Ground State Energy of the One-Dimensional Electron System with Short-Range Interaction. I

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The ground state energy of the one-dimensional electron gas interacting via a delta-function potential is considered as a function of interaction ($c$) for the given magnetization. The integral equations of Gaudin and Yang are used. It is shown that the ground state energy can be analytically continued at $c=0$ for non-zero magnetization. This fact holds also for the one-dimensional Hubbard model except for the half-filled case.

§ 1. Introduction

In many-body theory the analytic properties of the ground state energy of the electron gas as a function of interaction are of great physical interest. As a soluble model we take a one-dimensional electron gas interacting via a delta-function potential. The Hamiltonian is

$$\mathcal{H} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 4c \sum_{i<j} \delta(x_i - x_j), \quad (1\cdot1)$$

where $N$ is the number of electrons. If the system is finite one can prove easily that the ground state energy is analytic as a function of $c$. But it is possible that this does not hold at the thermodynamic limit. To generalize the problem we consider the ground state energy for fixed magnetization. It is well known that in the one-dimensional system the ground state energy for zero magnetization is the true ground state energy.1) We define a function $f(n_t, n_d; c)$ as the ground state energy per unit length at the thermodynamic limit. Here $n_t$ and $n_d$ are the numbers of electrons with up spin and down spin per unit length, respectively. Using the exact theory of Gaudin2) and Yang3) one can obtain the function $f$ by solving a set of coupled integral equations. Integral equations for $c>0$ and $c<0$ are different as are shown in (2·1) and (2·2). Then one supposes that the ground state energy is singular at $c=0$ and cannot be analytically continued from $c>0$ to $c<0$. But as are shown later there is no singularity at $c=0$ and analytic continuation is possible except for the case of zero magnetization.

Let us consider the power series expansion of $f$ by $n_t$. Yang5) and Suzuki4) derived from the integral equations (2·1):

$$f(x, n-x; c) = \frac{\pi^2 n^3}{3} - x \left\{ 2 \left( \frac{c^2}{\pi} + \pi n^2 \right) \tan^{-1} \frac{\pi n}{c} - 2cn \right\}$$
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\[ f(x, n-x; c) = \frac{\pi^2 n^3}{3} - x \left\{ 2 \left( \frac{c^3}{\pi} + \pi n^2 \right) \left( \pi + \tan^{-1} \frac{\pi n}{c} \right) - 2cn \right\} + 4x^2 n \left( \pi + \tan^{-1} \frac{\pi n}{c} \right)^2 + O(x^3) \]  
\( (1.2a) \)

for \( c > 0 \). From (2.2) we have

\[ f(x, n-x; c) = \frac{\pi^2 n^3}{3} - x \left\{ 2 \left( \frac{c^3}{\pi} + \pi n^2 \right) \left( \frac{\pi}{2} - \tan^{-1} \frac{c}{\pi n} \right) - 2cn \right\} + 4x^2 n \left( \frac{\pi}{2} - \tan^{-1} \frac{c}{\pi n} \right)^2 + O(x^3) \]  
\( (1.2b) \)

for \( c < 0 \). Details of this calculation should be given in Appendix B. The first order term of \( x \) coincides with McGuire's result for \( c < 0 \). These formula can be rewritten to a unified form:

\[ f(x, n-x; c) = \frac{\pi^2 n^3}{3} - x \left\{ 2 \left( \frac{c^3}{\pi} + \pi n^2 \right) \left( \frac{\pi}{2} - \tan^{-1} \frac{c}{\pi n} \right) - 2cn \right\} + 4x^2 n \left( \frac{\pi}{2} - \tan^{-1} \frac{c}{\pi n} \right)^2 + O(x^3) \]  
\( (1.2c) \)

The zeroth-, first- and second-order coefficients of \( x \) are analytically continuous at \( c = 0 \) notwithstanding that the two sets of integral equations (2.1) and (2.2) seem to be quite different. This fact is very remarkable. We naturally have the question whether the higher order coefficients can be analytically continued at \( c = 0 \). In § 2 it is shown that the answer is "yes". We derive a unified form of integral equations irrespective of the sign of \( c \) and prove the following theorem.

Theorem 1.

a) \( f(n_i, n_i; c) \) is analytic on the real axis of \( c \) when \( n_i \neq n_i \).

b) \( f \) is analytic on the real axis of \( c \) except for \( c = 0 \) when \( n_i = n_i \).

