A Discrete Model of the Inverse Love Wave Problem

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Summary

A mathematical model is used in order to understand some unresolved aspects of the uniqueness of the inverse problem for the Earth normal modes. More specifically, the conjecture of Backus & Gilbert regarding the information content of the natural frequencies of modes associated with different angular numbers is examined. This is done for the geophysically artificial case of a vibrating system which is a discrete version of an elastic membrane in the shape of an infinite strip, with a density variation across the width. The question then becomes whether the dispersion relations for the waves propagating along this membrane are sufficient to determine the density uniquely. A definitive answer to this conjecture is still not available, but our results strengthen its validity.

1. Introduction

The proliferation of earth models capable of accounting for the existing data related to the Earth structure is familiar to all seismologists. But what are the implications of this state of affairs? Does it mean that more data are required in order to discriminate between the Earth proper and nearby models? Or does it mean that the solution of the inverse problem for the internal structure of the Earth is inherently non-unique?

This central question of the uniqueness of the solution of the inverse problem has not been settled yet. Results related to simpler inverse problems have been invoked to argue both for and against uniqueness. For example, within the framework of ray theory, the classical Wiechert–Herglotz method provides a means for finding the speed of sound in an elastic medium from travel-time vs distance data. Uniqueness has been established for the case in which the speed of sound is a monotonic function of depth. More recently, Gerver & Markushevich (1972) have shown how the non-uniqueness introduced by the presence of low velocity zones can be removed by means of additional data from sources below these zones.

On the other hand, certain results from the theory of inverse eigenvalue problems can be mustered by the opposite side. I am referring, in particular, to Borg’s theorem (see e.g. Hochstadt 1973) which plays a central part in the theory of inverse Sturm–Liouville problems. This theorem can be started as follows:

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Given the regular Sturm–Liouville problem

\[ u'' + (\lambda - q(x)) u = 0, \]
\[ u(0) \sin \alpha + u'(0) \cos \alpha = 0, \]
\[ u(1) \sin \beta + u'(1) \cos \beta = 0, \]

where \( \alpha \) and \( \beta \) are parameters, then the spectra associated with two values of \( \alpha \), say \( \{\lambda_n(\alpha_1)\} \) and \( \{\lambda_n(\alpha_2)\} \), are required in order to determine \( q(x) \) uniquely.

In fact, as a consequence of this theorem, it is possible to construct an infinite number of distinct functions \( q(x) \) which are associated with the same spectrum. Arguing by analogy, it is tempting to say that the Earth density, bulk and shear moduli cannot be determined uniquely from the natural frequencies.

The present paper is devoted to a study of some aspects of this general question of uniqueness. Much of the original impetus for this work comes from a remark by Backus \& Gilbert (1968). Being aware of Borg's theorem and of its potential implications for the earth normal mode problem, they conjectured that the dependence of the natural frequencies on the angular number could be exploited to obtain uniqueness. In so doing, Backus \& Gilbert drew attention upon an aspect of the problem which tends to be overlooked. Namely, the mathematical formulation of the problem for the earth normal modes does not lead to a one-dimensional eigenvalue problem but rather to a one-parameter family of eigenvalue problems labelled by the angular number (see e.g. Alterman, Jarosch \& Pekeris 1959; Takeuchi \& Saito 1972). This is a direct consequence of the fact that the basic eigenvalue problem is a three-dimensional problem which is invariant under rotation and not a genuine one-dimensional problem. The invariance comes about from the fact that the density and Lamé parameters are functions of a single variable (namely, depth); in other words, from the spherical symmetry. Consequently, the relevance of the results from the one-dimensional theory, such as Borg's theorem, may be questioned. It is this aspect of the inverse problem with which the present paper is concerned.

Finally, in closing this introduction, I would like to acknowledge the influence of a paper by Kac (1972) and one by Gerver \& Kazdan (1972). These papers are devoted to two specific inverse eigenvalue problems which are invariant under rotation or translation. Kac's paper considers the problem of reconstructing the shape of an axially-symmetric body of revolution from a knowledge of its natural frequencies of vibration. The title of the paper by Gerver \& Kazdan, namely, 'Finding a velocity profile from a Love wave dispersion curve: problems of uniqueness ' is self-explanatory and could have served equally well as the title of my paper. I shall have several occasions to refer to these two papers.

2. Mathematical model

A brief summary of the salient results of Gerver \& Kazdan will be useful in the sequel. To that effect, it is preferable to refer to an older paper of theirs (Gerver \& Kazdan 1967) in which the results, divorced from the seismological context, appear in a simpler form. They considered the following eigenvalue problem:

\[ u'' + (pp(x) - s) u = 0, \]
\[ u'(0) = u'(1) = 0, \]

which is very similar to that arising in the theory of propagation of Love waves in the Earth mantle. We shall however look upon (2.1)–(2.2) as arising from an investigation of the propagation of waves along an elastic membrane filling the strip
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$0 < x < 1, \ -\infty < y < \infty$, whose superficial density $\rho$ is solely a function of $x$. These waves (which are either standing or travelling) have frequency $\omega = p^t$ and wave number $l = s^t$. The boundary conditions (2.2) imply that the edges are 'free'. With this interpretation, $u(x) \exp \left[ i(p^t t - s^t y) \right]$ is the transverse displacement of the membrane at the point $(x, y)$ at time $t$.

