Models of the Lateral Heterogeneity of the Earth Consistent with Eigenfrequency Splitting Data

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Summary

Let \( \omega_k, k = 1, 2, \ldots, 2l+1 \), denote the \( 2l+1 \) non-degenerate split eigenfrequencies associated with some isolated elastic-gravitational normal mode multiplet of the Earth, and let

\[ \omega_0 = \frac{1}{2l+1} \sum_{k=1}^{2l+1} \omega_k \]

denote the corresponding degenerate eigenfrequency of the spherically averaged and non-rotating terrestrial monopole. The multiplet variance

\[ \Delta \omega^2 = \frac{1}{2l+1} \sum_{k=1}^{2l+1} (\omega_k - \omega_0)^2 \]

is a quantity which, correct to first order, depends only upon the small deviations of the Earth away from the terrestrial monopole, i.e. upon the slow angular rotation, the hydrostatic ellipsoidal shape, and the lateral heterogeneity. If it is assumed that the \( N \) observed Fourier spectral peaks \( \omega_i, i = 1, 2, \ldots, N \), which have been assigned to any multiplet constitute a random sampling with replacement of the \( 2l+1 \) split eigenfrequencies \( \omega_k, k = 1, 2, \ldots, 2l+1 \), then the sample mean

\[ \bar{\omega} = \frac{1}{N} \sum_{i=1}^{N} \omega_i \]

and the sample variance

\[ \Delta \bar{\omega}^2 = \frac{1}{N-1} \sum_{i=1}^{N} (\omega_i - \bar{\omega})^2 \]

provide estimates of \( \omega_0 \) and \( \Delta \omega^2 \), respectively. This paper analyses the recently compiled sample variances for the fundamental toroidal and spheroidal multiplets \( oT_l \) and \( oS_l \) in an attempt to infer certain gross features and properties of the lateral heterogeneity of the Earth. This requires a preliminary correction of the raw sample variance data for the bias introduced by random errors in the observed Fourier spectral peaks \( \omega_i, i = 1, 2, \ldots, N \), as well as an \textit{a priori} estimation of the contribution to the splitting from the rotation and the hydrostatic ellipsoidal shape of the Earth. These adjustments are sufficiently large and uncertain that any formal inversion of fundamental mode multiplet variance data seems unwarranted at this time; instead, direct calculations have been performed

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for a number of hypothetical models of the lateral heterogeneity of the Earth. A comparison of the results of these calculations with observed multiplet variances points strongly to two main conclusions; the first is that there must be significant lateral heterogeneity in the lower mantle, and the second is that oceanic-continental differences must persist to depths of at least several hundred kilometres.

1. Introduction

This paper is a report on some preliminary attempts we have made to use normal mode multiplet eigenfrequency splitting data to infer the magnitude and extent of the lateral heterogeneity of the Earth. The theoretical foundation for this study is contained in a previous publication (Dahlen 1974); we make use here of most of the notation which was introduced there, with a few minor changes and simplifications.

For all but two or three very low frequency fundamental mode multiplets, we possess at the present time only rather indirect evidence concerning the degree of eigenfrequency splitting or fine structure. To cope with this lack of any direct information, we have resorted here to a statistical approach which is a straightforward extension of that suggested by Gilbert (1971), in connection with the estimation of multiplet degenerate eigenfrequencies. For many normal mode multiplets, especially those on the fundamental branch, what we do possess at the present time is a large number of observed Fourier spectral peak locations, each of which presumably lies somewhere within the corresponding band of split eigenfrequencies. Following Gilbert (1971), we shall make the basic assumption that the statistics of these various observations for a given multiplet can be used to deduce certain simple statistical properties of the distribution of split eigenfrequencies within that multiplet. In particular we shall suppose that the variance of the various measured peak locations for a given multiplet about their mean can be used to provide an estimate of the variance of the various split eigenfrequencies of that multiplet about the multiplet degenerate eigenfrequency; the square root of this multiplet variance is, in a familiar sense, essentially a measure of the width due to splitting of a multiplet. In Dahlen (1974) we have discussed specifically the problem of relating the variance, or alternatively the splitting width, of a multiplet to the lateral heterogeneity of the Earth. We here attempt to apply that theory to the recent compilation by Dziewonski & Gilbert (1972) of multiplet variances for the fundamental mode multiplets $oT_i$ and $oS_i$.

When this study was begun the initial intent was to undertake a formal inversion of this data; it has been shown in Dahlen (1974) that the inverse problem for multiplet variance data can be transformed into a strictly linear inverse problem if we are content to invert not for the lateral heterogeneity directly, but rather for its co-variance. A number of difficulties have however conspired to render such a formal inversion rather impractical at the present time. These difficulties will be discussed in some detail during the course of this paper, since they are certain to arise in any future study of this type. One of the problems encountered in the present investigation is the simple fact that the usual statistical uncertainties which corrupt any geophysical data are, as might be expected, in this instance fairly pronounced; this however is a problem which in principle will become reduced as we accumulate more observations. The other difficulties which have arisen here may not be so easily circumvented; these include the need to correct multiplet variance data for a bias due to noise, as well as the need to account for the theoretical contribution to the splitting from the rotation and hydrostatic ellipsoidal shape of the Earth. In light of these difficulties, we have deferred any formal inversion attempt, and have limited this study to a direct computation of the theoretical fundamental mode splitting variances due to a
number of prescribed models of the Earth's lateral heterogeneity. We have furthermore restricted consideration only to models in which the lateral heterogeneity is confined to differences in the structure of the upper mantle beneath oceanic and continental areas. We introduce this as a sort of null hypothesis, subject to test by a comparison of the theoretical eigenfrequency splitting for such models with the data of Dziewonski & Gilbert (1972). We are able to show, with reasonable assurance, that this hypothesis is in fact untenable, primarily because the observed splitting widths of the lower degree fundamental mode multiplets $o_T$ and $o_S$, $4 \leq l \leq 20$, are generally larger, by about a factor of two to three, than any of the corresponding theoretical widths we have computed. We believe that this result is a strong argument in favour of the existence of a relatively profound lateral heterogeneity of the Earth's lower mantle. We also make use of the data for the higher degree multiplets $o_T$ and $o_S$, $20 \leq l \leq 50$, to see if we may draw any conclusions about the depth to which significant differences beneath oceans and continents must persist. To do this, we take advantage of the important new observation, made by Sipkin & Jordan (1975), that the one-way vertical travel time of elastic shear body waves through the oceanic crust and upper mantle is less than that through the continental crust and upper mantle by about 5 s. By comparing the observed multiplet splitting widths of the fundamental mode multiplets $o_T$ and $o_S$, $20 \leq l \leq 50$, with the corresponding theoretical widths for models which are consistent with the datum of Sipkin & Jordan (1975), we are able to show that significant differences in the shear wave velocity structure beneath oceans and continents must persist to at least about 400 km; in fact, the model we obtain which is most consistent with the data of Dziewonski & Gilbert (1972) has differences which persist to 670 km depth. Any conclusions which are drawn from a study of this type are necessarily somewhat tentative, because of the various uncertainties which have been alluded to above. We feel however that both the conclusion that there must be significant deep mantle lateral heterogeneity, as well as the conclusion that oceanic-continental differences must persist to a relatively great depth, are strongly indicated by present-day multiplet splitting width data; the latter conclusion, in contrast to the former, clearly depends critically upon the veracity of the observations of Sipkin & Jordan (1975).

It is of interest to compare the results of this study with the corresponding results obtained in the two previous numerical investigations of normal mode multiplet eigenfrequency splitting due to prescribed models of the lateral heterogeneity of the Earth. The first such study is that of Madariaga & Aki (1972). These authors considered only fundamental toroidal multiplets $o_T$, for which the algebraic complexity of the perturbation theoretical formalism is relatively slight, and they did not include in their calculations the influence of the Earth's rotation or hydrostatic ellipsoidal shape. Furthermore, they did not have available for the purposes of comparison the extensive compilation of fundamental mode multiplet variance data of Dziewonski & Gilbert (1972) which we have used here. In general, the principal interest of Madariaga & Aki (1972) was not in utilizing their calculations to test hypotheses about the lateral heterogeneity of the Earth, but rather in examining the implications of normal mode multiplet splitting for the interpretation of great circular Love and Rayleigh surface wave dispersion data. The second study which has preceded this one is that of Luh (1974). This investigation does treat spheroidal as well as toroidal multiplets, and it also includes the contribution to the splitting of the Earth's rotation and hydrostatic ellipsoidal structure; the extent to which this latter contribution is uncertain due to a lack of knowledge of the details of the spherically averaged structure of the Earth's upper mantle is not however examined, and we show that to be a fairly important consideration here. Luh (1974) does attempt to compare his calculations to the measured multiplet variances compiled by Dziewonski & Gilbert (1972), but only in a rough order of magnitude way. His comparison is in any case somewhat vitiated by a minor error. The factor of two which he arbitrarily employs in order to
compare the observations of Dziewonski & Gilbert (1972) with the theoretically derived total split width of a multiplet has in fact been omitted from his Table 2. When this error is rectified, the conclusions which might be drawn from the results of Luh (1972) are generally consistent with the conclusions reached in the present study, particularly with regard to the indicated presence of lateral heterogeneity in the Earth's deep mantle. Both Madariaga & Aki (1972) and Luh (1974) have performed computations which enable the determination of each of the 2l+1 split eigenfrequencies which correspond to a given multiplet, or $S_l$; each such calculation requires the determination of the eigenvalues of a 2l+1 by 2l+1 complex Hermitian matrix, if the effects of the Earth's rotation are to be included. In the present study, we are interested only in the computation of multiplet splitting variances, since these are the only data we possess at the present time. Dahlen (1974) has shown how multiplet variances can be computed directly without having to solve, for every multiplet, the full perturbation theoretical eigenvalue problem. We have employed this method of direct calculation here, thereby saving a great deal of computational labour.

2. Review of the theory

We shall presume that, for the purpose of discussing its infinitesimal elastic-gravitational free oscillations, the Earth may be completely prescribed in terms of its mass density $\rho_0(r)$, incompressibility $\kappa(r)$, and rigidity $\mu(r)$, as well as its angular velocity $\Omega$ of diurnal rotation about its centre of mass. We shall use the shorthand notation $m(r; \Omega)$ to denote a model of the Earth which is characterized by the ordered triple $(\rho_0(r), \kappa(r), \mu(r))$ and which is uniformly rotating with angular velocity $\Omega$; the position vector $\mathbf{r}$ is here taken to be measured in the uniformly rotating reference frame from an origin $\mathbf{0}$ at the centre of mass. An alternative and equivalent prescription is in terms of the mass density $\rho_0(r)$, the compressional elastic wave speed $v_p(r)$, and the shear elastic wave speed $v_s(r)$, where $v_p^2 = (\kappa + \frac{4}{3} \mu)/\rho_0$ and $v_s^2 = \mu/\rho_0$; in either of these two equivalent prescriptions, we are implicitly ignoring the effects of any sort of anisotropy (Dahlen 1972; Walton 1974).

