Linear Response Theory in the Vortex State of Dirty Type-II Superconductors

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The vortex motion and the flux flow resistivity in dirty type-II superconductors are re-examined from the point of view of the linear response theory with respect to the applied electric field. A time dependent Ginzburg-Landau equation for the order parameter is different from that given by Caroli and Maki. The ordering of non-commutative operators should have been improper in their work. The time dependent order parameter shows the uniform flux flow. The resistivity is calculated in the gauge invariant way. It has a temperature dependence different from that by Caroli and Maki. Other transport coefficients in the vortex state are also reexamined. Thermal conductivity and the thermomagnetic coefficient agree with the results by Caroli and Maki, but the Hall coefficient does not. The microwave surface resistance, calculated numerically, is in agreement with the experiment by Fischer et al.

§ 1. Introduction

It is well known that transport phenomena in the vortex states of type-II superconductors often show behaviors very different from those of ordinary superconductors. If a type-II superconductor is under a magnetic field, $H$, whose direction is taken as $z$ axis, it is in the vortex state when $H_{c1} < H < H_{c2}$, where $H_{c1}$ and $H_{c2}$ are the lower and the upper critical fields, respectively. An electric field $E$ or a temperature gradient $dT$, applied perpendicularly to $H$ (along the $x$ axis), induces the stationary electric current and thermal current in the $x$ and $y$ directions. One can observe various effects such as Hall effect, Ettingshausen effect, Nernst effect and so on. The observation of the resistive electrical conduction first led us to the concept of the vortex motion. This paper is mainly devoted to the theoretical treatment of the electrical conductivity, i.e. the flux flow resistivity. The arguments naturally tell us how to calculate the other transport coefficients, except for the Hall effect.

Our discussions will be restricted to the case of dirty superconductors and high magnetic fields close to $H_{c2}$. The order parameter $\Delta$ is small and proportional to $(H_{c2} - H)^{1/2}$. Physical quantities will be expanded with respect to $\Delta$ up to its second order. Under an electric field, $\Delta$ is a space and time dependent pair potential. It is determined by a self-consistent equation, called the time dependent Ginzburg-Landau (TDGL) equation. The equation is of a diffusion type, because the excitation spectrum of the quasiparticles is gapless. The internal magnetic and electric fields are the same as the external fields, so far as the lowest order term with respect to $\Delta$ is concerned.
Table I. The works on the flux flow resistivity, which are classified by the gauge of an applied field, used in the explicit calculation. The equations in the present paper are quoted to refer their TDGL equations. Two equations in the respective rows are shown to be gauge covariant (see § 3).

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Several papers have already been presented along these lines. To clarify the purpose of the present paper, we shall shortly review them. They are listed up in Table I. First, Caroli and Maki\(^2\) (hereafter referred to as CMI) have calculated the microwave surface impedance, using the linear response theory. The change of a physical quantity between the normal and superconducting states is proportional to \(|\mathcal{A}|^2\). It is a sum of two parts. One is directly induced by the perturbational field (such as an electric field or a temperature gradient), the order parameter being unchanged from its equilibrium value. The other part arises from a linear modulation of the order parameter, called a fluctuation, induced by the field. For the electric currents, we call the former \(J^u\) and the latter \(J^n\). In their calculation of \(J^n\), an error leads to a curious result, i.e. the conductivity in the configuration \(H \perp E\) diverges in the static limit contrary to the observations.\(^k\) Recently, Thompson\(^k\) has taken account of the diagrams which were neglected in CMI, and has obtained a finite conductivity.

Caroli and Maki have discussed, in another paper\(^6\) (hereafter referred to as CMII), the time dependence of \(\mathcal{A}(r, t)\) induced by a static electric potential \(\phi(r)\). They have introduced the potential dependence simply replacing \(\omega\) by \(\omega - 2e\phi\) in the field-free equation

\[
\left\{ \psi \left( \frac{1}{2} + \frac{D(q - 2eA)^2 - i\omega}{4\pi T} \right) - \psi \left( \frac{1}{2} + \log \frac{T}{T_c} \right) \right\} \mathcal{A}(r, t) = 0.
\]