Solving the direct problem consists in finding the dispersion curves

$$\omega_n = \Omega_n(l)$$

or equivalently

$$s = \sigma_n(p), \ n = 1, 2, \ldots$$

(2.3)

The converse question is the one of interest here, namely, given a dispersion curve, say $s = \sigma_1(p)$, can $\rho(x)$ be recovered uniquely? Since $\rho(x)$ and $\rho(1-x)$ give rise to the same set of dispersion curves, we shall disregard this trivial multiplicity due to a symmetry about the centre.

This fundamental question of uniqueness has not been settled yet. What Gerver & Kazdan were able to show is that the function

$$h(r) = \text{mes} \{x : \rho(x) > r\},$$

(2.4)

i.e. the length of the interval(s) over which $\rho(x) > r$, can be deduced from the asymptotic behaviour of $\sigma_n(p)$ for $p \to \infty$. A similar result was also obtained by Kac by very different means.

**Fig. 1.** A density profile and associated $h$-function.
Fig. 1 illustrates the relationship between \( \rho(x) \) and \( h(r) \). If \( \rho(x) \) is a monotonic function of \( x \), then \( \rho \) and \( h \) are related thus:

\[
h(\rho(x)) = \begin{cases} 
x & \text{if } \rho \text{ is increasing,} \\
1 - x & \text{if } \rho \text{ is decreasing.}
\end{cases}
\] (2.5)

In seismological parlance, this case corresponds to an absence of low velocity zones.

In order to pursue the investigation of the general question of uniqueness without having to make use of sophisticated tools from the theory of functions of several complex variables, it is profitable to re-examine the work of Gerver & Kazdan in the context of a simpler physical problem. The particular problem which we shall consider can be viewed as arising from a discretization in the \( x \)-direction of the original problem for the membrane. More specifically, we shall consider \( N \) horizontal 'ribs' of linear density \( m_1, m_2, \ldots, m_N \) held together by means of massless strips of width \( l_1, l_2, \ldots, l_N \) (see Fig. 2). Thus, the density \( \rho(x) \) consists of \( N \) delta-functions of amplitudes \( m_i (i = 1, 2, \ldots, N) \), located a distance \( l_i \) apart. The obvious advantage of this discrete system lies in the fact that it has a finite number of modes. It is the two-dimensional analogue of the point-mass system which Krein (1952) has used so successfully in his investigation of the inverse problem for a vibrating string. Our hope is that, just as for the one-dimensional problem, the results of this discrete version of the two-dimensional problem will have a direct analogue in the continuous version.

If \( U_k \exp [i (p t - s y)] \) represents the displacement of the \( k \)-th rib, then Newton's law implies that

\[
-m_k \rho U_k = \frac{U_{k+1} - U_k}{l_k} - \frac{U_k - U_{k-1}}{l_{k-1}} - s U_k.
\] (2.6)

Departing slightly from the problem considered by Gerver & Kazdan, we shall consider that the edges of this ribbed structure are held fixed, namely

\[
U_0 = U_{N+1} = 0.
\] (2.7)

Denoting the \( N \) branches of the dispersion relation by \( \sigma_k (p), k = 1, 2, \ldots, N \) we shall inquire into whether the structure \( \{m_k\}_1^N \) and \( \{l_k\}_0^N \) can be inferred from the know-

\[
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\uparrow \\
\downarrow \\
\downarrow \\
\hline
1 \\
l_N \\
l_2 \\
l_1 \\
l_0
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\downarrow \\
\uparrow \\
\downarrow \\
\downarrow \\
\hline
m_N \\
m_3 \\
m_2 \\
m_1
\end{array}
\end{array}
\]

Fig. 2. The 'membrane' consists of ribs of linear densities \( \{m_i\}_1^N \) connected by massless strips of width \( \{l_i\}_0^N \).
ledge of one such branch, say $\sigma_1(p)$. We should emphasize that we are interested solely in the uniqueness of the solution of this inverse problem and not in its existence. We shall therefore always assume that $\sigma_k(p)$ are *bona fide* dispersion curves associated with at least one ribbed structure.

### 3. Some preliminary results

In order to solve the direct problem, i.e., to find the dispersion relations $\sigma_k(p)$, we must find the zeros of the following characteristic polynomial:

$$
D(s, p) = \begin{vmatrix}
 s - m_1 p + \frac{1}{l_0} + \frac{1}{l_1} & - \frac{1}{l_1} & \cdots & 0 \\
- \frac{1}{l_1} & s - m_2 p + \frac{1}{l_1} + \frac{1}{l_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & s - m_N p + \frac{1}{l_{N-1}} + \frac{1}{l_N}
\end{vmatrix}
$$

(3.1)

Clearly $D(s, p)$ is a polynomial of degree $N$ in both $s$ and $p$. Because of its importance, it is useful at this stage to obtain three different representations of this polynomial.