In § 3 we investigate the ground state energy of one-dimensional Hubbard model using the integral equations of Lieb and Wu. The Hamiltonian is

\[ \mathcal{H} = -\sum_{\langle ij \rangle} \sum_{\sigma} c_{i\sigma}^* c_{j\sigma} + 4U \sum_i n_i n_i. \]  
\( (1.3) \)

Theorem 1 holds also for this system unless the electron number is equal to the atomic site number. We define \( f_H(n_i, n_i; U) \) as the ground state energy per atomic site when numbers of electrons with up spin and down spin per atomic site are \( n_i \) and \( n_i \) respectively. We can regard the electrons in the continuum as the electrons in the lattice which has a very small lattice distance. Therefore we have a simple relation between \( f \) and \( f_H \).

Lemma 1.

\[ f(n_i, n_i; c) = \lim_{d \to 0} d^{-2} \left[ f_H(dn_i, dn_i; cd) + 2d (n_i + n_i) \right]. \]  
\( (1.4) \)

\(^{(*)}\) The analytic continuity of the first order coefficient was pointed out by Mc'Guire.}

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A detailed proof will be given in Appendix A.

§ 2. Proof of theorem 1

A) Integral equations of Gaudin and Yang

In the case $c>0$ the integral equations of Gaudin\(^{(b)}\) and Yang\(^{(b)}\) are

$$2\pi\rho(k) = 1 + \int_{-\infty}^{\infty} \frac{2c\sigma(A) dA}{\epsilon^2 + (k-A)^2}, \quad (2.1a)$$

$$2\pi\sigma(A) + \int_{-\infty}^{\infty} \frac{4c\sigma(A') dA'}{4\epsilon^2 + (A-A')^2} = \int_{-\infty}^{\infty} \frac{2c\rho(k) dk}{\epsilon^2 + (A-k)^2}, \quad (2.1b)$$

where $B$ and $Q$ are determined by

$$n = \int_{-\infty}^{\infty} \rho(k) dk, \quad (2.1c)$$

$$x = \int_{-\infty}^{\infty} \sigma(A) dA, \quad (2.1d)$$

and the ground state energy is given by

$$f(x, n-x; c) = \int_{-\infty}^{\infty} k^2\rho(k) dk. \quad (2.1e)$$

On the other hand the integral equations for $c<0$\(^{(b)}\) are

$$2\pi\rho(k) = 1 + \int_{-\infty}^{\infty} \frac{2c\sigma(A) dA}{\epsilon^2 + (k-A)^2}, \quad (2.2a)$$

$$2\pi\sigma(A) - \int_{-\infty}^{\infty} \frac{4c\sigma(A') dA'}{4\epsilon^2 + (A-A')^2} = 2 + \int_{-\infty}^{\infty} \frac{2c\rho(k) dk}{\epsilon^2 + (A-k)^2}, \quad (2.2b)$$

$$n = \int_{-\infty}^{\infty} \rho(k) dk + 2 \int_{-\infty}^{\infty} \sigma(A) dA, \quad (2.2c)$$

$$x = \int_{-\infty}^{\infty} \sigma(A) dA, \quad (2.2d)$$

$$f(x, n-x; c) = \int_{-\infty}^{\infty} k^2\rho(k) dk + \int_{-\infty}^{\infty} 2(A^2 - c^2)\sigma(A) dA. \quad (2.2e)$$

B) A unified form of integral equations

Using (2.1a) or (2.2a) we can eliminate $\rho(k)$. With respect to the relation

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\(^{(b)}\) One can easily derive these equations from (1·4) and (3·2). Put $A\rightarrow dA$, $k\rightarrow dk$ and $U\rightarrow cd$. This derivation is very simple. But Gaudin gave a more orthodox derivation using exact wave function. For details see his thesis or "Excitonic Insulator in One Dimension", M. Takahashi, Prog. Theor. Phys. 43 (1970), 917, Appendix B.
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\[
\frac{4|c|}{4c^2 + (A-A')^2} = \frac{2c^2}{\pi} \int_{-\infty}^{\infty} \frac{dx}{\{c^2 + (A-x)^2\} \{c^2 + (x-A')^2\}},
\]

we obtain an integral equation for \( \sigma(A) \),

\[
\sigma(A) + \int_{-B}^{B} K(A, A') \sigma(A') dA' = \frac{1}{2\pi} g(A), \tag{2.3a}
\]