We shall further suppose that every such Earth model $m(r; \Omega)$ can be unambiguously decomposed into three parts: a non-rotating spherically averaged part $m_0(r)$, a small perturbation $\delta m_{\text{rot}}(r; \Omega; h_j)$ which takes into account the diurnal angular rotation $\Omega$ as well as the hydrostatically supported ellipsoidal deviations of Earth structure which are a consequence of this rotation, and a further perturbation $\delta m_{\text{het}}(r; h_j)$ associated with the additional non-hydrostatic lateral heterogeneity of the Earth. The non-rotating spherically symmetric part $m_0(r)$ can be completely defined by an ordered triple $(\rho_0(r), \kappa(r), \mu(r))$ where $r = |\mathbf{r}|$ is the radius, $0 \leq r \leq a$. Jump discontinuities in any or all of these three functions of radius are allowed at a finite number of spherical shells $a_1 < a_2 < \ldots < a_n \equiv a$. Both the hydrostatic ellipsoidal and the additional lateral heterogeneity structure perturbations $\delta m_{\text{rot}}(r; \Omega; h_j)$ and $\delta m_{\text{het}}(r; h_j)$ are then specified in terms of the volume perturbations $\{\delta \rho_0(r), \delta \kappa(r), \delta \mu(r)\}$ within the spherical volume $0 \leq r \leq a$, as well as the perturbations $h_1(\mathbf{r}), h_2(\mathbf{r}), \ldots, h_n(\mathbf{r})$ in the locations of each of the discontinuity surfaces; $\mathbf{r} = |\mathbf{r}|$ is the unit radial vector. Unlike the lateral heterogeneity $\delta m_{\text{het}}(r; h_j)$, the hydrostatic ellipsoidal structure perturbation $\delta m_{\text{rot}}(r; \Omega; h_j)$ is not independent of the spherically symmetric part $m_0(r)$; in fact $\delta m_{\text{rot}}(r; \Omega; h_j)$ may be determined completely in terms of $m_0(r)$ and $\Omega$ by making use of the hydrostatic figure theory of Clairaut. Let $r, \theta, \phi$ be spherical polar co-ordinates centred on $\mathbf{0}$ and with the polar axis aligned along $\Omega$, and let $e(r)$ denote the ellipticity of the ellipsoidal surfaces of constant density which result if the spherically symmetric Earth model $m_0(r)$ is allowed to rotate uniformly with angular velocity $\Omega$ while required to remain in hydrostatic equilibrium. As long as the scale lengths of any continuous changes in the spherically
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symmetric functions \( \rho_0(r), \kappa(r), \mu(r) \) are large compared to \( r \varepsilon(r) \), \( \delta m_{\text{ell}}(r; \Omega; h_j) \) may be given simply in terms of \( \Omega, \varepsilon(r) \), and the radial derivatives of \( m_0(r) \), namely

\[
\begin{align*}
\delta \rho_0(r, \theta) &= \frac{2}{3} r \varepsilon(r) \mu'(r) P_2(\cos \theta) \\
\delta \kappa(r, \theta) &= \frac{2}{3} r \varepsilon(r) \kappa'(r) P_2(\cos \theta) \\
\delta \mu(r, \theta) &= \frac{2}{3} r \varepsilon(r) \mu'(r) P_2(\cos \theta) \\
h_j(\theta) &= -a_j \varepsilon(a_j) P_2(\cos \theta), \quad j = 1, 2, \ldots, n,
\end{align*}
\]

(1)

where \( P_2(\cos \theta) \) is the Legendre polynomial \( P_2(\cos \theta) = \frac{1}{2}(3 \cos^2 \theta - 1) \). It has here been tacitly assumed that the elastic moduli \( \kappa(r) + \delta \kappa(r, \theta) \) and \( \mu(r) + \delta \mu(r, \theta) \) of the rotating, hydrostatic ellipsoidal Earth model \( m_0(r) + \delta m_{\text{ell}}(r; \Omega; h_j) \) are constant on the same ellipsoidal surfaces as the density \( \rho_0(r) + \delta \rho_0(r, \theta) \). Symbolically, we can write equations (1) in the shorthand notation

\[
\delta m_{\text{ell}}(r; \Omega; h_j) = \frac{2}{3} r \varepsilon(r) m_0'(r) P_2(\cos \theta)
\]

(2)

to indicate that the hydrostatic ellipsoidal structure perturbation depends essentially on the radial gradients \( m_0'(r) \) of the spherically averaged model; the dependence is in fact approximately linear, since the Radau approximation (Jeffreys 1970) demonstrates that the dependence of \( \varepsilon(r) \) upon the details of \( m_0(r) \) is weak. It is convenient to express the additional lateral heterogeneity structure perturbation \( \delta m_{\text{het}}(r; h_j) \) as an expansion in terms of surface spherical harmonics, similar to (1); we make use of the fully normalized complex surface spherical harmonics \( Y_l^m(\theta, \phi) \) defined, e.g. by Edmonds (1957);

\[
\begin{align*}
\delta \rho_0(r, \theta, \phi) &= \sum_{s=1}^{\infty} \sum_{t=-s}^{s} \delta \rho_s'(r) Y_s^t(\theta, \phi) \\
\delta \kappa(r, \theta, \phi) &= \sum_{s=1}^{\infty} \sum_{t=-s}^{s} \delta \kappa_s'(r) Y_s^t(\theta, \phi) \\
\delta \mu(r, \theta, \phi) &= \sum_{s=1}^{\infty} \sum_{t=-s}^{s} \delta \mu_s'(r) Y_s^t(\theta, \phi) \\
h_j(\theta, \phi) &= \sum_{s=1}^{\infty} \sum_{t=-s}^{s} h_s^j Y_s^t(\theta, \phi), \quad j = 1, 2, \ldots, n.
\end{align*}
\]

(3)

Each of the sums in equations (3) begins at degree \( s = 1 \), since \( m_0(r) \) has been defined by means of an averaging over spherical shells of the properties of the rotating, ellipsoidal, laterally heterogeneous Earth model \( m(r; \Omega) \). Symbolically, we write

\[
\delta m_{\text{het}}(r; h_j) = \sum_{s=1}^{\infty} \sum_{t=-s}^{s} \delta m_s'(r) Y_s^t(\theta, \phi)
\]

(4)

and

\[
m(r; \Omega) = m_0(r) + \delta m_{\text{ell}}(r; \Omega; h_j) + \delta m_{\text{het}}(r; h_j)
\]

(5)

where

\[
m_0(r) = \int_{\Omega} m(r; \Omega) dA,
\]

(6)

\( \Omega \), being the sphere of radius \( r \) centred on \( 0 \). The method of treating discontinuity surfaces in the Earth model \( m(r; \Omega) \) in making the decomposition (5) is not only in most instances fairly natural (e.g. the major differences between oceanic and continental crustal structure are the differences in the elevation of the outer surface and in the
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depth to the Mohorovicic discontinuity), but it also leads to a very convenient perturbation theoretical formalism for the normal mode eigenfrequencies and eigenfunctions.

Consider now an isolated primarily toroidal or primarily spheroidal multiplet \(_{n} T_{l} \) or \(_{n} S_{l} \) of the Earth model \( m(r; \Omega) \). Let \( \omega_{k}, k = 1, 2, \ldots, 2l + 1, \) denote the \( 2l + 1 \) non-degenerate split eigenfrequencies associated with this multiplet, and let \( \omega_{0} \) denote the corresponding degenerate eigenfrequency of the non-rotating, spherically averaged model \( m_{0}(r) \). If the eigenfrequency splitting perturbations \( \delta \omega_{k} = \omega_{k} - \omega_{0}, k = 1, 2, \ldots, 2l + 1, \) are small, they may be computed in terms of the Earth model perturbations \( \delta m_{\text{ell}}(r; \Omega, h_{j}) \) and \( \delta m_{\text{het}}(r; h_{j}) \) by making use of degenerate perturbation theory; the results we quote and make use of here are only accurate to first order in \( \delta m_{\text{ell}}(r; \Omega, h_{j}) \) and \( \delta m_{\text{het}}(r; h_{j}) \). An important result is the diagonal sum rule (Gilbert 1971), which states that

\[
\omega_{0} = \frac{1}{2l + 1} \sum_{k=1}^{2l+1} \omega_{k};
\]

i.e. the mean of the split eigenfrequencies \( \omega_{k} \) associated with any isolated multiplet of a rotating, ellipsoidal, and laterally heterogeneous Earth model \( m(r; \Omega) \) is precisely the corresponding degenerate eigenfrequency \( \omega_{0} \) of its non-rotating spherically symmetric part \( m_{0}(r) \). This result is an immediate consequence of equation (6). We will be concerned not with multiplet mean frequencies \( \omega_{0} \), but rather with multiplet splitting variances \( \Delta \omega^{2} \), defined by

\[
\Delta \omega^{2} = \frac{1}{2l + 1} \sum_{k=1}^{2l+1} (\omega_{k} - \omega_{0})^{2}.
\]

Let \( \beta \) and \( \alpha \) be, respectively, the rotational and elliptical multiplet splitting parameters defined by Backus & Gilbert (1961) and Dahlen (1968, 1974); the latter is a linear functional of the hydrostatic ellipsoidal structure perturbation \( \delta m_{\text{ell}}(r; \Omega, h_{j}) \). Let

\[
C_{s}(r_{1}, r_{2}) = \sum_{s=2}^{2l+1} \delta m_{s}(r_{1}) \delta m_{s}^{*}(r_{2})
\]

denote the \( s \)th degree variance of the lateral heterogeneity structure perturbation \( \delta m_{\text{het}}(r; h_{j}) \). Dahlen (1974) has shown that the multiplet splitting variance \( \Delta \omega^{2} \) of an isolated multiplet \(_{n} T_{l} \) or \(_{n} S_{l} \) may be written in terms of \( \beta, \alpha, \) and the even degree variances \( C_{s}(r_{1}, r_{2}) \), \( s = 2, 4, \ldots, 2l, \) in the form

\[
\Delta \omega^{2} = \frac{1}{4} l(l+1) \beta^{2} \Omega^{2} + \frac{1}{2} \frac{(2l+3)(2l-1)}{l(l+1)} \alpha^{2} \varepsilon_{n}^{2} \omega_{0}^{2} + \frac{2l+1}{4\pi} \sum_{s=2}^{2l} \left( \begin{array}{c} l \cr 0 \end{array} \right) \left( \begin{array}{c} s \cr 0 \end{array} \right) ^{2} \frac{1}{r_{1}^{2} r_{2}^{2}} \int_{0}^{a} \int_{0}^{a} r_{1}^{2} d r_{1} d r_{2}
\]

\[
\times \mathcal{M}_{s}(r_{1}), \mathcal{M}_{s}(r_{2}) C_{s}(r_{1}, r_{2}),
\]

where \( \varepsilon_{n} = \varepsilon(a) \), and where \( \left( \begin{array}{c} l \cr 0 \end{array} \right) \left( \begin{array}{c} s \cr 0 \end{array} \right) ^{2} \) is the Wigner 3-\( j \) symbol (Edmonds 1957). The kernel functions \( \mathcal{M}_{s}(r) \) which appear in (10) are given explicitly by Dahlen (1974) as well as by Luh (1974) and will not be repeated here.
In this paper, we investigate only that lateral heterogeneity which is associated with differences in the structure beneath oceanic and continental areas. Suppose that the structure beneath oceanic areas deviates from that of the spherically averaged Earth model \( m_0(r) \) by amounts \( \delta \rho_{\text{ocean}}(r), \delta \kappa_{\text{ocean}}(r), \delta \mu_{\text{ocean}}(r), 0 \leq r \leq a, \) and \( h^\text{ocean}, j = 1, 2, \ldots, n \), and that the structure beneath continents deviates from that of \( m_0(r) \) by amounts \( \delta \rho_{\text{cont}}(r), \delta \kappa_{\text{cont}}(r), \delta \mu_{\text{cont}}(r), 0 \leq r \leq a, \) and \( h^\text{cont}, j = 1, 2, \ldots, n \). We will use the shorthand notation \( A_{\text{mo}}(r) \) to denote the differences between the structure underlying oceans and that underlying continents; these oceanic-continental differences are defined by

\[
A_{\text{mo}}(r) = \delta \rho_{\text{ocean}}(r) - \delta \rho_{\text{cont}}(r), \quad A_{\text{mc}}(r) = \delta \kappa_{\text{ocean}}(r) - \delta \kappa_{\text{cont}}(r), \quad A_{\text{mu}}(r) = \delta \mu_{\text{ocean}}(r) - \delta \mu_{\text{cont}}(r), \quad 0 \leq r \leq a,
\]

and \( \Delta h^j, j = 1, 2, \ldots, n \).