From the very definition of $\sigma_k(p)$ and from the fact that the coefficient of $s^N$ is unity, we can obviously write

$$
D(s, p) = \prod_{k=1}^{N} (s - \sigma_k(p)).
$$

(3.2)

This product representation is very practical whenever the dispersion curves are known.

As is frequently the case in the theory of algebraic curves (see e.g. Primrose 1955), it is convenient to introduce a third 'homogeneous' variable, typically denoted by $z$, and to consider instead of $D(s, p)$ the following *homogeneous* polynomial of degree $N$

$$
D(s, p, z) = \begin{vmatrix}
 s - m_1 p + z q_1 & - \frac{z}{l_1} & \cdots & 0 \\
- \frac{z}{l_1} & s - m_2 p + z q_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & s - m_N p + z q_N
\end{vmatrix}
$$

(3.3)

where

$$
q_k = \frac{1}{l_{k-1}} + \frac{1}{l_k}, \quad k = 1, 2, \ldots, N.
$$

(3.4)

Clearly

$$
D(s, p, 1) = D(s, p).
$$

Looked upon as a polynomial in $z$, we can write $D(s, p, z)$ as follows:

$$
D(s, p, z) = d_N(s, p) + z d_{N-1}(s, p) + \ldots + z^N d_0.
$$

(3.5)
This second representation is particularly useful for investigating the asymptotic properties of \( D(s, p) = 0 \) as \( p \to \infty \). Whenever we shall want to emphasize the fact that \( D(s, p) \) is associated with the \( N \)-rib case, we shall make use of a superscript and write

\[
D^{(N)}(s, p, z) = \sum_{k=0}^{N} z^k d_k^{(N)}(s, p).
\]  

(3.6)

By developing the determinant (3.3) about the last row or column, we can easily see that

\[
D^{(N)}(s, p, z) = (s - m_N p + z q_N) D^{(N-1)}(s, p, z) - \frac{z^2}{l_{N-1}} D^{(N-2)}(s, p, z).
\]  

(3.7)

This recurrence relation can, in turn, be used to get recurrence relations for the \( d_k^{(N)} \). Indeed, substituting (3.6) on (3.7) and equating powers of \( z \), we deduce that

\[
d_k^{(N)}(s, p) = q_N d_k^{(N-1)}(s, p) + (s - m_N p) d_k^{(N-1)}(s, p) - \frac{1}{l_{N-1}^2} d_k^{(N-2)}(s, p).
\]  

(3.8)

Even though \( k \) ranges only between 0 and \( N \), it is convenient to define:

\[
d_{N+1}^{(N)} = d_{N+2}^{(N)} = 0
\]  

(3.9)

\[
d_0^{(N)} = 1.
\]  

(3.10)

We shall primarily be concerned with the terms in \( D(s, p) \) of degree \( N \), \( N-1 \) and \( N-2 \) i.e. with \( d_N^{(N)}(s, p), d_{N-1}^{(N)}(s, p) \) and \( d_{N-2}^{(N)}(s, p) \). By means of (3.8), we can deduce that

\[
d_N(s, p) = \prod_{k=1}^{N} (s - m_k p),
\]  

(3.11)

\[
d_{N-1}(s, p) = \sum_{j=1}^{N} q_j \prod_{k=j}^{N} (s - m_k p),
\]  

(3.12)

\[
d_{N-2}(s, p) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} q_i q_j \prod_{k \neq i, j}^{N} (s - m_k p) - \sum_{j=1}^{N-1} \frac{1}{j^2} \prod_{k \neq j, j+1}^{N} (s - m_k p).
\]  

(3.13)

Finally, the third representation comes simply from looking upon \( D(s, p) \) as a polynomial in both \( s \) and \( p \), namely

\[
D(s, p) = \sum_{k=0}^{N} \sum_{j=0}^{k} (-1)^j C_j^{(k)} s^k - j p^j
\]  

(3.14)

where

\[
C_0^{(k)} = 1.
\]

Note that \( D(s, p) \) contains \( \frac{1}{2} N(N+3) \) coefficients.

Looked upon as an algebraic curve, \( D(s, p) \) has several important properties which have physical interpretations. For instance, the branches \( s = \sigma_k(p) \) are monotonic increasing functions of \( p \). Indeed, from the identity

\[
D(\sigma_k(p), p) = 0
\]
we deduce that

$$\frac{d\sigma_k}{dp} = -\left(\frac{\partial D}{\partial s}\right)_{s=\sigma_k(p)} \; \frac{\partial p}{\partial s}$$

(3.15)

Now

$$\frac{\partial D}{\partial s} = \begin{vmatrix}
1 & -1/l_1 & \ldots & 0 \\
0 & s-m_2 p+q_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & s-m_N p+q_N
\end{vmatrix} + \ldots
$$

where $\Delta_j(s, p)$ is the minor associated with the $j$-th diagonal term of $D(s, p)$.