where

\[
K(A, A') = \frac{c^2}{\pi^3} \int_{|-Q|}^{Q} \frac{dx}{\{c^2 + (A-x)^2\} \{c^2 + (x-A')^2\}},
\]

\[
g(A) = \begin{cases} 
\frac{1}{\pi} \left( \frac{\tan^{-1}Q+A}{c} + \frac{\tan^{-1}Q-A}{c} \right) & \text{for } c > 0, \\
2 + \frac{1}{\pi} \left( \frac{\tan^{-1}Q+A}{c} + \frac{\tan^{-1}Q-A}{c} \right) & \text{for } c < 0,
\end{cases}
\]

or

\[
= \frac{1}{\pi} \text{Im} \ln \left( \frac{A+Q+ic}{A-Q-ic} \right),
\]

where we put \( 2\pi > \text{Im} \ln (\cdots) \geq 0 \). Equations (2.1c) and (2.2c) are rewritten as

\[
n(\pi) = \frac{Q}{\pi} + \int_{-B}^{B} dA g(A) \sigma(A). \tag{2.3b}
\]

Equations (2.1e) and (2.2e) are rewritten as

\[
f = \frac{Q^3}{3\pi} + \int_{-B}^{B} \frac{dA}{\pi} \left\{ 2Qc - Ac \log \frac{c^2 + (Q+A)^2}{c^2 + (Q-A)^2} + \pi (A^2 - c^2) g(A) \right\} \sigma(A). \tag{2.3c}
\]

One can write (2.3a) as follows:

\[
2\pi \sigma = g - Kg + KKg - KKKg + \cdots,
\]

where \( K \) is an operator defined by

\[
K \cdots = \int_{-B}^{B} dA' K(A, A') \cdots.
\]

This series is uniformly convergent in a wider sense and each term is analytic as a function of \( Q, B, c \) and \( A \) \((-B \leq A \leq B\) except the region

\[
c = 0, \quad Q < B. \quad (c = 0, x = n/2) \tag{2.4}
\]

Lemma 2. \( f, n \) and \( x \) are analytic as functions of \( Q, B \) and \( c \) except the region (2.4).

Consider the transformation from \((Q, B, c)\) space to \((n, x, c)\) space.

Lemma 3. Except for the region
\[ c \geq 0, \quad x = n/2, \quad (2.5) \]
\[ D = \frac{\partial (n, x, c)}{\partial (Q, B, c)} = \frac{\partial n}{\partial Q} \frac{\partial x}{\partial B} - \frac{\partial x}{\partial Q} \frac{\partial n}{\partial B} > 0. \quad (2.6) \]

In the region \((2.5)\) \(D\) equals zero.

The proof will be given in Appendix C. This means that the transformation from \((Q, B, c)\) space to \((n, x, c)\) space is one to one mapping.

C) **Proof of theorem 1**

One can easily prove that
\[
\frac{\partial^i f(x, n-x; c)}{\partial c^i} = \sum_{i,j,k \geq 0} \frac{i!}{i!} \frac{j!}{j!} \frac{k!}{k!} \left( \frac{\partial Q}{\partial c} \right)_{n,x} \left( \frac{\partial B}{\partial c} \right)_{n,x} \frac{\partial^i f(Q, B; c)}{\partial c^i \partial Q^j \partial B^k} ,
\]
where
\[
\left( \frac{\partial Q}{\partial c} \right)_{n,x} = \left( \frac{\partial n}{\partial B} \frac{\partial x}{\partial c} - \frac{\partial x}{\partial B} \frac{\partial n}{\partial c} \right) / D , \quad \left( \frac{\partial B}{\partial c} \right)_{n,x} = \left( \frac{\partial n}{\partial Q} \frac{\partial x}{\partial c} - \frac{\partial x}{\partial Q} \frac{\partial n}{\partial c} \right) / D .
\]

From lemma 2 and lemma 3 we have that \(\partial^i f / \partial c^i\) is finite except for \(c \geq 0\) and \(x = n/2\).