Let \( C(\theta, \phi) \) denote the oceanic projection function, namely \( C(\theta, \phi) = 1 \) where there are oceans and \( C(\theta, \phi) = 0 \) where there are continents; we may express \( C(\theta, \phi) \) in a surface spherical harmonic expansion,

\[
C(\theta, \phi) = \sum_{s=0}^{\infty} \sum_{r=-s}^{s} c_s^r Y_s^r(\theta, \phi).
\]  

(11)

It is now easy to show that in the special case where the only lateral heterogeneity is associated with oceanic-continental differences, equation (10) reduces to

\[
\Delta \omega^2 = \frac{1}{4} l(l+1) \beta^2 \Omega^2
\]

\[
+ \frac{1}{2} \frac{(2l+3)(2l-1)}{l(l+1)} \tau^2 e_h^2 \omega_0^2
\]

\[
+ \frac{2l+1}{4\pi} \sum_{s=2 \text{ even}}^{2l} \left( \begin{array}{ccc} l & l & s \\ 0 & 0 & 0 \end{array} \right)^2 \sigma_s^2 \left[ \int_0^a r^2 dr \Delta m(r) \mathcal{M}_s(r) \right]^2,
\]  

(12)

where \( \tau^2, s = 2, 4, \ldots, 2l, \) are the even degree variances of the oceanic projection function \( C(\theta, \phi) \), i.e.

\[
\sigma_s^2 = \sum_{i=-s}^{s} c_s^r c_s^r.
\]  

(13)

Equation (12) is the principal result upon which all numerical investigation in this paper will be based. Actually, in order to facilitate comparison between the multiplet variances of different multiplets, it is convenient to consider for each multiplet not \( \Delta \omega^2 \) directly, but rather the normalized, dimensionless quantity \( \Delta \omega/\omega_0 \), where \( \Delta \omega = (\Delta \omega^2)^{1/2} \); we shall, for brevity refer to \( \Delta \omega/\omega_0 \) simply as the width of an isolated split multiplet \( _nT_i \) or \( _nS_i \).

3. Fundamental mode data

At the present time, there has been very little progress made upon the general problem of resolving the split eigenfrequencies \( \omega_i, k = 1, 2, \ldots, 2l+1, \) of the normal mode multiplets of the Earth; only a few of the very lowest frequency multiplets have been even partially resolved. In the absence of any direct measurements of eigenfrequency splitting, we are obliged to adopt a statistical approach to the estimation of multiplet mean frequencies \( \omega_0 \) and multiplet variances \( \Delta \omega^2 \) or widths \( \Delta \omega/\omega_0 \). Suppose that we possess \( N \) observed Fourier amplitude or energy spectra, \( |F_i(\omega)| \) or \( |F_i(\omega)|^2 \), \( i = 1, 2, \ldots, N \), corresponding to some particular unresolved multiplet. Each of these will exhibit a peak which, in the absence of any seismic noise, will be located somewhere within the band of frequencies spanned by the \( 2l+1 \) split eigenfrequencies; we denote these \( N \) observed Fourier spectral peaks by \( \omega_i, i = 1, 2, \ldots, N \). The basis of the statistical approach is the fundamental assumption that the \( N \) observed Fourier
spectral peaks \( \omega_i, i = 1, 2, \ldots, N \) corresponding to any multiplet constitute a random and uniform sampling of the overall distribution within the multiplet of the \( 2I + 1 \) split eigenfrequencies \( \omega_k, k = 1, 2, \ldots, 2I + 1 \). If this assumption is correct, then the sample mean

\[
\bar{\omega}_0 = \frac{1}{N} \sum_{i=1}^{N} \omega_i
\]

and the sample variance

\[
\Delta\omega^2 = \frac{1}{N-1} \sum_{i=1}^{N} (\omega_i - \bar{\omega}_0)^2
\]

provide estimates of the true multiplet mean \( \omega_0 \) and the true multiplet variance \( \Delta\omega^2 \), respectively. The sample mean \( \bar{\omega}_0 \) of the various observed Fourier spectral peaks corresponding to a given multiplet has been generally employed as the best available estimate of the degenerate eigenfrequency \( \omega_0 \) of that multiplet, even before Gilbert (1971) showed that the diagonal sum rule (7) provides a justification for its use. We shall make use here of the multiplet sample width, defined by \( \Delta\omega/\bar{\omega}_0 \) where \( \Delta\omega = (\Delta\omega^2)^{1/2} \), as a simple measure of the general magnitude of eigenfrequency splitting induced by the Earth’s rotation, hydrostatic ellipsoidal structure, and lateral heterogeneity.

The most recent and complete compilation of observed multiplet sample means \( \bar{\omega}_0 \) and sample variances \( \Delta\omega^2 \) is that of Dziewonski & Gilbert (1972). We shall only consider here the fundamental normal mode multiplets \( oT \) and \( oS \). The statistical approach necessarily yields an estimate of the multiplet width \( \Delta\omega/\bar{\omega}_0 \) which will be considerably less certain than the corresponding estimate of the mean or degenerate eigenfrequency \( \omega_0 \); in general, only for the fundamental mode multiplets do we possess a sufficiently large number of observations to make a statistical approach to the estimation of the multiplet splitting width viable at this time. Fig. 1 shows the raw multiplet sample widths \( \Delta\omega/\bar{\omega}_0 \) for fundamental toroidal multiplets \( oT \), \( 4 \leq l \leq 46 \), and for fundamental spheroidal multiplets \( oS \), \( 4 \leq l \leq 50 \), plotted against the degree \( l \); these have been derived from the ‘all data’ sample means and variances listed in Tables 2 and 3 of Dziewonski & Gilbert (1972).

Two independent procedures may be used to estimate the uncertainty which should be assigned to each of the multiplet width estimates \( \Delta\omega/\bar{\omega}_0 \) shown in Fig. 1. First, we can estimate the formal uncertainty of each estimate \( \Delta\omega/\bar{\omega}_0 \) separately. Histograms of the observed Fourier spectral peaks \( \omega_i, i = 1, 2, \ldots, N \), associated with each of the fundamental normal mode multiplets appear to be generally consistent with the hypothesis that the density distribution of split eigenfrequencies within a multiplet may be reasonably well approximated by a normal distribution. It is therefore not unreasonable to postulate that each of the sample variances \( \Delta\omega^2 \) has a probability distribution which may be approximated by a \( \chi^2 \) distribution with \( N-1 \) degrees of freedom. We show in Fig. 1 for each multiplet, as well as the point estimate \( \Delta\omega/\bar{\omega}_0 \) of splitting width defined by equations (14) and (15), a 95 per cent confidence interval estimate which has been computed under the above assumption. Each of these 95 per cent confidence intervals gives, in principle, an indication of the magnitude of the uncertainty we must expect simply because we do not possess an infinitely large sample of observed Fourier spectral peaks. This formal uncertainty is considerably greater for toroidal multiplets than for spheroidal multiplets. We possess fewer toroidal observations than spheroidal observations, mainly because toroidal motion can only be sensed by horizontal seismic instruments; the average number \( N \) of observed Fourier spectral peaks for spheroidal multiplets \( oS \), \( 4 \leq l \leq 50 \), is 75, whereas the corresponding number \( N \) for toroidal multiplets \( oT \), \( 4 \leq l \leq 46 \), is only 14.
Models of the lateral heterogeneity of the Earth

Second, we may obtain a rough but independent idea of the uncertainty by examining the amount of scatter in Fig. 1 between the point estimates $\Delta \omega/\omega_0$ of multiplet splitting width for nearby degrees $l$. In general, it can be seen that this scatter is roughly consistent with the formal uncertainty of each point estimate. We take this as an indication that the scatter in Fig. 1, which is especially severe for the more sparsely sampled toroidal multiplets, is for the most part a result of inadequate sampling; further sampling in the future should improve the quality of multiplet splitting width data.

There are a few specific multiplets which appear to exhibit excessive scatter, notably $0T_{11}, 0T_{19}, 0T_{39}, 0T_{45}, 0T_{46}$, and $0S_{40}, 0S_{41}, 0S_{47}$; also the group $0S_{14}-0S_{16}$ appears to be somewhat inconsistent with the group $0S_{17}-0S_{22}$. We attribute these occasional discrepancies to interference between adjacent multiplets. We will not attempt to deal in any way with these occasional data which appear to be biased as a result of such interference. The theoretical development which has been outlined in Section 2 is only appropriate for those normal mode multiplets which are reasonably well isolated in the eigenfrequency spectrum; this of course precludes its application to any multiplet which appears to have been interfered with by one which is adjacent. We shall continue to plot these discrepant and presumably biased data in the diagrams to follow, but we shall not insist that any theoretical calculations be consistent with them. We note, in passing, that if the splitting width estimate $\Delta \omega/\omega_0$ of any multiplet has been biased by interference, it is likely that the corresponding degenerate eigenfrequency estimate $\omega_0$ has been as well; the assigned degenerate eigenfrequencies of those multiplets which appear to exhibit undue scatter in Fig. 1 should perhaps be scrutinized for any such effect.
As well as the occasional bias of a few data due to interference, there is also the possibility of a systematic bias which could affect all the data in Fig. 1; there are at least two possible sources of such a systematic bias. The first of these is non-uniform sampling. The statistical approach will only provide unbiased estimates of either \( \omega_0 \) or \( \Delta \omega/\omega_0 \) if the sampling of the split eigenfrequencies \( \omega_k, k = 1, 2, ..., 2l+1 \), is uniform; this in turn depends on having a fairly uniform distribution of both earthquake sources and seismographic instruments upon the surface of the Earth. In practice, this is not the case. Dziewonski & Gilbert (1972) have investigated the extent to which non-uniform sampling is likely to have systematically biased degenerate eigenfrequency estimates \( \omega_0 \), and have concluded that it is small. This does not prove, but it at least makes plausible, the assumption that such bias is small in the splitting width estimates \( \Delta \omega/\omega_0 \) of Fig. 1 also; in the absence of any contradictory evidence, we shall make such an assumption.