Similarly

$$\frac{\partial D}{\partial p} = -\sum_{j=1}^{N} m_j \Delta_j(s, p).$$

(3.16)

But $\Delta_j(s, p)$ can be considered as the characteristic polynomial of the stiffer system obtained by constraining the $j$-th rib to be fixed. Consequently, according to Rayleigh's principle, the $N-1$ eigenfrequencies of this stiffer system must interlace the eigenfrequencies of the original system. Therefore, if $\sigma_j^{(j)}(p)$ denote the zeros of $\Delta_j(s, p)$, then

$$\sigma_1(p) > \sigma_1^{(1)}(p) > \ldots > \sigma_1^{(N-1)}(p) > \sigma_1^{(N)}(p).$$

(3.18)

In other words, the branch $s = \sigma_k(p)$ lies between the curves $\sigma_1^{(j)}(p)$ and $\sigma_2^{(j)}(j = 1, 2, \ldots, N)$ and hence all the $\Delta_j(\sigma_k(p), p)$ have the same sign. It therefore follows that

$$\frac{d\sigma_k}{dp} = \sum_{j=1}^{N} m_j \Delta_j(\sigma_k, p) > 0.$$ 

(3.19)

This result is not surprising and has a direct analogue in the theory of Love waves.

Since we shall not be concerned with the question of existence of solutions and with the related problems of conditions which arbitrary curves $\{\sigma_i(p)\}_{i=1}^{N}$ must satisfy in order to be genuine branches of a dispersion relation, we shall not pursue this investigation any further.
4. Probing via short waves: the asymptotics \( p \to \infty \)

As \( p \to \infty \), the branches \( \sigma_k(p) \) tend to infinity. This region of \((s, p)\) plane corresponds to high frequency waves of short horizontal wave number. These waves, which are reflected back and forth by the boundaries \( x = 0, x = 1 \) are ideally suited for probing the fine structure of the density profile.

In order to examine this region of the \((s, p)\) plane, we must first find the asymptotes of the \( N \)-branches of the dispersion curve, i.e. whether \( s/p \) tends to a limit as \( p \to \infty \). Making use of the representation (3.6), we start by writing

\[
\frac{D(s, p)}{p^N} = d_N \left( \frac{s}{p}, 1 \right) + \frac{1}{p} d_{N-1} \left( \frac{s}{p}, 1 \right) + \cdots + \frac{d_0}{p^N}.
\]

Clearly, as \( p \to \infty \) the direction of the asymptotes is given by the zeros of

\[
d_N \left( \frac{s}{p}, 1 \right) = 0.
\]

From (3.11), we deduce that these zeros are simply the densities \( m_i \) of the ribs. In other words, from the slopes of the asymptotes of the dispersion curves \( \sigma_k(p) \) we can deduce the densities \( \{m_k\}^N_1 \) (see Fig. 3). Note that we cannot infer the location of the ribs, or for that matter, the order in which these ribs of known density are arranged. To emphasize the limitation of our knowledge at this stage, we shall introduce a column matrix \( \mathbf{M} \) made up by the zeros of (4.2) arranged in decreasing order of magnitude, namely

\[
\mathbf{M}^T = (M_1, M_2, \ldots, M_N),
\]

where ' \( T \) ' stands for transpose. Then, if

\[
\mathbf{m}^T = (m_1, m_2, \ldots, m_N)
\]

![Fig. 3. Dispersion curve in the \((s, p)\) plane. The slope of the asymptotes are equal to \( M_1, M_2, \ldots, M_N \). The corresponding s-intercepts are equal to \(-Q_1, -Q_2, \ldots, -Q_N\).](https://academic.oup.com/gji/article-abstract/44/1/61/665639)
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\[ m = PM \]  

(4.5)

where \( P \) is an \( N \) by \( N \) permutation matrix. Note that if we knew, \textit{a priori}, that the density distribution was a monotonic decreasing function of height, then \( P \) would be the identity matrix and the asymptotic behaviour of the dispersion relation would uniquely determine \( \{m_i\}^N_{i=1} \).

As we shall see, some complications occur whenever two or more of the \( m_i \)'s are equal. In order to avoid these difficulties, we shall assume that all the \( m_i \)'s are distinct, i.e. that

\[ M_1 > M_2 > \ldots > M_N. \]  

(4.6)

The results we have just obtained can be looked upon as the first term of the asymptotic expansion of \( \sigma_k(p) \). In order to get the next term, we write

\[ \sigma_k(p) = M_k p - Q_k + 0(p^{-1}) \]  

(4.7)

and substitute it in (4.1), namely

\[ d_N \left( M_k - \frac{Q_k}{p}, 1 \right) + \frac{1}{p} d_{N-1}(M_k, 1) + 0(p^{-2}) = 0. \]  

(4.8)

Making use of the explicit expressions for \( d_N(s, p) \) and \( d_{N-1}(s, p) \) given in (3.11) and (3.12) we deduce that

\[ \prod_{i=1}^{N} \left( M_k - \frac{Q_k}{p} - m_i \right) + \frac{1}{p} \sum_{j=1}^{N} q_j \prod_{i \neq j}^{N} (M_k - m_i) + 0(p^{-2}) = 0, \]

i.e.

\[ Q_k = \frac{\sum_{j=1}^{N} q_j \prod_{i \neq j}^{N} (M_k - m_i)}{\prod_{i=1, i \neq k}^{N} (M_k - M_i)}. \]  

(4.9)

The above expression looks more complicated than it really is. Indeed, if the \( k \)-th heaviest rib is in the \( i \)-th location, i.e. if

\[ m_i = M_k, \]

then (4.9) becomes

\[ Q_k = q_i \]  

(4.10)

More generally, \( Q \) is the column matrix whose entries are the negatives of the intercepts of the asymptotes along the \( s \)-axis, then

\[ q = PQ \]  

(4.11)

where

\[ q^T = (q_1, q_2, \ldots, q_N), \]  

(4.12)

and \( P \) is the same permutation previously encountered (see Fig. 3). Once again, the values of \( \{q_i\}^N_{i=1} \) are known but not their order.