In the case \(c \geq 0\), and \(x = n/2\) one can show
\[
B = \infty, \quad Q = \text{finite and } D = 0 . \quad (2.7)
\]

Then in this case it is possible that \(\partial^i f / \partial c^i\) diverges. But we prove that it is not the case except for the limit \(c \to 0\). Using the Fourier transformation we obtain for \(B = \infty\) and \(c \geq 0\):
\[
\rho(k) = \int_{-\infty}^{\infty} R \left( \frac{k-k'}{c} \right) \rho(k') dk' = \frac{1}{2\pi} ,
\]
where
\[
R(x) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{1}{1 + (x-y)^2} \text{sech} \frac{\pi y}{2} dy .
\]
Integration kernel is analytic except for \(c = 0\). Then \(f\) and \(n\) are analytic as functions of \(Q\) and \(c\). Considering \(\partial n / \partial Q > 0\) we see that \(\partial^i f / \partial c^i\) is finite except for \(c = 0\). [Q.E.D.]

We propose the following conjecture.

**Conjecture 1:** \(f\) is singular at \(c = 0\) when \(n_1 = n_j\).

If this conjecture is true we should recognize that the circumstances are very similar to those of the phase transitions of the Ising ferromagnet. In this case the free energy as a function of temperature is analytic in the non-zero magnetic field and singular at curie temperature in zero magnetic field.
§ 3. Hubbard model

Lieb and Wu\(^6\) applied the method of Gaudin and Yang to the one-dimensional Hubbard model described by the Hamiltonian (1·3). Corresponding to theorem 1 we prove the following theorem.

Theorem 2.

a) In the case \( n \neq 1 \), \( f_H \) is analytic on the real axis of \( U \) if \( n_1 \neq n_\cdot \)

b) In the case \( n \neq 1 \), \( f_H \) is analytic except for \( U = 0 \) if \( n_1 = n_\cdot \).

And we propose the following conjecture.

Conjecture 2: \( f_H \) is singular at \( U = 0 \) when \( n_1 = n_\cdot \).

Integral equations for \( U > 0 \) and \( n_1 + n_\cdot \leq 1 \) are

\[
2\pi \rho(k) = 1 + \cos k \int_{-\pi}^{\pi} \frac{2U\sigma(A) dA}{\sqrt{4U^2 + (\sin k - A)^2}},
\]

\[
2\pi \sigma(A) + \int_{-\pi}^{\pi} \frac{4U\sigma(A') dA'}{2U^2 + (A - A')^2} = \int_{-\pi}^{\pi} \frac{2U\rho(k) dk}{\sqrt{4U^2 + (\sin k - A)^2}},
\]

\[
n_1 + n_\cdot = \int_{\pi}^{\pi} \rho(k) dk,
\]

\[
n_1 = \int_{-\pi}^{\pi} \sigma(A) dA,
\]

\[
f_H(n_1, n_\cdot; U) = -\int_{\pi}^{\pi} 2 \cos k\rho(k) dk.
\]

Integral equations for \( U < 0 \) are

\[
2\pi \rho(k) = 1 + \cos k \int_{-\pi}^{\pi} \frac{2U\sigma(A) dA}{\sqrt{4U^2 + (\sin k - A)^2}},
\]

\[
2\pi \sigma(A) - \int_{-\pi}^{\pi} \frac{4U\sigma(A') dA'}{2U^2 + (A - A')^2} = \Re \frac{1}{\sqrt{1 -(A - Ui)^2}} + \int_{-\pi}^{\pi} \frac{2U\rho(k) dk}{\sqrt{4U^2 + (\sin k - A)^2}},
\]

\[
n_1 + n_\cdot = \int_{\pi}^{\pi} \rho(k) dk + 2 \int_{-\pi}^{\pi} \sigma(A) dA,
\]

\[
n_1 = \int_{-\pi}^{\pi} \sigma(A) dA,
\]

\[
f_H(n_1, n_\cdot; U) = -2 \int_{-\pi}^{\pi} \cos k\rho(k) dk - 4 \int_{-\pi}^{\pi} \Re \frac{1}{\sqrt{1 -(A - Ui)^2}} \sigma(A) dA,
\]

in replace of (3·1b), (3·1c), (3·1e). Equations (3·1a) and (3·1d) hold also in this case. These equations can be rewritten as follows:

\[
\sigma(A) + \int_{-\pi}^{\pi} K(A, A') \sigma(A') dA'
\]
\[ \frac{1}{2\pi^2} \text{Im} \left[ \frac{1}{\sqrt{1-(A-Ui)^2}} \ln \left\{ \left( \frac{A-Ui}{\sqrt{1-(A-Ui)^2}} + \tan Q \right) \left( \frac{A-Ui}{\sqrt{1-(A-Ui)^2}} - \tan Q \right) \right\} \right], \]