The second source of a possible systematic bias in the estimates \( \Delta \omega/\omega_0 \) is the presence of random noise in the Fourier spectra \( |F_i(\omega)| \) or \( |F_i(\omega)|^2, i = 1, 2, ..., N \). Any such noise will necessarily lead to an uncertainty in the locations of the spectral peaks \( \omega_i, i = 1, 2, ..., N \); i.e. each of the measured peak locations which correspond to a given multiplet must be treated as a random variable. We shall denote these random variables by \( \tilde{\omega}_i, i = 1, 2, ..., N \). In general, the properties of each of the random variables \( \tilde{\omega}_i \) will differ, since the properties of the random noise in each of the \( |F_i(\omega)| \) or \( |F_i(\omega)|^2 \) need not be the same. To simplify matters, we shall however neglect this complication. We shall assume that each of the \( \tilde{\omega}_i, i = 1, 2, ..., N \) has a mean value which is unbiased; i.e. \( E[\tilde{\omega}_i] = \omega_i \), \( i = 1, 2, ..., N \), and that the variance of each about its mean is the same, i.e. \( E[(\tilde{\omega}_i - \omega_i)^2] = E[(\tilde{\omega}_j - \omega_j)^2] \), \( i, j = 1, 2, ..., N \); we shall denote the value of this common variance by \( \sigma^2 \). Under these circumstances, it is clear that the sample mean \( \bar{\omega}_0 \) will be an unbiased estimate of the multiplet degenerate eigenfrequency \( \omega_0 \), i.e. \( E[\bar{\omega}_0] = \omega_0 \), but that the sample variance \( \Delta \bar{\omega}^2 \) will be biased with respect to the multiplet variance \( \Delta \omega^2 \) by an amount \( \sigma^2 \), i.e. \( E[\Delta \bar{\omega}^2] = \Delta \omega^2 + \sigma^2 \). To obtain an unbiased estimate of \( \Delta \omega^2 \), we must subtract the noise variance \( \sigma^2 \).

We consider the estimation of the noise variance bias \( \sigma^2 \) for a given multiplet in Appendix 1. The problem which we treat there is actually that of determining the noise variance in the estimation of the angular frequency of oscillation of a single decaying sinusoid in the presence of stationary random noise. The results of that study are not strictly applicable to the present situation, since none of the observed Fourier spectra \( |F_i(\omega)| \) or \( |F_i(\omega)|^2, i = 1, 2, ..., N \), which correspond to a given multiplet will really consist of only a single spectral line at one of the split eigenfrequencies \( \omega_k, k = 1, 2, ..., 2l+1 \). In the absence of any more general result, we shall however make use of the results of Appendix 1 here. Its use may be to a certain extent justified by the fact that each of the \( |F_i(\omega)| \) or \( |F_i(\omega)|^2, i = 1, 2, ..., N \), generally will consist predominantly of only a few lines, all of which will be adjacent to one another, it is in fact this circumstance which makes the various observed spectral peaks \( \omega_i, i = 1, 2, ..., N \), differ from each other in the first place. We shall furthermore, in applying the results of Appendix 1, ignore the small differences between any of the \( 2l+1 \) split eigenfrequencies \( \omega_k, k = 1, 2, ..., 2l+1 \), and the corresponding degenerate eigenfrequency, and we shall assume that every normal mode in a given multiplet has the same attenuation factor \( Q^{-1} \), as well as that every seismogram which has been analysed to obtain the Fourier spectra \( |F_i(\omega)| \) or \( |<F_i(\omega)>|^2, i = 1, 2, ..., N \), has the same duration \( T \). We are thus led to adopt as the best readily determined estimate \( \bar{\sigma}^2 \) of the noise variance corresponding to a given multiplet the quantity

\[
\bar{\sigma}^2 = \frac{R(\omega_0 T/Q)}{T^2 \langle \text{snr}(\omega_0) \rangle},
\]

(16)
where $\langle \text{snr}(\omega_0) \rangle$ is the average signal/noise ratio in the Fourier energy spectra $|F_i(\omega)|^2$, $i = 1, 2, \ldots, N$, at the degenerate eigenfrequency of the multiplet, and where $R(\omega_0 T/Q)$ is a rather complicated function which is given explicitly in Appendix 1; we note that the factor $\omega_0 T/Q$ is a measure of the number of successive times the elastic-gravitational energy of a particular normal mode has been attenuated by a factor $1/e$ during the duration $T$ of seismic recording.

All of the quantities $\omega_0$, $T$, $Q$, and $\langle \text{snr}(\omega_0) \rangle$ which appear in the expression (16) may be at least roughly estimated for all of the fundamental normal mode multiplets displayed in Fig. 1. Since more than half the total number of observed Fourier spectral peaks which have been used in the determination of the raw estimates $\Delta \omega/\omega_0$ which are shown in Fig. 1 have been obtained from the 84 Fourier spectra of WWSSN recordings of the 1964 Alaskan earthquake analyzed by Dziewonski & Gilbert (1972, 1973), it is reasonable to estimate the two parameters $T$ and $\langle \text{snr}(\omega_0) \rangle$ using only those spectra as a basis. The 84 recordings listed in Table 1 of Dziewonski & Gilbert (1972) have an average length of 19.5 hr; 29 of these are horizontal component recordings, and these have an average length of 17.8 hr. We have thus chosen $T = 19.5\text{ hr}$ and $T = 17.8\text{ hr}$ for spheroidal and toroidal multiplets, respectively. We have estimated, as well as possible, the signal/noise ratios $\langle \text{snr}(\omega_0) \rangle$ of the fundamental normal mode multiplets by inspection of the average energy spectra in Figs 5 and 6 of Dziewonski & Gilbert (1973). These estimates are necessarily somewhat subjective. We have, in general, considered as noise all the energy below a smooth line which connects consecutive minima in any of the spectra; this seems to be an appropriate definition of noise in the present context even if we do thereby include mostly signal-related noise (Dziewonski & Gilbert 1973), since even signal-related noise will presumably contribute to the uncertainty in the location of an individual spectral peak, and hence to the noise variance bias $\sigma^2$. We have made a rough estimate of the attenuation factors $Q$ corresponding to the fundamental toroidal and spheroidal normal mode multiplets by means of a simple computation. We assume that only elastic shear energy is dissipated, and we employ a simple parabolic function (suggested by Gilbert, private communication, c. 1972) for the spherically symmetric inverse shear attenuation factor $Q^{-1}(r)$, namely

$$Q^{-1}(r) = 0.005 + 0.038 \left( \frac{r}{a} - \frac{1}{2} \right)^2.$$  \hspace{1cm} (17)

<table>
<thead>
<tr>
<th>degree $l$</th>
<th>$Q$ of $\delta T_1$</th>
<th>$Q$ of $\delta S_1$</th>
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<tr>
<td>4</td>
<td>206</td>
<td>483</td>
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<tr>
<td>6</td>
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This model gives toroidal and spheroidal multiplet attenuation factors which are shown in Table 1; we note, in passing, that these values are at least reasonably consistent with the recent compilation of fundamental normal mode attenuation data which has been reviewed by Smith (1972).

Values of the quantity $\delta/\delta_0$, where $\delta = (\delta^2)^{1/2}$, determined by the procedure described above, are shown superimposed upon the raw multiplet splitting width estimates $\Delta \omega/\omega_0$ in Fig. 1; these values fall along the smooth curves labelled NOISE CONTRIBUTION; we note that the apparent relative magnitude of the noise contribution, as displayed in Fig. 1, is slightly misleading, as it is really the square of each of the splitting width estimates $(\Delta \omega/\omega_0)^2$ which is biased by the square of the noise contribution $(\delta/\delta_0)^2$. It is difficult to assign any uncertainty to the estimated noise bias shown in Fig. 1, because of the necessarily rather subjective and approximate

![Fig. 2. A comparison of the upper mantle structure $v_p(r)$, $v_s(r)$, and $\rho_0(r)$ for Earth models 1066A and 1066B of Gilbert & Dziewonski (1975), as well as for the improvised model 1066B (50), which has been constructed by replacing each of the sharp upper mantle discontinuities in Model 1066B by smoothed transition zones with a width of 50 km.](https://academic.oup.com/gji/article-abstract/44/1/77/665825)
manner in which it has been estimated. We shall therefore, in what follows, make use solely of corrected point estimates of multiplet splitting width $\Delta \omega / \omega_0$ in comparing the results of theoretical calculation. We display these corrected point estimates in Figs 3, 4, 6, 7, and 8, all of which will be discussed in more detail in Sections 4 and 5; each of the splitting width estimates which are plotted in these diagrams has been computed according to the formula $[(\Delta \omega / \omega_0)^2 - (\delta^2 / \omega_0)^2]^{1/2}$. As we are unable to assign, with any assurance, an uncertainty to these corrected values, no confidence intervals have been shown; subjectively, we feel that it is a fair statement that the overall quality of these corrected point estimates can be judged adequately in terms of their scatter, when considered as a function of degree $l$. This point of view has been adopted implicitly in the considerations which follow.
FIG. 5. The difference $\Delta v_S(r) = v_S^{\text{ocean}}(r) - v_S^{\text{continental}}(r)$ in the upper mantle shear wave velocity structure beneath continents and oceans for the model PEM-OC compared with that for the two improvised models 420 and 670. Above a depth of 120 km, the models 420 and 670 have differences $\Delta \rho_0(r)$, $\Delta \rho_6(r)$, $\Delta \rho_8(r)$, and $\Delta h_j, j = 1, 2, \ldots, n$, which are identical to those for the model PEM-OC. The rather large difference in ocean and continental crustal structure is not shown.

FIG. 6. The theoretical fundamental mode multiplet splitting widths $\Delta \omega/\omega_0$ computed for each of three models of oceanic-continental differences PEM-OC, 670, and 420, superimposed upon the rotational and elliptical splitting width contribution of model 1066A, and compared with the corresponding corrected point estimates $(\Delta \omega^2 - \delta^2)^{1/2}/\omega_0$ of the observed splitting widths.
Certain gross characteristics of the multiplet splitting width, when considered as a function of degree $l$ for the fundamental normal mode multiplets $oT_l$ and $oS_l$, we take to be fairly well established by the corrected data shown in Figs 3, 4, 6, 7 and 8. Consider, e.g. the splitting width of those multiplets with degrees $l \geq 20$, i.e. with degenerate periods $2\pi/\omega_0$ less than about 350 s. The overall quality of the corrected splitting width data for the generally inadequately sampled toroidal multiplets is sufficiently low that its gross characteristics, as a function of degree $l \geq 20$, can be reasonably well summarized in terms of a single number; the average toroidal multiplet splitting width $\Delta \omega/\omega_0$ for $oT_l$, $20 \leq l \leq 46$, appears to be about 0.0025. The corrected splitting width data for spheroidal multiplets $oS_l$, for $l \geq 20$, is of higher quality; a clear increasing trend from a value of $\Delta \omega/\omega_0$ of about 0.0015 at $l = 20$ to about 0.0022 at $l = 50$ can be discerned. Throughout the range $20 \leq l \leq 50$, the splitting width of the fundamental spheroidal multiplets $oS_1$ appears to be less than that of the fundamental toroidal multiplets $oT_1$. 