The first two terms of the asymptotic expansion of \( \sigma_k(p) \) are insensitive to the permutation matrix \( P \). Thus, if one had no other information than those two terms, then the question of uniqueness could not be resolved. However, the third terms in the expansions of \( \sigma_k(p) \) do discriminate between the permutations \( P \). To see this, let us write

\[ \sigma_k(p) = M_k p - Q_k - \Lambda_k p^{-1} + 0(p^{-2}). \]  

(4.13)
Following the same procedure as before, we write
\[
d_N \left( M_k - \frac{Q_k}{p} - \frac{\Lambda_k}{p^2}, 1 \right) + \frac{1}{p} d_{N-1} \left( M_k - \frac{Q_k}{p}, 1 \right) \]
\[p^{1+} d_{N-2}(M_k, 1) + O(p^{-3}) = 0. \quad (4.14)\]

Omitting the intermediary algebra, we see that
\[
Q_k^2 \sum_{j=1}^{N} \prod_{i \neq k} (M_k - M_i) - \Lambda_k \prod_{i \neq k} (M_k - M_i) 
- Q_k \sum_{j=1}^{N} \prod_{i \neq k} (M_k - M_i) - \Lambda_k \prod_{i \neq k} (M_k - M_i).
\]
\[+ \frac{1}{2} \sum_{h=1}^{N} \sum_{j=1}^{N} Q_h Q_j \prod_{i \neq h, j} (M_k - M_i) - \sum_{j=1}^{N-1} \prod_{i \neq j, j+1} (M_k - M_i) = 0\]
or better still
\[\Lambda_k = - \frac{\sum_{j=1}^{N} \prod_{i \neq j, j+1} (M_k - M_i)}{\prod_{i \neq k} (M_k - M_i)}. \quad (4.15)\]

The sum appearing in the numerator contains at most two terms. More importantly, the unknowns \(\{l_k\}_0\) and \(\{m_1\}_0\) do not enter in the above expression in a symmetric manner. This feature will become clearer in the next section.

5. Information content of a single dispersion branch

In the previous paragraph we saw that we can retrieve \(\{M_i\}_0\) and \(\{Q_i\}_0\) from an investigation of the asymptotic behaviour of \(\sigma_k(p)\), \(k = 1, 2, \ldots, N\). This result is the discrete analogue of that obtained by Kac and Gerver & Kazdan. In the present section we shall rederive this same result from a different starting point, namely we shall assume that the data at hand consists of a single branch of the dispersion curve, say \(s = \sigma_1(p)\). In particular, we shall assume that \(\sigma_1(p)\) is known explicitly. Furthermore, we shall examine whether the permutation matrix \(P\) can also be retrieved.

Since \(s = \sigma_1(p)\) is the branch of an algebraic curve, it is possible to free this relationship from radicals and obtain a polynomial in \(p\) and \(s\). We shall say that the branch \(\sigma_1(p)\) is not degenerate if the polynomial thus obtained is of degree \(N\), i.e. is in fact the characteristic polynomial \(D(s, p) = 0\). For the non-degenerate case, knowing \(\sigma_1(p)\) is equivalent to knowing \(D(s, p)\), and the coefficients \(C_j^{(k)}\) entering in the representation (3.14). In particular, from the terms of degree \(N\), we deduce that
\[
\begin{align*}
m_1 + m_2 + \ldots + m_N &= C_1^{(N)} ,
m_1 m_2 + m_1 m_3 + \ldots + m_{N-1} m_N &= C_1^{(N)} ,
\vdots
m_1 m_2 \ldots m_N &= C_N^{(N)}.
\end{align*}
\quad (5.1)
\]

* Degeneracy seems to be related to the case in which two or more \(m_i\)'s are equal.
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Just as previously we can solve for the vector \( \mathbf{M} \) which is related to \( \mathbf{m} \) as in (4.5).