(3.3a)

\[ K(A, A') = \frac{U^3}{\pi^2} \int_{|x| \sin q} dx \frac{(A-x)^3 + U^3}{((x-A')^3 + U^3)}, \]

(3.3b)

\[ n = \frac{Q}{\pi} + \int_{-n}^{n} \frac{1}{\sigma(A)} \text{Im} \left\{ \ln \left( \frac{A-Ui + \sin Q}{A-Ui - \sin Q} \right) \right\} dA, \]

(3.3c)

\[ x = \int_{-n}^{n} \sigma(A) dA, \]

(3.3d)

\[ f_H = -\frac{2}{\pi} \sin Q + \int_{-n}^{n} \frac{4QU}{\pi} - \frac{2}{\pi} \text{Im} \left\{ \frac{1}{\sqrt{1-(A-Ui)^2}} \right\} \ln \left( \left( \frac{A-Ui}{\sqrt{1-(A-Ui)^2}} + \tan Q \right) \left( \frac{A-Ui}{\sqrt{1-(A-Ui)^2}} - \tan Q \right) \right) \sigma(A) dA. \]

(3.3e)

Here we take the branches of root and logarithm as \( \pi/2 > \text{Arg} ... > -\pi/2 \) and \( 2\pi > \text{Im} \ln(\cdots) > 0 \). If \( |A| < \sin Q \) the arguments of logarithm do not cross the branch cut when \( U \) changes its sign. Therefore except for the region

\[ U = 0, \quad \sin Q < B, \quad (U = 0, x = n/2 \text{ or } n = 1) \]

(3.4)

\( f_H, n \) and \( x \) are analytic as functions of \( B, Q \) and \( U \).

Corresponding to lemma 3 we have lemma 4.

Lemma 4. Except for the region

\[ U \geq 0, \quad x = n/2 \quad \text{and} \quad U \leq 0, \quad n = 1, \]

(3.5)

\[ D = \frac{\partial n}{\partial Q} \frac{\partial x}{\partial B} - \frac{\partial n}{\partial B} \frac{\partial x}{\partial Q} > 0. \]

(3.6)

In the region (3.5) \( D \) equals zero.

Then except for the region (3.5) \( f_H(x, n-x; U) \) is analytic as a function of \( U \).

Proof of theorem 2b) can be made in the same way as in theorem 1b).

From (3.1) Suzuki derived the expansion of \( f_H \) with \( n \) for \( U > 0 \):

\[ f_H(x, n-x; U) = -\frac{2}{\pi} \sin \pi n + x \left[ 4U n - \frac{4}{\pi} \sqrt{U^2 + 1} \tan^{-1} \left( \frac{\sqrt{U^2 + 1} \tan \pi n}{U} \right) \right. \]

\[ + \frac{4}{\pi} \cos \pi n \tan^{-1} \left( \frac{\sin \pi n}{U} \right) \left] + \frac{4x^2}{\pi} \sin \pi n \left( \tan^{-1} \frac{\sin \pi n}{U} \right)^2 + O(x^3). \right. \]

(3.7a)

Using Theorem 2 we have the expansion which holds for arbitrary value of \( U \) corresponding to (1.2c):

\[ f_H(x, n-x; U) = -\frac{2}{\pi} \sin \pi n + x \left[ 4U n - \frac{4}{\pi} \sqrt{U^2 + 1} \tan^{-1} \left( \frac{\pi}{2} - \tan^{-1} \frac{U}{\sqrt{U^2 + 1} \tan \pi n} \right) \right. \]

(3.7b)
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\[ + \frac{4}{\pi} \cos \pi n \left( \frac{\pi}{2} - \tan^{-1} \frac{U}{\sin \pi n} \right) + \frac{4x^n}{\pi} \sin \pi n \left( \frac{\pi}{2} - \tan^{-1} \frac{U}{\sin \pi n} \right)^2 + O(x^n). \]

(3.7b)

The expansion to the third order of \( x \) at \( n=1 \) was obtained by the author:

\[ f_H(x, 1-x; U) = \begin{cases} 4(\sqrt{1+U^2}-U)x \pi^2 \frac{1}{\sqrt{U^2+1}}x^3 + O(x^4), & U>0, \\ 4(1-U)x + \frac{8\pi^2}{3}x^3 + O(x^4), & U<0. \end{cases} \]

(3.8)

It is apparent that each of the coefficients cannot be analytically continued at \( U=0 \). This leads us to the following conjecture.