**Fig. 7.** Same as Fig. 6, with model 1066B used for the computation of the contribution from the Earth's rotation and ellipticity.

**Fig. 8.** Same as Figs 6 and 7, with model 1066B (50) used for the computation of the contribution from the Earth's rotation and ellipticity.
4. Results of numerical computation

We seek now to investigate what implications, if any, these rather rough generalizations concerning the measured splitting widths of the fundamental normal mode multiplets might have for the lateral heterogeneity of the Earth. The first problem we must deal with in this regard is the fact that the theoretical splitting width of a given multiplet depends not only upon the nature of the lateral heterogeneity $\delta m_{\text{het}}(r; h_j)$, but also upon the diurnal rotation $\Omega$ and the hydrostatic ellipsoidal structure perturbation $\delta m_{\text{ell}}(r; \Omega; h_j)$. The rotational and elliptical splitting parameters $\beta$ and $\alpha$ which express this dependence in equations (10) and (12) may be computed, but only if the spherically averaged structure of the Earth $m_0(r)$ is assumed to be well known.

A reasonably reliable estimation of the fundamental mode rotational splitting parameters $\beta$ presents no difficulties. The rotational splitting parameter $\beta$ for any toroidal multiplet $^nT_l$ is simply $1/(l+1)$, independent of $m_0(r)$; the rotational splitting parameter of a spheroidal multiplet $^pS_l$ does depend on $m_0(r)$, but it may be shown that at least for the fundamental mode multiplets $^pS_1$, $l \geq 4$, $\beta$ is not particularly sensitive to the relatively small differences between present-day Earth models $m_0(r)$. The relative contribution of the Earth's rotation to the splitting of a fundamental normal mode multiplet is in any case fairly insignificant, except at a very low degree $l$. The fundamental mode elliptical splitting parameters $\alpha$ cannot, on the other hand, be computed with any confidence at this time. The reason for this is that the hydrostatic ellipsoidal structure perturbation $\delta m_{\text{ell}}(r; \Omega; h_j)$ depends, according to equations (1) or (2), on the radial gradients $m_0'(r)$ of the properties of the spherically averaged Earth model. Because of the rather limited resolving power of conventional gross Earth data, these radial gradients are not well constrained, particularly in the upper mantle.

Any inability to compute, with some assurance, the fundamental mode elliptical splitting parameters $\alpha$ will necessarily detract from a program aimed at utilizing measured fundamental mode multiplet splitting widths $\Delta \phi/\phi_0$ to infer the nature of the Earth's lateral heterogeneity $\delta m_{\text{het}}(r; h_j)$. To appraise this situation, we have explored, in a limited way, the uncertainty in the parameters $\alpha$ in the light of present-day knowledge about the Earth's spherically averaged structure. We consider first the two Earth models 1066A and 1066B of Gilbert & Dziewonski (1975), both of which are consistent with all currently available estimates of the Earth's degenerate eigenfrequencies as well as with body wave travel-time data. These two models differ distinctly in the upper mantle, as shown in Fig. 2. Model 1066B has two sharp discontinuities in upper mantle structure at depths of 420 and 670 km; the upper mantle structure of model 1066A is on the other hand relatively smooth. In general, it is impossible to discriminate between spherically averaged Earth models $m_0(r)$ which either have, or do not have, sharp discontinuities in upper mantle structure by making use only of the set of conventional gross Earth data considered by Gilbert & Dziewonski (1975). The elliptical splitting parameters $\alpha$ of Earth models whose gradients $m_0'(r)$ differ from each other by as much as those of models 1066A and 1066B can be expected to disagree considerably; any sharp discontinuities in structure, such as those of model 1066B, will make a substantial contribution to the parameters $\alpha$ by virtue of their displacement $h_j(\ell)$, given in equations (1). In Fig. 3 we show, superimposed upon the corrected fundamental mode multiplet splitting width estimates $\Delta \phi/\phi_0$, the theoretical splitting widths due only to rotation and the hydrostatic ellipsoidal structure $\delta m_{\text{ell}}(r; \Omega; h_j)$ for models 1066A and 1066B; these theoretical widths are given by $\Delta \omega/\omega_0$, where $\Delta \omega = (\Delta \omega^2)^{1/2}$ is computed using equation (10) with $\delta m_{\text{het}}(r; h_j) = 0$, i.e. $C_s(r_1, r_2) = 0$, $s = 2, 4, \ldots, 2l$. The steep rise of the theoretical splitting width curves for $l \leq 7-8$, particularly apparent for the spheroidal modes, is due almost entirely to rotational splitting; in this low degree region where the splitting is dominated by rotation, the theoretical results are very nearly the same for all present-day
Earth models \( m_0(r) \) because of the relative insensitivity of the parameters \( \beta \) to these models. For \( l \geq 7-8 \), the effect of rotation becomes small compared to that of the hydrostatic ellipsoidal structure perturbation; the theoretical curves for models 1066A and 1066B begin to diverge at this point. The divergence for toroidal modes is not great, especially when compared to the present-day uncertainty in the toroidal splitting width data \( \Delta \tilde{\omega}/\omega_0 \); the divergence for spheroidal modes is on the other hand considerable, amounting to almost a factor of three near \( l = 20-30 \). The reason for this difference in the sensitivities of toroidal and spheroidal elliptical splitting to the presence or absence of upper mantle discontinuities is related to the lack of a vertical component of particle motion in a toroidal displacement.

Models 1066A and 1066B represent, in a sense, two extreme examples of the general features which a spherically averaged upper mantle structure \( m_0(r) \) may display and still be compatible with the set of conventional gross Earth data, i.e. with observed degenerate eigenfrequencies and travel-time observations. We shall presume in what follows that the two theoretical curves for models 1066A and 1066B in Fig. 3 may be taken as generally indicative of the extent to which the ellipticity contribution to fundamental mode multiplet splitting is uncertain, because of uncertainties in the gradients \( m'_0(r) \) of the spherically averaged Earth. A number of recent studies have attempted to increase the degree of resolution with which upper mantle structure may be delineated, using methods which provide information supplementary to that contained in the conventional gross Earth data set. A notable example is the recent study of Helmberger \& Engen (1975), in which a model of upper mantle shear velocity structure \( u_s(r) \) has been concocted to be compatible not only with conventional shear wave travel-time data, but also with the observed shape of long period \( SH \) waveforms. Since fundamental mode elliptical splitting parameters \( \alpha \) are generally more sensitive to the details of \( u_s(r) \) than to \( v_p(r) \) or \( \rho_0'(r) \), the results of this study are particularly relevant. Model SHR14, the upper mantle shear wave velocity model preferred by Helmberger \& Engen (1975), is relatively smooth except for two regions near 420 and 670 km where the gradient \( v_s'(r) \) is very large. Steep gradient regions in upper mantle models such as SHR14, as well as the analogous but more severe sharp discontinuities in models such as 1066B, are both customarily interpreted in terms of phase changes in the materials which comprise the upper mantle. The width of both the 420 km and the 670 km phase transition regions in model SHR14 is about 50 km. Phase transition regions of about this same width appear to be indicated as well by large array studies of \( dT/d\Delta \) for upper mantle shear wave arrivals, such as that of Robinson \& Kovach (1972). On the other hand it should be noted that a width of 50 km for the phase transition at 670 km does not appear to be consistent with the reported short period reflection of \( PKIKP \) phases from this zone (Engdahl \& Flinn 1969; Richards 1972). We have not investigated the ellipticity splitting for model SHR14, but we have investigated that of a model whose radial gradients \( m'_0(r) \) have the same general property. We have contrived a model, which we shall call model 1066B (50), by simply smoothing both the 420-km and the 670-km discontinuities of model 1066B over a width of 50 km, as shown in Fig. 2. This smoothing process has essentially a negligible effect upon the calculated degenerate fundamental toroidal and spheroidal eigenfrequencies \( \omega_0 \); there can be little doubt that it should be possible to find a spherically averaged Earth model \( m_0(r) \) with all of the general characteristics of model 1066B (50) which is fully compatible with the entire gross Earth data set. The computed multiplet splitting widths \( \Delta \omega/\Delta \omega_0 \), due only to rotation and ellipticity, for model 1066B (50) are shown in Fig. 3, together with those for models 1066A and 1066B. As might have been expected, the process of smoothing the upper mantle structure of model 1066B has reduced the magnitude of the elliptical splitting of spheroidal multiplets \( \sigma S_l \), especially in the range \( 10 \leq l \leq 50 \); the relative insensitivity of the elliptical splitting of toroidal multiplets to the details of \( m_0(r) \) is confirmed. We note, in passing, that the simple
formulae (1) or (2) for determining the hydrostatic ellipsoidal structure perturbation \( \delta m_{\text{ell}}(r; \Omega; h_j) \) in terms of \( m_0(r) \) is not applicable to Earth models which have, instead of actual discontinuities, steep radial gradients \( m'(r) \) like those displayed by model 10663 (50). Those formulae, although they have a strong intuitive appeal, are in fact only an approximation to the full Clairaut theory, and they become invalid whenever the width of any steep gradient region at a radius \( r \) exceeds \( \frac{2r_0(r)}{I_0} \); in Appendix 2 we give the more complete formulae which we have used in lieu of (1) or (2) in the numerical computation for model 1066B (50).

The major difficulties which, at the present time, impede progress on the problem of using observations of fundamental mode multiplet splitting to infer the nature and extent of the Earth’s lateral heterogeneity are apparent upon examining Fig. 3. For the fundamental toroidal multiplets \( _o T_l, \, 4 \leq l \leq 46 \), we may compute with some confidence the contribution to the splitting of the Earth’s rotation and ellipticity, but the observed splitting width estimates are highly uncertain, largely because of inadequate sampling. The splitting width estimates for the fundamental spheroidal multiplets \( _s S_l, \, 4 \leq l \leq 50 \), are smoother as a function of degree \( l \), but the contribution from rotation and ellipticity is not well known. It is because of these difficulties that we have not attempted any formal inversion of the data shown in Fig. 3, although the strict linearity of this inverse problem makes it an attractive one. Instead, in what follows, we shall report on some direct multiplet splitting width calculations for a few prescribed models of the lateral heterogeneity \( \delta m_{\text{het}}(r; h_j) \) of the Earth. As indicated in Sections 1 and 2, we only consider lateral heterogeneity which is confined to the upper mantle, i.e. above 670 km, and we furthermore suppose that the dominant lateral heterogeneity in the upper mantle is that associated with differences in the structure beneath oceans and continents. We shall compute total multiplet splitting widths \( \Delta \omega/\omega_o \) for comparison with the corresponding splitting width estimates \( \Delta \omega/\omega_o \), using as a spherically averaged base each of the three models 1066A, 1066B, 1066B (50) in order to gauge the robustness with respect to uncertainties in \( \delta m_{\text{het}}(r; \Omega; h_j) \) of any conclusions we might draw.