By considering the terms of degree \( N-1 \), we see that

\[
\begin{align*}
q_1 + q_2 + \ldots + q_N &= C_1^{(N-1)} \\
(m_2 + m_3 + \ldots + m_N)q_1 + \ldots + (m_1 + m_2 + \ldots + m_{N-1})q_N &= C_2^{(N-1)} \\
& \quad \vdots \\
m_2 m_3 \ldots m_N q_1 + \ldots + m_1 m_2 \ldots m_{N-1} q_N &= C_N^{(N-1)}.
\end{align*}
\]  

(5.2)

If we define \( \mathbf{Q} \) thus:

\[ \mathbf{q} = \mathbf{PQ}, \]

(5.3)

i.e. as in (4.11), then we can rewrite (5.2) as follows:

\[
\begin{align*}
Q_1 + Q_2 + \ldots + Q_N &= C_1^{(N-1)} \\
(M_2 + M_3 + \ldots + M_N)Q_1 + \ldots + (M_1 + M_2 + \ldots + M_{N-1})Q_N &= C_2^{(N-1)} \\
M_2 M_3 \ldots M_N Q_1 + \ldots + M_1 M_2 \ldots M_{N-1} Q_N &= C_N^{(N-1)}
\end{align*}
\]  

(5.4)

Knowing \( \mathbf{M} \), we can therefore solve for \( \mathbf{Q} \).

We now turn our attention to the terms of degree \( N-2 \), and note that \( d_{N-2}(s, p) \) can be expressed partly in terms of the known quantities \( \{M_i\} \) and \( \{Q_i\} \), namely

\[
d_{N-2}(s, p) = -\sum_{j=1}^{N-1} \frac{1}{l_j^2} \prod_{k=j+1}^{N} (s - m_k) + \frac{1}{2} \sum_{j=1}^{N-1} \sum_{k\neq j} Q_j Q_k \prod_{k\neq j} (s - M_k).
\]

(5.5)

By considering the coefficients of the various terms \( s^p N^{-2-n} \), we can derive \( N-1 \) linear equations for \( l_i^{-2}, i = 1, 2, \ldots, N-1 \), namely

\[
\begin{align*}
\frac{1}{l_1^2} + \frac{1}{l_2^2} + \ldots + \frac{1}{l_{N-1}^2} &= \Gamma_1, \\
(m_3 + m_4 + \ldots + m_N) \frac{1}{l_1^2} \\
+ (m_1 + m_4 + \ldots + m_N) \frac{1}{l_2^2} + \ldots + (m_1 + m_2 + \ldots + m_{N-2}) \frac{1}{l_{N-1}^2} &= \Gamma_2 \\
m_3 m_4 \ldots m_N \frac{1}{l_1^2} + m_1 m_4 \ldots m_N \frac{1}{l_2^2} + \ldots + m_1 m_2 \ldots m_{N-2} \frac{1}{l_{N-1}^2} &= \Gamma_{N-1}
\end{align*}
\]

(5.6)

The coefficients \( \Gamma_i \) which are related to \( C_i^{(N-2)} \) as well as to \( C_j^{(N)} \) and \( C_j^{(N-1)} \) are known. This set of equation is not invariant with respect to the permutation matrices \( \mathbf{P} \). Therefore, it could be used to reduce the manifold of potential solutions arising from the group of the \( NxN \) permutation matrices. Indeed, at least in principle, we could solve (5.6) for each one of the permutation matrix in that group. In order for a particular matrix to be acceptable, the corresponding solution of (5.6) should be such that

(i) \( l_i^2 > 0 \) for \( i = 1, 2, \ldots, N-1 \);

(ii) \( \frac{1}{l_i} + \frac{1}{l_{i+1}}, (i = -1, 2, \ldots, N-2) \) be compatible with the appropriate values of \( \{Q_j\} \).

Let us illustrate this procedure by means of the simplest of the \( N = 4 \) cases. The set of equations corresponding to (5.6) becomes:
We shall assume that the polynomial
\[ R(z) = \Gamma_1 z^2 - \Gamma_2 z + \Gamma_3 \]  
has no real zeros. Finally, let us assume that the vectors \( \mathbf{M} \) and \( \mathbf{Q} \) have already been determined. There are \( 4! = 24 \) permutation matrices. However, we have agreed to count only once those pairs of matrices \( \mathbf{P}, \mathbf{P}' \) which are 'per-symmetric', i.e. such that
\[ (\mathbf{P}_{ij}) = (\mathbf{P}_{N-j, N-i}). \]

As a result we only need to consider 12 permutation matrices. Following the convention of placing the heavier rib at the bottom, we use the following notations to label the arrangements corresponding to the various permutations:

\[
\begin{align*}
P_1 &= \{M_1, M_2, M_3, M_4\} & P_7 &= \{M_2, M_1, M_3, M_4\} \\
P_2 &= \{M_1, M_2, M_4, M_3\} & P_8 &= \{M_2, M_1, M_4, M_3\} \\
P_3 &= \{M_1, M_3, M_2, M_4\} & P_9 &= \{M_2, M_3, M_1, M_4\} \\
P_4 &= \{M_1, M_3, M_4, M_2\} & P_{10} &= \{M_2, M_4, M_1, M_3\} \\
P_5 &= \{M_1, M_4, M_2, M_3\} & P_{11} &= \{M_3, M_1, M_2, M_4\} \\
P_6 &= \{M_1, M_4, M_3, M_2\} & P_{12} &= \{M_3, M_2, M_1, M_4\}.
\end{align*}
\]

From (5.7) we can obtain a formula for \( l_1^{-2} \) and for \( l_3^{-2} \):
\[
\begin{align*}
l_1^{-2} &= \frac{R(m_1)}{(m_1-m_3)(m_1-m_4)} \quad (5.9) \\
l_3^{-2} &= \frac{R(m_4)}{(m_4-m_2)(m_4-m_1)}. \quad (5.10)
\end{align*}
\]

