)}_{ab} (\delta, \delta'; \varphi, \vartheta, \chi). \quad (A·18)
\]
Equations (A·13)∼(A·18) are sufficient to reduce the integral domain of the variables \( \varphi \) and \( \chi \) from \([0, 2\pi]\) to \([0, \pi/2]\). Here we define the second kind of the overlap and energy kernels, \( n^{(1)} \) and \( k^{(1)} \), by
\[
n^{(1)} (\delta, \delta'; \varphi \theta \chi) = N(\delta) N(\delta') \exp \left[ \frac{1}{2} \text{Tr} \{ \ln (1 + M^{(1)} (\delta, \delta' ; \Omega)) \} \right], \quad (A·19)
\]
\[
k^{(1)} (\delta, \delta'; \varphi \theta \chi) = h^{(1)} (\delta, \delta'; \varphi \theta \chi) / n^{(1)} (\delta, \delta'; \varphi \theta \chi)
\]
\[
= \sum_{\alpha \beta} \epsilon_{\alpha\beta} \theta^{(5)}_{\alpha\beta} (\delta, \delta'; \varphi \theta \chi) + \sum_{\alpha \beta \gamma} V_{\alpha\beta\gamma}
\]
\[
\times \left[ \frac{1}{2} \theta^{(5)}_{\alpha\gamma} (\delta, \delta'; \varphi \theta \chi) \rho_{\alpha\gamma}^{(5)} (\delta, \delta'; \varphi \theta \chi) + \frac{1}{2} \kappa^{(5)}_{\alpha\beta} (\delta, \delta'; \varphi \theta \chi) \kappa^{(5)}_{\alpha\beta} (\delta, \delta'; \varphi \theta \chi) \right]. \quad (A·20)
\]
Then, the projected kernels of the integral equation (61) can be written as
\[
H_{KK'} (\delta, \delta') - E^*_n N_{KK'} (\delta, \delta')
\]
\[
= 2 \int_0^{\pi} \sin \theta d\theta dK K' (\delta) \left[ \int_0^{\pi/2} d\varphi e^{iK' \varphi} \int_0^{\pi/2} d\chi 
\right.
\]
\[
\times \left\{ (k (\delta, \delta'; \varphi \theta \chi) - E^{(1)}_n) n (\delta, \delta'; \varphi \theta \chi) e^{iK' \chi} 
\right.
\]
\[
+ (k^{(1)} (\delta, \delta'; \varphi \theta \chi) - E^{(1)}_n) n^{(1)} (\delta, \delta'; \varphi \theta \chi) e^{-iK' \chi} \right\} + \text{c.c.} \right]. \quad (A·21)
\]
Next, we make the reduction of the domain of integration about the variable \( \theta \). First, let us prove the following equation,
\[
M_{ab} (\delta, \delta'; \varphi, \pi - \vartheta, \chi) = M^{(1)}_{ab} (\delta, \delta'; \varphi, \vartheta, \chi) \cdot (-1)^{m_a - m_b}. \quad (A·22)
\]
Proof: From the definition of \( M_{ab} \) the left-hand side of Eq. (A·22) is written as
\[
M_{ab} (\delta, \delta'; \varphi, \pi - \vartheta, \chi) = - \sum_{\gamma} F_{ab} (\gamma; \pi - \vartheta, \chi) G_{\gamma \delta} (\delta; \varphi).
\]
Making use of Eq. (A·10), one can further transform the right-hand side as
\[
= - \sum_{\gamma} (-1)^{m_a + m_{\gamma}} F_{a\delta}^{*} (\gamma; \vartheta, \chi) G_{\gamma \delta} (\delta; \varphi)
\]
\[
= (-1)^{m_a - m_b} M^{(1)}_{ab} (\delta, \delta'; \varphi, \vartheta, \chi).
\]
since \(m_a + m_b = \text{even}\), for non-vanishing \(G_{a\beta}(\delta; \varphi)\). Q.E.D.