To compute multiplet splitting widths \( \Delta \omega/\omega_o \) due to differences between the structures underlying oceans and continents, we need to know, according to equation (12), the degree variances \( \sigma_s^2, \, s = 2, 4, \ldots \) of the oceanic projection function \( C(\ell) \) as well as the differences \( \Delta m(r) \); recall that \( \Delta m(r) \) is a shorthand notation not only for the volume differences \( \Delta \rho_0(r) = \rho_0^{\text{ocean}}(r) - \rho_0^{\text{continent}}(r) \), \( \Delta \kappa(r) = \kappa^{\text{ocean}}(r) - \kappa^{\text{continent}}(r) \), and \( \Delta \mu(r) = \mu^{\text{ocean}}(r) - \mu^{\text{continent}}(r) \), but also for the differences \( \Delta h_j = h_j^{\text{ocean}} - h_j^{\text{continent}}, \, j = 1, 2, \ldots, N \), in the radial locations of any major discontinuities. The degree variances \( \sigma_s^2 \) for \( s \leq 36 \) have been computed by and were obtained from Balmino, Lambeck & Kaula (1973). For any multiplet \( _o T_l \) or \( _s S_l \) with \( l > 18 \), we have simply truncated the sum which appears in equation (12) at degree \( s = 36 \); the error due to this truncation is in general insignificant. In fact we have found, in every computation we have performed, that the theoretical multiplet variance of all fundamental mode multipoles can be computed with reasonable accuracy by considering only the very lowest degree terms \( s = 2, 4 \) and \( 6 \); the contributions from higher degrees \( s \) generally decay rapidly with increasing \( s \). We conjecture that this might be a fairly universal result; if so, it has immediate consequences for any future attack, using more adequate data, on the lateral heterogeneity inverse problem.

We consider first a recent model for the differences in structure beneath oceanic and continental areas due to Dziewonski, Hales & Lapwood (1975). These investigators have presented three parametrically defined spherically symmetric models of the Earth, PEM-A, PEM-O, and PEM-C. The first of these is a representation of the spherically averaged Earth, whereas the latter two are intended to represent the radial variation in the properties of the Earth beneath oceans and continents, respectively. These latter two models PEM-O and PEM-C have been specifically constructed in order to be consistent with the generally observed patterns of oceanic and continental
group velocity dispersion for relatively short period (20–100 s) Rayleigh surface waves, as well as with the set of conventional gross Earth data. The difference between these two spherically symmetric models defines a particular model \( \Delta m(r) \), which we shall denote by PEM-OC. The models PEM-O and PEM-C are identical to each other and to model PEM-A below a depth of 420 km. Model PEM-OC is thus identically zero below 420 km; it is, as well, reasonably small below a depth of about 200 km. Model PEM-OC is in both these regards rather typical of the differences between oceanic and continental crustal structures which have been suggested by a number of studies which have focused principally upon the observed differences in oceanic and continental surface wave dispersion data; the principal differences are confined to the upper few hundred kilometres and are generally associated with the presence of a relatively more profound low elastic shear wave velocity zone beneath oceans than beneath continents. We shall, for the purposes of this discussion, refer to any such model \( \Delta m(r) \) as a conventional model; the particular conventional model for which we have performed numerical computations is model PEM-OC. In the interest of completeness, we have computed theoretical multiplet splitting widths \( \Delta \omega / \omega_0 \) for oceanic-continental differences PEM-OC superimposed upon the rotating, hydrostatic ellipsoidal Earth model which is defined by model PEM-A, the spherically averaged counterpart to models PEM-O and PEM-C. The results of this computation are shown in Fig. 4, once again together with the corrected multiplet splitting width estimates \( \Delta \delta / \delta_0 \); the curves labelled PEM-A in Fig. 4 represent the splitting due to rotation and ellipticity alone, while those labelled PEM-OC represent the total splitting due to oceanic-continental differences as well. There are two primary features which are of interest in Fig. 4. The first is that the elliptical splitting of model PEM-A is exceedingly large; in fact the theoretical splitting widths \( \Delta \omega / \omega_0 \) of the fundamental spheroidal multiplets \( \Theta_1^1 \) due to rotation and ellipticity alone exceed the total estimated splitting widths \( \Delta \delta / \delta_0 \) throughout the entire intermediate range \( l = 13–33 \). It thus appears that model PEM-A, although consistent with the full set of normal mode degenerate eigenfrequency data, is inconsistent with fundamental normal mode multiplet splitting data. These very large ellipticity splitting widths of the parametrically defined Earth model PEM-A are a consequence of the presence of a relatively large number of sharp discontinuities in the structure of the crust and upper mantle. Model PEM-A could undoubtedly be made consistent with the eigenfrequency splitting data by the expedient of smoothing out some of these discontinuities, but any such procedure would be difficult to parameterize, and much of the appeal and value of the model would be lost. The second obvious feature is the fact that the additional splitting induced by the oceanic-continental differences of model PEM-OC is considerably greater for fundamental toroidal multiplets than for fundamental spheroidal multiplets, by as much as a factor of seven at degree \( l = 50 \). Luh (1974) found a similar effect, although not quite so pronounced, in his numerical computations for a different but also conventional model of oceanic-continental differences. We suspect that this is a fairly universal result, valid for any conventional model with essentially negligible oceanic-continental differences below a few hundred kilometres.

Conventional models of the differences in structure beneath oceans and continents are generally not consistent with the important new datum obtained by Sipkin & Jordan (1975). These authors have established, in a well-designed study of the travel times of ScS phases, that the one-way travel time of elastic shear waves travelling vertically under normal oceanic basins is about 5 s greater than that of elastic shear waves travelling vertically under continents. Since the fairly well-established differences in oceanic and continental crustal structure give rise to such a difference of about 1 s but of the opposite sign, this observation implies that there must in fact be a net difference of about 6 s in the mantle, i.e. below about 35 km depth. The corresponding difference for model PEM-OC, for a shear wave travelling vertically from anywhere
below 420 km depth to 35 km depth is only 2.5 s; most other conventional models will give rise to a difference of about this magnitude as well. Oceanic-continental shear wave velocity differences which are substantially greater than those in conventional models such as PEM-OC appear to be required by the observation of Sipkin & Jordan (1975). A very important question which arises in this regard is: to what depth must significant oceanic-continental differences persist in order to be consistent with a one-way vertical shear wave travel-time difference below 35 km of about 6 s, as well as with other geophysical data? This question, with its various tectonic implications, has been discussed in some detail by Jordan (1975); we shall consider this question here, making use of fundamental mode multiplet splitting widths as the 'other geophysical data'.

We have performed theoretical splitting width computations for two hypothetical models \( \Delta \rho_0(r) \) which are consistent with the observation of Sipkin & Jordan (1975). We have taken both these models to have the same oceanic-continental differences as model PEM-OC above a depth of 120 km in view of the generally good qualitative agreement of that model with short and intermediate period surface wave dispersion data. Below 120 km depth, we have taken \( \Delta \rho_0(r) \) to be constant down to a depth of either 420 or 670 km, but in both cases consistent with a 6 s one-way vertical travel-time difference below 35 km. We have furthermore taken both \( \Delta \rho_0(r) \) and \( \Delta \rho_0(r) \), as well as any \( \Delta h_j, j = 1, 2, \ldots, N \), to be zero below 120 km; since theoretical fundamental mode multiplet splitting widths are primarily sensitive to \( \Delta \rho_0(r) \), or alternatively to differences \( \Delta h_j, j = 1, 2, \ldots, N \), in the elevations of any sharp discontinuities in \( v_s(r) \), this latter assumption is of little consequence. To verify this we have computed the theoretical splitting widths for model PEM-OC with both \( \Delta \rho_0(r) \) and \( \Delta \rho_0(r) \) taken to be zero below 120 km depth; the results are insensibly different from those displayed in Fig. 4. In Fig. 5 we display \( \Delta \rho_0(r) \) for model PEM-OC, as well as for the two hypothetical models, which we shall call model 420 and model 670, respectively. We have performed splitting width computations for each of the three models PEM-OC, 420, and 670 shown in Fig. 5, using each of the three spherically averaged models 1066A, 1066B, and 1066B (50) for the determination of the contribution from the Earth's rotation and ellipticity. The results are shown in Figs 6, 7 and 8. In each of these diagrams, the lowest theoretical curve represents the effect of rotation and ellipticity only, as already displayed in Fig. 3; the upper curves then show the additional influence of the three models for oceanic-continental differences. As before, the theoretical curves have been plotted superimposed upon the corrected observed splitting widths \( \Delta \omega/\omega_0 \).

5. Conclusions

There are two general conclusions which can be drawn from an inspection of Figs 6, 7 and 8; we shall argue that both of these conclusions are a necessary consequence of the observed multiplet splitting widths of Dziewonski & Gilbert (1972). The first conclusion which we feel is indicated is that there must be significant lateral heterogeneity in the deep mantle of the Earth. All of the computations displayed in Figs 6, 7 and 8 are for models of the lateral heterogeneity which are confined to the upper mantle, above a depth of 670 km. It is apparent that for both toroidal and spheroidal multiplets \( qT_l \) and \( qS_l \), throughout the range of low degrees \( 5 \leq l \leq 26 \), the observed splitting widths \( \Delta \omega/\omega_0 \) are systematically greater than any of the corresponding theoretical values. This discrepancy is especially severe for the lower degrees \( l \), where it amounts on the average to almost a factor of two. The fact that there is such a discrepancy for all of the theoretical models clearly indicates that some significant lateral heterogeneity other than that we have considered must be present somewhere within the Earth. As we shall discuss in more detail below, the measured
splitting widths for the higher degree multiplets $oT_l$ and $oS_l$, $20 \leq l \leq 50$, are, in contrast to the low degree multiplets, generally consistent with the hypothesis that oceanic-continental differences in the upper mantle are the dominant lateral heterogeneity contributing to the splitting. This suggests strongly that the location of the additional lateral heterogeneity must be in the lower mantle. The splitting of the higher degree fundamental multiplets would be relatively unaffected by any deep-seated lateral heterogeneity, because the associated kernel functions $M_s(r)$, $s = 2$, $4$, ..., $2l$, in equation (10) are relatively small at great depths; the lowest degree multiplets would be the most strongly affected. We have not performed any numerical computations for any hypothetical models which include lateral heterogeneity in the lower mantle, in spite of the evidence we have found here which suggests its presence, in part because we have very little idea what form it might take. Independent evidence for the existence of lateral heterogeneity in the lower mantle has been accumulating recently, in a number of recent studies of the travel times of elastic body waves (Julian & Sengupta 1973; Jordan & Lynn 1974). The evidence we have presented here, based on the observed splitting widths of the low degree fundamental normal mode multiplets, is complementary to this travel-time evidence, since it provides direct information about the large scale, global nature of lower mantle lateral heterogeneity, whereas travel-time studies are necessarily able to sample only selected and relatively small portions of the lower mantle.