More generally
\[
\begin{align*}
l_1^{-2} &= \frac{R(m_1)}{\prod_{k=2}^{N-1} (m_4-m_1+k)} \quad (5.11) \\
l_2^{-2} &= \frac{R(m_2)}{\prod_{k=2}^{N-1} (m_4-m_2+k)} \\
l_3^{-2} &= \frac{R(m_3)}{\prod_{k=2}^{N-1} (m_4-m_3+k)} \\
l_4^{-2} &= \frac{R(m_4)}{\prod_{k=2}^{N-1} (m_4-m_4+k)}
\end{align*}
\]

if we redefine \( R(z) \) thus:

* That such a case does arise can be seen by considering the example \( m_1 = 3, m_2 = 4, m_3 = 1 \), \( m_4 = 2; l_0 = 1/4, l_1 = 1/3, l_2 = 1, l_3 = 1/2, l_4 = 1 \) for which \( R(z) = 14z^2 - 60z + 72 \).
Now, since $R(m_l)$ is positive, it follows from (5.9) and (5.10) that

\[(m_4 - m_3)(m_1 - m_4) > 0, \quad (5.13a)\]
\[(m_4 - m_2)(m_4 - m_1) > 0. \quad (5.13b)\]

But, on account of the adopted convention

\[m_1 > m_4. \quad (5.14)\]

Therefore (5.13) implies that

\[m_1 > m_3, \quad (5.15a)\]
\[m_2 > m_4. \quad (5.15b)\]

(5.15a) eliminates the arrangements $P_{12}$, $P_{11}$, $P_{10}$ and $P_{9}$ while (5.15b) eliminates $P_4$, $P_5$ and $P_6$. The remaining five arrangements are guaranteed to yield positive values for $l^2_1$ and $l^2_3$ but not necessarily for $l^2_2$. In particular, $P_3$ must be eliminated on the grounds that it gives rise to a negative $l^2_2$. Indeed for the arrangement $P_3$,

\[I_2^{-2} = \frac{\mathcal{N}}{(M_1 - M_2)(M_3 - M_4)(M_1 - M_4)} \quad (5.16)\]

where

\[\mathcal{N} = -\Gamma_1 C + \Gamma_2 B - \Gamma_3 A \quad (5.18)\]

and

\[
\begin{aligned}
A &= M_1 + M_3 - M_2 - M_4, \\
B &= M_1 M_3 - M_2 M_4, \\
C &= M_1 M_3 (M_2 + M_4) - M_2 M_4 (M_1 + M_3).
\end{aligned}
\quad (5.19)\]

Since $M_1 > M_2 > M_3 > M_4$, it follows that $\mathcal{N}$ determines the sign of $l^2_2$. Furthermore

\[A, B, C > 0. \quad (5.20)\]

The fact that $R(z)$ has no real zeros implies that

\[\Gamma_2 < 2\Gamma_1 \Gamma_3^\dagger \quad (5.21)\]

and hence

\[\mathcal{N} < -\Gamma_1 C + 2\Gamma_1 \Gamma_3^\dagger B - \Gamma_3 A\]
\[= -\left((\Gamma_1 \dagger C^\dagger - \Gamma_3 \dagger A^\dagger)^2 - 2\Gamma_1 \dagger \Gamma_3^\dagger (\sqrt{(AC)} - B)\right) \quad (5.22)\]

It is an easy matter to check that

\[AC - B^2 = (M_1 - M_2)(M_1 - M_4)(M_2 - M_3)(M_3 - M_4) > 0,\]

i.e.,

\[\sqrt{(AC)} > B^2. \quad (5.23)\]

Thus the arrangement $P_3$ must be eliminated.

Therefore we are left with four possible arrangements, namely $P_1$, $P_2$, $P_7$ and $P_8$. Throughout this analysis we have assumed the existence of at least one solution to our inverse problem. Let us say that $P_8$ is that solution. The question then is: Can $P_1$, $P_2$ or $P_7$ be also solutions? In general, the answer is no.

Let us first prove that $P_2$ and $P_8$ are two solutions if and only if

\[
\begin{aligned}
M_1 &= M_2, \\
Q_1 &= Q_2.
\end{aligned}
\quad (5.23)\]
That the above conditions are sufficient is obvious. Let us therefore concentrate on their necessity. Since $P_8$ is a solution (5.7) becomes:

$$
\begin{align*}
\frac{1}{l_1^2} + \frac{1}{l_2^2} + \frac{1}{l_3^2} &= \Gamma_1 \\
\frac{M_3 + M_4}{l_1^2} + \frac{M_2 + M_3}{l_2^2} + \frac{M_1 + M_2}{l_3^2} &= \Gamma_2 \\
\frac{M_3 M_4}{l_1^2} + \frac{M_2 M_3}{l_2^2} + \frac{M_1 M_2}{l_3^2} &= \Gamma_3
\end{align*}
$$

(5.24a)