Then it is easy to prove the relation

\[
\rho_{a\beta}(\delta, \delta'; \varphi, \pi - \vartheta, \chi) = (-1)^{m_a - m_b} \rho_{a\beta}^{(1)}(\delta, \delta'; \varphi, \theta, \chi). \tag{A·23}
\]

Proof: We expand the left-hand side of Eq. (A·23) in a power series,

\[
\rho_{a\beta}(\delta, \delta'; \varphi, \pi - \vartheta, \chi) = \sum_{n=1}^{\infty} (-1)^n \sum_{k_1, \ldots, k_n} M_{k_1 \ldots k_n}(\delta, \delta'; \varphi, \pi - \vartheta, \chi) 
\]

then inserting the relation (A·22) into the right-hand side,

\[
= \sum_{n=1}^{\infty} (-1)^n \sum_{k_1, \ldots, k_n} (-1)^{m_a - m_b} \rho_{a\beta}^{(1)}(\delta, \delta'; \varphi, \theta, \chi) M_{k_1 \ldots k_n}(\delta, \delta'; \varphi, \theta, \chi) 
\]

Thus the proof is completed.

We can prove the similar relations for the pair matrices just in the same way:

\[
\kappa_{a\beta}(\delta, \delta'; \varphi, \pi - \vartheta, \chi) = (-1)^{m_a + m_b} \kappa_{a\beta}^{(1)}(\delta, \delta'; \varphi, \theta, \chi) \tag{A·24}
\]

and

\[
\kappa_{a\beta}(\delta, \delta'; \varphi, \pi - \vartheta, \chi) = (-1)^{m_a + m_b} \kappa_{a\beta}^{(1)}(\delta, \delta'; \varphi, \theta, \chi). \tag{A·25}
\]

The last step is to prove the relations

\[
k(\delta, \delta'; \varphi, \pi - \vartheta, \chi) = k^{(1)}(\delta, \delta'; \varphi, \theta, \chi) \tag{A·26}
\]

and

\[
n(\delta, \delta'; \varphi, \pi - \vartheta, \chi) = n^{(1)}(\delta, \delta'; \varphi, \theta, \chi). \tag{A·27}
\]

The proof of the second relation is easy. We prove here the first one.

Proof: Using Eqs. (A·23) \sim (A·25) the left-hand side of Eq. (A·26) can be written as

\[
k(\delta, \delta'; \varphi, \pi - \vartheta, \chi) = \sum_{a\beta} \epsilon_{a\beta} \rho_{a\beta}^{(1)}(\delta, \delta'; \varphi, \theta, \chi) (-1)^{m_a - m_b}
\]

\[
+ \sum_{a\sigma\tau} V_{a\sigma\tau} \{ \frac{1}{2} (-1)^{m_a - m_{\sigma\tau}} \rho_{a\sigma}^{(1)}(\delta, \delta'; \varphi, \theta, \chi) (-1)^{m_{\sigma\tau} - m_\beta} \rho_{\sigma\beta}^{(1)}(\delta, \delta'; \varphi, \theta, \chi) 
\]

\[
+ \frac{1}{2} (-1)^{m_a + m_{\sigma\tau}} \kappa_{a\sigma}^{(1)}(\delta, \delta'; \varphi, \theta, \chi) (-1)^{m_{\sigma\tau} + m_\beta} \kappa_{\sigma\beta}^{(1)}(\delta, \delta'; \varphi, \theta, \chi) \}
\]

Since the matrix elements \(\epsilon_{a\beta}\) and \(V_{a\sigma\tau}\) conserve the magnetic quantum number, all the phase factors appearing on the right-hand side disappear, and Eq. (A·26) is shown to be valid.

Now, the kernel in the integral equation becomes

\[
[H_{KK'}(\delta, \delta') - E_n \mathcal{N}_{KK'}(\delta, \delta')] 
\]

\[
= 2 \int_{0}^{E/2} \sin \vartheta d\vartheta [d_{KK'}^{(1)}(\vartheta) \mathcal{A}_{KK'}(\vartheta) + (-1)^{1/2} d_{K'-K}^{(1)}(\vartheta) \mathcal{A}_{K'-K'}(\vartheta)], \tag{A·28}
\]
where
\[
\mathcal{A}_{EE'}(\hat{\theta}) = \int_0^{\pi/2} d\varphi e^{i\varphi} \int_0^{\pi/2} d\chi \left[ (k(\hat{\theta}, \hat{\theta}'; \varphi \varphi \chi) - E_n') n(\hat{\theta}, \hat{\theta}'; \varphi \varphi \chi) e^{iK'x} \right. \\
+ (k^{(1)}(\hat{\theta}, \hat{\theta}'; \varphi \varphi \chi) - E_n) n^{(1)}(\hat{\theta}, \hat{\theta}'; \varphi \varphi \chi) e^{-iK'x} + c.c.].
\] (A·29)

The computation time may be largely reduced by using (A·28) and (A·29) in place of the original form, Eqs. (63) and (64).

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