Conjecture 3: In the case \( n=1 \), \( f_H \) is singular at \( U=0 \).

§ 4. Discussion

Perhaps the most interesting result of the present work is that the two cases \( e>0 \) (\( U>0 \)) and \( e<0 \) (\( U<0 \)) can be treated in unified integral equations (2.3) (3.3). This explains the analytic continuity of \( (\partial f/\partial x)|_{x=\pm} \) at \( e=0 \) (\( U=0 \)) which was appointed by Mc’Guire.

Theorem 1 guarantees that the perturbation expansion of \( f(n_r, n_i; c) \) with \( c \) does not show divergence when \( n_r \neq n_i \). Theorem 2 guarantees the same fact when \( n_r \neq n_i \) and \( n \neq 1 \). This does not hold for a three-dimensional electron gas interacting via Coulomb potential in the positive charge back ground. In this case the second order term of the perturbation expansion diverges. But we can regard the Coulomb potential as a special case. Because in the case \( c<0 \) the problem is equivalent to the gravitation and we cannot take the thermodynamic limit. A rough estimation shows that the ground state energy is proportional to \( N^{7/4} \) and not to \( N \) for fixed density if fermions interact via gravitational potential. It is very probable that the ground state energy for non-zero magnetization is analytic at \( c=0 \) for other kinds of potential if the thermodynamic limit exists at \( c>0 \) and \( c<0 \).

We also investigated \( f \) and \( f_H \) as functions of magnetization \( s = (n_1 - n_i)/2 \). These functions are singular at \( s=0 \) and analytic at \( n/2 \leq |s| > 0 \). The details will be given in a later paper.

Appendix A

Proof of lemma 1

We put \( d \) is the lattice distance and \( L \) is the length of the system. Equations (1.1) and (1.3) can be rewritten as

\[ (1.1) = \sum_{k, \xi} k^2 c_{k_\xi} c_{k_\xi} + \frac{4c}{L} \sum_{k_1, k_2, k_3} c_{k_1} c_{k_2} c_{k_3} c_{k_4} \delta (k_1 + k_2 - k_3 - k_4), \]

\[ (1.3) = -2 \cos dk c_{k_\xi} c_{k_\xi} + \frac{4Ut}{L} \sum_{k_1, k_2, k_3} c_{k_1} c_{k_2} c_{k_3} c_{k_4} \delta (k_1 + k_2 - k_3 - k_4), \]
where $k = 2\pi \times \text{integer}/L$ and

$$\delta(x) = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$A(x) = \begin{cases} 1 & \text{for } x = \text{integer } \times 2\pi/d \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$(1.1) = \lim_{d \to 0} d^{-2} [(1.3) + 2 \sum_{k \neq 0} c_k c_{-k}],$$

when $U = cd$. Then we obtain lemma 1. [Q.E.D.]

**Appendix B**

Here we derive (1.2b). Put $B \ll |c|$ and $Q$. From (2.2) we have

$$x = \int_{-\pi}^{\pi} \sigma(A) \, dA = 2B\delta(0) + O(B^2), \quad (B1)$$

$$n = \frac{Q}{\pi} + \frac{4B\delta(0)}{\pi} \left[ \pi + \tan^{-1} \frac{Q}{c} \right] + O(B^2), \quad (B2)$$

$$f = \frac{Q^3}{3\pi} + \frac{4B\delta(0)}{\pi} \left[ Qc - \frac{c^3}{\pi} \left( \pi + \tan^{-1} \frac{Q}{c} \right) \right] + O(B^2). \quad (B3)$$

Eliminating $2B\delta(0)$ in (B2) and (B3) one has

$$Q = n - 2x \left[ \pi + \tan^{-1} \frac{Q}{c} \right] + O(x^3), \quad (B4)$$

$$f = \frac{Q^3}{3\pi} + \frac{x}{\pi} \left[ 2Qc - \frac{2c^3}{\pi} \left( \pi + \tan^{-1} \frac{Q}{c} \right) \right] + O(x^3). \quad (B5)$$

Using (B4) one can eliminate $Q$:

$$f = \frac{\pi^2 n^3}{3} - x \left[ 2 \left( \pi n^2 + \frac{c^3}{\pi} \right) \left( \pi + \tan^{-1} \frac{Q}{c} \right) - 2cn \right] + x^3 \left[ 4n \left( \pi + \tan^{-1} \frac{Q}{c} \right)^2 - \frac{4c}{\pi} \left( \pi + \tan^{-1} \frac{Q}{c} \right) \right] + O(x^3). \quad (B6)$$