The second conclusion which we feel may be drawn from an inspection of Figs 6, 7 and 8 is that differences in upper mantle structure beneath continents and oceans must persist at least to depths below about 400 km. We base this conclusion primarily upon a comparison of the observed splitting widths of both toroidal and spheroidal multiplets $oT_l$ and $oS_l$ of fairly high degree, say $35 \leq l \leq 50$, with the corresponding theoretical splitting widths computed for model 420, which has oceanic-continental differences consistent with the observation of Sipkin & Jordan (1975) confined above 420 km depth. The observed splitting widths are systematically less than the corresponding theoretical splitting widths for model 420 throughout this entire range $35 \leq l \leq 50$, even in Fig. 6 where the ellipticity contribution, corresponding to the very smooth upper mantle model 1066A, is the least. This systematic discrepancy is a strong indication that oceanic-continental differences must extend below a depth of 420 km. This argument does not depend upon the fact that we have not considered any upper mantle lateral heterogeneity other than the differences in structure beneath continents and oceans. There is presumably such additional upper mantle lateral heterogeneity not associated with oceanic-continental differences, but it is clear that its inclusion can only increase the theoretical curves in Figs 6, 7 and 8, because of the fact that multiplet splitting variances depend upon the covariance of the lateral heterogeneity. We note that a vital role has been played in the argument we have made here that oceanic-continental differences must extend to fairly great depths by the single datum obtained by Sipkin & Jordan (1975). It is true that on the basis of the information displayed in Figs 6, 7 and 8, we probably could mount a persuasive argument that some lateral heterogeneity in the upper mantle in addition to that of the conventional model PEM-OC must be present, in view of the relatively slight but distinctly systematic discrepancy between all of the theoretical curves for model PEM-OC for the higher degree fundamental poloidal multiplets $oS_l$, $35 \leq l \leq 50$. We could not however establish from this evidence alone whether or not this additional lateral heterogeneity is primarily associated with oceanic-continental differences, nor that these differences need be as profound as is indicated by the observation of Sipkin & Jordan (1975). We remark finally upon the model we have considered which is most consistent with the observations. It may be seen that the theoretical curves for model 670 superimposed upon the ellipticity contribution due to model 1066B (50), displayed in Fig. 8, constitute quite a respectable fit to both the toroidal and the spheroidal multiplet splitting width estimates throughout the entire
range of higher degrees, $25 \leq l \leq 50$; in particular, the clearly indicated increasing trend of the much less scattered estimates for the spheroidal multiplets from about $-0.015$ at $l = 25$ to $0.022$ at $l = 50$ is duplicated. It is probably a fair statement that model 1066B (50), with two phase transition zones of about 50 km width each, yields, at least approximately, the best estimate of the ellipticity splitting contribution which we can make at the present time. In this sense, 670 km might then be said to represent the corresponding best estimate which we are able to make on the basis of present-day splitting width data of the depth to which ocean-continent differences must persist. It is intriguing that the best rough estimate we can make for this depth coincides with the depth to the 670 km phase transition.

Sipkin & Jordan (1975) and Jordan (1975) have considered the question of the depth to which oceanic-continental differences must persist by using as the 'other geophysical data' the differences in long period oceanic and continental Love and Rayleigh wave phase velocity dispersion which have been inferred by Kanamori (1970) by means of a decomposition of measured great circular phase velocity dispersion data. We would like to comment briefly on the relative merits of that approach, compared to the one which has been used here. It is apparent that these two approaches are very similar, both taking advantage of slightly different aspects of what is essentially the same information. The 300–150 s fundamental mode Love and Rayleigh surface waves analysed by Kanamori (1970) correspond precisely to the fundamental toroidal and spheroidal mode multiplets $0T_i$ and $0S_i$, with $l$ between about 25 and 60, which have been used here to argue for deep-seated oceanic-continental differences. Decomposed great circular surface wave dispersion data does have one distinct advantage over the multiplet splitting width data employed here, namely that it should not be subject to any systematic bias due either to noise or to the effect of the Earth's hydrostatic ellipsoidal structure. The hydrostatic ellipsoidal structure of the Earth does have an effect upon great circular surface waves and, as pointed out by Dahlen (1975), failure to incorporate this effect in considering a decomposition will have a corresponding effect on the inferred oceanic and continental dispersion values; this effect should not however have any particular bias, as long as the sample of great circular paths, considered is a fairly representative one. Both the noise variance bias and the splitting contribution of the Earth's hydrostatic ellipsoidal shape have, on the other hand, played an important role in the present study in limiting the certainty with which any conclusions might be drawn. If it were not these difficulties, it is fairly clear that the method employed here to examine oceanic-continental differences should be preferred to the method employed by Sipkin & Jordan (1975) and Jordan (1975), because of the fact that the degenerate eigenfrequency splitting theory upon which this study has been based is a more accurate theory than the geometrical optics approximation which is used in decomposing great circular surface wave phase velocity observations. This method has been the subject of some controversy ever since Dziewonski (1971) employed an identical regionalization scheme to that used by Kanamori (1970) in the decomposition of an independent set of great circular Rayleigh wave phase velocity measurements, and obtained different results. Madariaga & Aki (1971), in seeking to explain this disagreement, have made use of an asymptotic version of normal mode splitting theory in order to demonstrate that the application of the geometrical optics approximation is not, at least formally, valid for 150–300 s fundamental mode surface waves. If this is the case, then the considerations we have made here might well represent the strongest seismic evidence we have at the present time for the existence of deep-seated differences in the upper mantle structure beneath oceans and continents.

In summary, we contend that the theoretical calculations of the splitting widths of the fundamental normal mode multiplets which have been performed here support two important conclusions concerning the lateral heterogeneity of the Earth. The first is that there must be a significant lateral heterogeneity present in the Earth's lower mantle, and the second is that the lateral heterogeneity in the upper mantle
Models of the lateral heterogeneity of the Earth

associated with the differences in structure beneath oceans and continents must persist to depths of at least a few hundred kilometres. We have been led to these conclusions by the consideration of a number of models of the lateral heterogeneity which did not display either of these features; we were able to show thereby that models which do not have these features quite generally predict splitting widths which are incompatible with the observed splitting width. In both cases the discrepancy is a fairly large one, and is systematic over a large range of degrees \( l \) for both toroidal and spheroidal multiplets. Neither conclusion is very strongly affected by the uncertainty which prevails in the contribution from the Earth's hydrostatic ellipsoidal structure, since in both instances the systematic discrepancy between the theoretical and observed values which forces the conclusion exists for all of the models 1066A, 1066B, 1066B (50). Neither of these two conclusions is of course new, and there is other geophysical evidence which supports each one. In principle at least, multiplet splitting width estimates can be improved by simply adding to the number of independently observed Fourier spectral peaks assigned to each multiplet. If their overall quality can be so improved, observed multiplet splitting widths should provide a valuable future tool not only in the study of the Earth's lateral heterogeneity, but also in the study of the spherically averaged steep gradient regions in the Earth's upper mantle.

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References


**Appendix 1**

**Linear least squares estimation of the frequency of a decaying sinusoid**

Suppose that we have a recording in which the geophysical signal of interest is known to be a single decaying sinusoid with unknown parameters, which we seek to estimate by an analysis of the record we possess. In any real situation, the recording $s(t)$ will have a finite duration $0 \leq t \leq T$, and will inevitably be contaminated by random noise $n(t)$ arising from a variety of geophysical and instrumental sources.
We intend here to investigate the effects of both finite record length $T$ and random noise $n(t)$ upon the estimation process. The treatment we give here of this problem is modelled after the treatment by Munk & Hasselmann (1964) of a vaguely similar spectral resolution problem which is of interest in the analysis of tidal records.

Let $\omega_0 > 0$, $Q > 0$ and $A$ denote, respectively, the unknown angular frequency of oscillation, the attenuation factor, and the complex amplitude of the decaying signal, i.e.

$$s(t) = Ae^{\omega_0 t} e^{-\omega_0 t/2Q} + A* e^{-\omega_0 t} e^{-\omega_0 t/2Q} + n(t), \quad 0 \leq t \leq T.$$  \hfill (18)

We shall presume that $n(t)$ may be satisfactorily considered to be a single truncated realization of an infinite stationary random process with zero mean and with a power spectrum $N(\omega)$, i.e.

$$\langle n(t) \rangle = 0, \quad -\infty < t < \infty$$  \hfill (19)

and

$$\langle n^2(t) \rangle = \int_{-\infty}^{\infty} N(\omega) d\omega, \quad -\infty < t < \infty$$  \hfill (20)

where $\langle \, . \, \rangle$ has been used to denote an ensemble average. It is convenient to represent $n(t)$ as a Fourier-Stieltjes integral

$$n(t) = \int_{-\infty}^{\infty} dN(\omega) e^{i\omega t};$$  \hfill (21)

equations (19) and (20) then imply

$$\langle dN(\omega) \rangle = 0$$  \hfill (22)

and

$$\langle dN(\omega) dN^*(\omega') \rangle = N(\omega) \delta(\omega - \omega') d\omega d\omega'.$$  \hfill (23)

To be specific, we shall estimate the parameters $\omega_0$, $Q$, and $A$ by an application of the method of least squares; for convenience we shall suppose that the estimation procedure is carried out in the time domain, although in any real application it probably would not be. First, in order to simplify the algebraic details, we shall let $\tau_0 = \omega_0 (1 + i/2Q)$, in which case equation (18) becomes

$$s(t) = Ae^{i\omega_0 t} + A* e^{-i\omega_0 t} + n(t), \quad 0 \leq t \leq T.$$  \hfill (24)

We consider a corresponding noise-free signal

$$\check{s}(t) = \check{A}e^{i\omega_0 t} + \check{A}^* e^{-i\omega_0 t}, \quad 0 \leq t \leq T,$$  \hfill (25)

and we seek to determine the complex frequency $\hat{\omega}_0$ and the complex amplitude $\hat{A}$ by minimizing the quantity

$$I = \int_{0}^{T} \langle [s(t) - \check{s}(t)]^2 \rangle dt.$$  \hfill (26)

Setting each of $dI/d\text{Re}\hat{\omega}_0$, $dI/d\text{Im}\hat{\omega}_0$, $dI/d\text{Re}\hat{A}$, and $dI/d\text{Im}\hat{A}$ equal to zero, we obtain four real non-linear algebraic equations for the four unknown estimates $\text{Re}\hat{\omega}_0$, $\text{Im}\hat{\omega}_0$, $\text{Re}\hat{A}$, and $\text{Im}\hat{A}$. To make further progress we shall introduce a number of approximations. First, we shall restrict attention to the case of a record length $T$ which is long enough to contain several cycles of oscillation, i.e.

$$\omega_0 T \gg 1.$$  \hfill (27)
Second, we shall write \( \delta_0 = \delta_0 + \delta \) and \( \hat{A} = A + \hat{a} \), and we shall presume that
\[
\begin{align*}
|\text{Re}\delta|/\text{Re}\sigma_0 & \ll 1 \\
|\text{Im}\delta|/\text{Im}\sigma_0 & \ll 1 \\
|\text{Re}\hat{A}|/|\text{Re}\hat{A}| & \ll 1 \\
|\text{Im}\hat{A}|/|\text{Im}\hat{A}| & \ll 1.
\end{align*}
\]

We employ the conditions (28) to obtain a linearized approximation to the four nonlinear equations for the four new unknowns \( \text{Re}\delta, \text{Im}\delta, \text{Re}\hat{A}, \) and \( \text{Im}\hat{A} \), and we employ the condition (27) to simplify further the resulting linearized equations; written as two complex linear equations, the final result of this linearization and reduction procedure is
\[
\begin{bmatrix}
2I_0 & 2iAI_1 \\
2AA^*I_1 & 2IAA^*I_2
\end{bmatrix}
\begin{bmatrix}
\hat{A} \\
\hat{a}
\end{bmatrix}
= 2 \begin{bmatrix}
\int_0^T e^{-i\omega_0 t} n(t) \, dt \\
\int_0^T e^{-i\omega_0 t} n(t) \, dt
\end{bmatrix}
= 2A^* \begin{bmatrix}
\int_0^T e^{-i\omega_0 t} n(t) \, dt \\
\int_0^T e^{-i\omega_0 t} n(t) \, dt
\end{bmatrix},
\]
where we have let
\[
I_n = \int_0^T t^n e^{-i\omega_0 t/4} \, dt.
\]

We are principally interested here in estimating the angular frequency of oscillation; upon solving the equations (29) for \( \text{Re}\delta \), which we shall also denote by \( \delta\omega_0 \), we obtain
\[
\delta\omega_0 = \frac{1}{(I_1^2/I_0 - I_2)} \frac{1}{\sqrt{(AA^*)}} \int_0^T \left( t - \frac{I_1}{I_0} \right) e^{-i\omega_0 t/4} \sin(\omega_0 t + \phi) n(t) \, dt,
\]
where \( \phi \) is the phase, i.e. \( A = \sqrt{(AA^*)} e^{i\phi} \).