The solutions to (5.24) must in addition satisfy the compatibility conditions:

$$
\begin{align*}
\frac{1}{l_1} + \frac{1}{l_2} &= Q_1, \\
\frac{1}{l_2} + \frac{1}{l_3} &= Q_4.
\end{align*}
$$

(5.25a)

Similarly, $P_2$ which is a solution associated with the width-distribution $\{l_i^*\}_0^N$, must satisfy the following relations:

$$
\begin{align*}
\frac{1}{l_1^*} + \frac{1}{l_2^*} + \frac{1}{l_3^*} &= \Gamma_1 \\
\frac{M_3 + M_4}{l_1^*^2} + \frac{M_2 + M_3}{l_2^*^2} + \frac{M_1 + M_2}{l_3^*^2} &= \Gamma_2 \\
\frac{M_3 M_4}{l_1^*^2} + \frac{M_2 M_3}{l_2^*^2} + \frac{M_1 M_2}{l_3^*^2} &= \Gamma_3
\end{align*}
$$

(5.24b)

and

$$
\begin{align*}
\frac{1}{l_1^*} + \frac{1}{l_2^*} &= Q_2, \\
\frac{1}{l_2^*} + \frac{1}{l_3^*} &= Q_4.
\end{align*}
$$

(5.25b)

From (5.24) we can deduce that

$$
\begin{align*}
\frac{1}{l_1^*^2} &= \frac{1}{l_1^2} + \frac{M_1 - M_2}{M_1 - M_4} \frac{1}{l_2^2}, \\
\frac{1}{l_2^*^2} &= \frac{M_2 - M_4}{M_1 - M_4} \frac{1}{l_2^2}, \\
\frac{1}{l_3^*^2} &= \frac{1}{l_3^2},
\end{align*}
$$

(5.26)
whereas from (5.25) we see that

\[
\frac{1}{l_2} + \frac{1}{l_3} = Q_4 = \frac{1}{l_2'} + \frac{1}{l_3'}.
\]  

(5.27)

Thus

\[
l_2 = l_2',
\]  

(5.28)

which together with (5.26) and (5.25) implies (5.23). As a result \(P_2\) and \(P_8\) are both solutions only if the corresponding arrangements are identical. Note that on account of the restriction (4.6) which we have adopted, \(P_2\) has to be eliminated.

By similar reasoning we can also eliminate the possibility that both \(P_7\) and \(P_8\) are solutions.

The case for \(P_1\) and \(P_8\) is resolved slightly differently. If we denote once again by primes the widths \(\{l_i\}_0^4\) associated with the 'second' solution, namely \(P_1\), then from (5.7)

\[
\begin{aligned}
\frac{1}{l_1'^2} &= \frac{1}{l_1^2} + \frac{M_1-M_2}{M_1-M_4 l_2^2}, \\
\frac{1}{l_2'^2} &= \frac{M_2-M_3}{M_1-M_4 l_2^2}, \\
\frac{1}{l_3'^2} &= \frac{1}{l_3^2} + \frac{M_3-M_4}{M_1-M_4 l_2^2}.
\end{aligned}
\]

(5.29)

On the other hand, from the constraints (5.3), namely

\[
\frac{1}{l_0} = Q_2 - \frac{1}{l_2'}, \\
\frac{1}{l_1'} = Q_3 - \frac{1}{l_2'}
\]

we can deduce that:

\[
\begin{aligned}
\frac{1}{l_1'} &= \frac{1}{l_0} + \frac{1}{l_1} - \sqrt{\left(\frac{M_2-M_3}{M_1-M_4}\right) \frac{1}{l_2}}, \\
\frac{1}{l_3'} &= \frac{1}{l_3} + \frac{1}{l_4} - \sqrt{\left(\frac{M_2-M_3}{M_1-M_4}\right) \frac{1}{l_2}}.
\end{aligned}
\]

(5.30)

The values of \(l_1'\) and \(l_3'\) computed from (5.29) are, in general,\(^*\) not compatible with those computed from (5.30). As a result, we must rule out the possibility that \(P_1\) is a second solution.

6. Concluding remarks

The tedious method of enumeration and elimination used in proving the uniqueness of the solution to the inverse problem for the case \(N = 4\) does not lend itself to generaliza-

\(^*\) The term 'in general' is used here as it is used in algebraic geometry (see e.g. Collidge 1931, p. 8)
tion. Nevertheless, this result strengthens the conjecture that the inverse two-
dimensional problem which we have considered has, in general, a unique solution.

We have also seen that in order to arrive at a unique solution, we must consider
several terms in the asymptotic expansion of the dispersion relation. In the problems
considered by Kac and Gerver & Kazdan, these terms were very difficult to compute.
Their results were based solely on the equivalent of our first two terms.

Finally, the problem we have just considered shares with the geophysical inverse
problem for the earth normal modes, the property that it is invariant under a group of
transformation. The fact that the appropriate group is the translation group rather
than the rotation one is not very significant. As we have seen, because of this invariance
the eigenvalue problem reduces to a seemingly one-dimensional problem. But, as the
case \( N = 4 \) shows, there are enough data to arrive at a unique solution. Thus, this
result also lends credence to the original conjecture of Backus & Gilbert.

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