The first order term of $x$ can be rewritten as

$$-x \left[ 2 \left( \pi n^2 + \frac{c^3}{\pi} \right) \left( \pi + \tan^{-1} \frac{\pi n}{c} \right) - 2x \left( \frac{d}{dQ} \tan^{-1} \frac{Q}{c} \right) \left( \pi + \tan^{-1} \frac{Q}{c} \right) - 2cn \right]$$

$$= -x \left[ 2 \left( \pi n^2 + \frac{c^3}{\pi} \right) \left( \pi + \tan^{-1} \frac{\pi n}{c} \right) - 2cn \right] + \frac{4c^2 x^3}{\pi} \left( \pi + \tan^{-1} \frac{Q}{c} \right) + O(x^3).$$

Then finally we have
Appendix C

Proof of lemmas 3 and 4

At first we prove lemma 3. From (2.1) one has for \( c \geq 0 \)

\[
\frac{\partial n}{\partial Q} = 2\rho(Q) + \int_{-B}^{B} \sigma_q(A) dA, \quad \frac{\partial n}{\partial B} = \int_{-B}^{B} \sigma_b(A) dA,
\]

and from (2.2) for \( c \leq 0 \)

\[
\frac{\partial n}{\partial Q} = 2\rho(Q) - \int_{-B}^{B} (2 - h(A)) \sigma_q(A) dA, \quad \frac{\partial n}{\partial B} = \int_{-B}^{B} (2 - h(A)) \sigma_b(A) dA,
\]

\[
\frac{\partial x}{\partial Q} = -\int_{-B}^{B} \sigma_q(A) dA, \quad \frac{\partial x}{\partial B} = \int_{-B}^{B} \sigma_b(A) dA.
\]

(C1)

Here

\[
h(A) = \int_{-q}^{q} dx \frac{|c|}{\pi} \frac{1}{c^2 + (A-x)^2},
\]

(C3)

\[
\sigma_b(A) = \sigma(B) \{ R(A, B - \varepsilon) + R(A, -B + \varepsilon) \},
\]

(C4)

\[
\sigma_q(A) = \frac{|c|}{\pi} \rho(Q) \int_{-B}^{B} dA' R(A, A') \left\{ \frac{1}{c^2 + (A'-Q)^2} + \frac{1}{c^2 + (A'+Q)^2} \right\},
\]

(C5)

and \( R \) is the integration kernel which is defined by

\[
R(A, A') = \int_{-B}^{B} \left\{ \int_{x>q} dx \frac{c^2}{\pi} \left\{ \frac{1}{c^2 + (A-x)^2} \right\} \left\{ \frac{1}{c^2 + (x-A')^2} \right\} \right\} \times R(A'', A') dA'' = \delta(A-A').
\]

Then one obtains for \( c \geq 0 \) and \( c < 0 \)

\[
D = \frac{\partial n}{\partial Q} \frac{\partial x}{\partial B} - \frac{\partial n}{\partial Q} \frac{\partial x}{\partial B} = 4\rho(Q) \sigma(B)
\]

\[
\times \left[ \left( 1 + \int_{-B}^{B} \frac{J(A)}{\pi} \frac{|c|dA}{c^2 + (A-Q)^2} \right) L(B) - \left( \int_{-B}^{B} \frac{L(A)}{\pi} \frac{|c|dA}{c^2 + (A-Q)^2} \right) J(B) \right],
\]

(C6)

where

\[
L(A) = \int_{-B}^{B} R(A, A') dA', \quad J(A) = \int_{-B}^{B} R(A, A') h(A') dA'.
\]
In the region (2·5), $\sigma(B)$ is equal to zero, namely, $D$ is zero. On the outside of this region one can show easily
\[ 1 + \int_{-b}^{b} \frac{J(A)}{\pi} \frac{|c|dA}{c^2 + (A - Q)^2} > \int_{-b}^{b} \frac{L(A)}{\pi} \frac{|c|dA}{c^2 + (A - Q)^2} > 0, \]
and
\[ L(B) > J(B) > 0, \]
and
\[ \sigma(B) > 0, \quad \rho(Q) > 0. \]
Then we have $D > 0$.
The proof of lemma 4 can be made in the parallel way.

References