We will make use of the result (31) to investigate the statistical properties of the estimated frequency of oscillation \( \hat{\omega}_0 = \omega_0 + \delta\omega_0 \). We note first that since the noise has zero mean, as expressed by equation (19), the estimate \( \hat{\omega}_0 \) is unbiased, i.e.
\[
\langle \delta\omega_0 \rangle = 0
\]
and thus
\[
\langle \hat{\omega}_0 \rangle = \omega_0.
\]

This result is of course intuitively obvious; a much more interesting quantity is the variance \( \langle \delta\omega_0^2 \rangle \) of the estimate \( \hat{\omega}_0 \). Upon substituting the Fourier–Stieltjes representation (21) into (31), and making use of the orthogonality relation (23) after an interchange of the order of integration, we obtain
\[
\langle \delta\omega_0^2 \rangle = \frac{1}{(I_1^2/I_0 - I_2)^2} \frac{1}{AA^*} \int_{-\infty}^{\infty} N(\omega) |\Phi(\omega)|^2 \, d\omega,
\]
where
\[
\Phi(\omega) = \int_0^T e^{i\omega t} \left( t - \frac{I_1}{I_0} \right) e^{-i\omega t/4} \sin(\omega_0 t + \phi) \, dt.
\]
For $Q \gg 1$, it may be shown that the kernel function $|\Phi(\omega)|^2$ in equation (34) is sharply peaked near $\omega = \pm \omega_0$; under these circumstances we may, with little error, replace $N(\omega)$ by $N(\omega_0) = N(-\omega_0)$, obtaining
\[
\langle \delta \omega_0^2 \rangle = \frac{1}{(I_1^2/I_0-I_2^2)} \frac{N(\omega_0)}{AA^*} \int_{-\infty}^{\infty} |\Phi(\omega)|^2 \, d\omega.
\] (36)

We are left with the evaluation of a number of integrals, in equations (30), (35), and (36); those that are not readily available in tables may be evaluated by an application of the theorem of residues. Upon neglecting terms of order $1/\omega T$, and after considerable reduction, equation (36) becomes
\[
\langle \delta \omega_0^2 \rangle = C(\omega_0 T/Q) \frac{N(\omega_0)}{AA^* T^3},
\] (37)
where
\[
C(x) = \frac{\pi x^3(1-e^{-x})}{1-e^{-x}(x^2+2)+e^{-2x}}.
\] (38)

For $x \ll 1$, it is easy to show that $C(x) \sim 12\pi$, and for $x \gg 1$, $C(x) \sim \pi x^3$, a plot of the function $C(x)$ is shown in Fig. 9. Equation (37) is the formal solution to the problem we have posed; it gives the variance $\langle \delta \omega_0^2 \rangle = \langle (\omega_0 - \omega_0)^2 \rangle$ due to the presence of noise as a function of the signal parameters $\omega_0, Q$, and $AA^*$, as well as the record length $T$ and the noise power spectral density $N(\omega_0)$; any dependence on the phase $\phi$ is of higher order in $1/\omega_0 T$. Since $\tau = Q/\omega_0$ is the time required for the energy of the sinusoidal signal to decay by a factor of $1/e$, the combination $\omega_0/T/Q$ is exactly the number of such $1/e$ energy decay times contained in the record of length $T$; we note that no restrictions have been placed on the magnitude of this parameter by any of the approximations we have made, and the result (37) is thus valid for all values of $\omega_0 T/Q$.

**Fig. 9.** Plots of the functions $C(\omega_0 T/Q)$ and $R(\omega_0 T/Q)$; also shown are the large argument asymptotes for each.
The result (37) is not in a particularly convenient form for the present application, since we have no direct information about either \(N(\omega_0)\) or \(AA^*\). We consider now an alternative prescription of the noise variance \(\langle \delta \omega_0^2 \rangle\) in terms of the signal/noise ratio at the frequency of oscillation \(\omega_0\) in a Fourier energy spectrum of the finite length recording \(s(t)\), \(0 \leq t \leq T\). Suppose first that we have a recording which consists only of noise, i.e. \(s(t) = n(t), 0 \leq t \leq T\). The Fourier energy spectrum at the frequency \(\omega_0\) is given by

\[
|F_n(\omega_0)|^2 = \left| \int_0^T n(t) e^{-i\omega_0 t} dt \right|^2. \tag{39}
\]

Upon substituting the Fourier–Stieltjes representation (21) into (39), and making use of the orthogonality relation (23), we obtain for the mean value

\[
\langle |F_n(\omega_0)|^2 \rangle = 2\pi T N(\omega_0); \tag{40}
\]

several terms of higher order in \(1/\omega_0 T\) have here been neglected. If instead we have a recording which consists only of the decaying sinusoidal signal, i.e.

\[
s(t) = A e^{i\omega_0 t} e^{-\omega_0 t/2Q} + A^* e^{-i\omega_0 t} e^{-\omega_0 t/2Q}, \quad 0 \leq t \leq T, \tag{41}
\]

the Fourier energy spectrum at \(\omega_0\) is given by

\[
|F_s(\omega_0)|^2 = \left| \int_0^T A e^{i\omega_0 t} e^{-\omega_0 t/2Q} + A^* e^{-i\omega_0 t} e^{-\omega_0 t/2Q} e^{-i\omega_0 t} dt \right|^2. \tag{42}
\]

Upon neglecting terms of higher order in \(1/\omega_0 T\), this reduces to

\[
|F_s(\omega_0)|^2 = \frac{4AA^*}{(\omega_0 T/Q)} \left[ 1 - e^{-\omega_0 T/2Q} \right]^2. \tag{43}
\]

The mean signal/noise ratio in an energy spectrum at the angular frequency of oscillation is defined by

\[
\langle \text{snr}(\omega_0) \rangle = \frac{|F_s(\omega_0)|^2}{\langle |F_n(\omega_0)|^2 \rangle}. \tag{44}
\]

Using the expressions (40) and (43) we may rewrite the result (37) in terms of \(\langle \text{snr}(\omega_0) \rangle\), namely

\[
\langle \delta \omega_0^2 \rangle = \frac{R(\omega_0 T/Q)}{T^2 \langle \text{snr}(\omega_0) \rangle} \tag{45}
\]

where

\[
R(x) = \frac{2x(1-e^{-x})(1-e^{-x/2})^2}{1-e^{-x}(x^2+2)+e^{-2x}}. \tag{46}
\]

For \(x \ll 1\), \(R(x) \sim 6\), and for \(x \gg 1\), \(R(x) \sim 2x\); a plot of the function \(R(x)\) is also shown in Fig. 9. The result (45) is the same as (16), and has been used in this study in order to determine the magnitude of the noise variance bias in normal mode multiplet splitting width data.

Finally, we consider briefly the validity of the linearization procedure which has been employed here; an \textit{a posteriori} check upon the first of the conditions (28) can be obtained from equation (45). We note that \(\langle \delta \omega_0^2 \rangle^2/\omega_0^2\) will be \(\ll 1\) if

\[
\langle \text{snr}(\omega_0) \rangle^2 \gg R^2(\omega_0 T/Q)/\omega_0 T. \tag{47}
\]
This is a necessary condition on the magnitude of the signal/noise ratio if the linearization procedure is to be valid; we have verified that this condition is reasonably well satisfied in the present application.

Appendix 2

Clairaut’s hydrostatic theory

The expressions (1) for the volume perturbations $\delta \rho_0(r, \theta)$, $\delta \kappa(r, \theta)$, and $\delta \mu(r, \theta)$ due to the hydrostatic ellipsoidal structure of the Earth are only approximate expressions, which lose their validity in any region of the Earth model where any of the radial gradients $\rho'_0(r)$, $\kappa'(r)$ or $\mu'(r)$ become large. The hydrostatic figure theory of Clairaut, in its most general form as developed by Jeffreys (1970), can be used to obtain a somewhat more complicated expression for $\delta \rho_0(r, \theta)$ which has an accuracy that is independent of the magnitude of $\rho'_0(r)$, except in so far as it affects the ellipticity $\varepsilon(r)$. Assuming, as before, that the ellipsoidal surfaces of constant density $\rho_0(r) + \delta \rho_0(r, \theta)$ are also surfaces of constant incompressibility $\kappa(r) + \delta \kappa(r, \theta)$ and rigidity $\mu(r) + \delta \mu(r, \theta)$, the more accurate expressions which should be used in place of (1) are

\[
\begin{align*}
\delta \rho_0(r, \theta) &= [\rho_0(r) - \rho_0(r - \frac{3}{2} r \varepsilon(r))] P_2(\cos \theta) \\
\delta \kappa(r, \theta) &= [\kappa(r) - \kappa(r - \frac{3}{2} r \varepsilon(r))] P_2(\cos \theta) \\
\delta \mu(r, \theta) &= [\mu(r) - \mu(r - \frac{3}{2} r \varepsilon(r))] P_2(\cos \theta) \\
\end{align*}
\]

(48)

Symbolically, we can write equations (48) in the form

\[
\delta m_{\text{ell}}(r; \Omega; \mathbf{h_j}) = [m_0(r) - m_0(r - \frac{3}{2} r \varepsilon(r))] P_2(\cos \theta),
\]

(49)

which corresponds to equation (2). The approximate equation (2) is just the first term in the Taylor series expansion in the radial variable $r$ of equation (49); the conditions in which this can be expected to be a good approximation are clear. In the present study, we have found the approximation (2) adequate for all of the Earth models 1066A, 1066B, and PEM-A, but not for the contrived model 1066B(50), which has two regions where the properties $m_0(r)$ change considerably over a distance which is on the order of $\frac{3}{2} r \varepsilon(r)$.