Here \(\psi(z)\) is the di-gamma function. The symbols \(i\mathbf{q}\) and \(-i\omega\) represent the differential operators \(\nabla\) and \(\partial/\partial t\), respectively. \(A\) is a vector potential. \(D\) is a diffusion constant \(\nu l/3\), \(\nu\) being the Fermi velocity and \(l\) the electron mean free path. Their solution is the eigenfunction of \(\mathcal{A}\) with the lowest eigenvalue, where \(\mathcal{A} = -i\tilde{\omega} + D(q - 2eA)^2\) and \(\tilde{\omega} = \omega - 2e\phi\). It shows explicitly the motion of the vortex pattern. The electric currents have been estimated using this function. But it comes into question why this approach does not lead to the same results as they obtained by using the purely perturbational method in CMI or as Thompson did, even in the weak field limit.
Recently, their treatments are extended by Fischer et al. (hereafter referred to as FM) to the problem of the finite frequency conductivity, using a different gauge of the potentials. The microwave surface impedance, thus obtained, differs from those by CMI and Thompson in a numerical factor. According to FM, the approximation in CMI might be a “sudden” switched-on perturbation and the one in CMII or FM would be an “adiabatic” perturbation, and only the latter should be appropriate for treating a bodily motion of the vortex pattern.

However, these terminologies lead to a needless confusion. In this paper, we shall show that the procedure of the introduction of electric potentials in CMII or FM are wrong, and the approach in CMI can intrinsically include the motion of the vortices. It is appropriate to treat the flux flow phenomena contrary to the interpretations in CMII. Since the flux flow resistivity seems to be purely ohmic and has no singularity when \( E \to 0 \), if the pinning effect is neglected, the linear response theory with respect to \( E \) should be well applied to the flux flow phenomena. It must also be pointed out that the procedure in CMII, the introduction of \( \phi \), may be ambiguous. In the slow modulation limit, the first term of Eq. (1) may be replaced by

\[
\phi \left( \frac{1}{2} + \frac{D(q-2eA)^2}{4\pi T} \right) - i\omega \frac{1}{4\pi T} \phi(1) \left( \frac{1}{2} + \frac{D(q-2eA)^2}{4\pi T} \right),
\]

where \( \phi(1) \) is the tri-gamma function. Replacement of \( \omega \) by \( \bar{\omega} \) gives a TDGL equation different from that of CMII, because the operators \( \bar{\omega} \) and \( (q-2eA)^2 \) do not commute with each other. In order to find a correct ordering of operators, we shall retain the wave number and frequency dependence of the scalar potential during the calculations, and finally take the static and uniform limit.

In § 2, we shall calculate the linear response of the order parameter \( \mathcal{A}(r, t) \), and obtain the TDGL equation up to the linear order of the perturbational field \( \phi \). In § 3 it is shown that the solution of this TDGL equation shows explicitly the vortex motion, qualitatively similar to that in CMII. However, there is a quantitative difference. Section 4 is devoted to calculations of the flux flow resistivity and the microwave surface impedance. The obtained results turn out to be identical with that by Thompson. The numerical result for the surface impedance is compared with the experiments by FM. The behavior of the microwave surface impedance is interesting at temperatures near \( T_c \), where the vortex motion cannot follow the rapid change of the applied field. In § 5 the physical interpretation of the obtained TDGL equation is discussed in connection with the gauge invariance of the transport currents. The general proof of the gauge invariance is performed in Appendix A. We also discuss in § 5 the heat transport by the procedure of CMI. We need the TDGL equation under a

\[^{a)}\text{The results have been reported recently in reference 6. Equations (5) and (9) in reference 6 involve some errors of numerical factors. They are corrected by the corresponding equations (27) and (37) in this paper.}\]
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The resulting transport coefficients turn out to be identical with those in CMII. Discussions of the Hall coefficient will also be given, using our formulation.

§ 2. Time dependent Ginzburg-Landau equation

The total Hamiltonian of the system is given by

\[ \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{BCS}} + \mathcal{H}_{\text{ext}}, \]

where

\[ \mathcal{H}_0 = \sum_e \int \left[ \frac{1}{2m} \left( \mathbf{p} - ieA \right) \psi_e^{\dagger}(x) \cdot \left( \mathbf{p} + ieA \right) \psi_e(x) + U_i(r) \psi_e^{\dagger}(x) \psi_e(x) \right] dr, \]  

\[ \mathcal{H}_{\text{BCS}} = g \int \Phi^{\dagger}(x) \Phi(x) dr, \]  

\[ \mathcal{H}_{\text{ext}} = \int \phi(x) \phi(x) dr. \]

Here, \( \Phi^{\dagger} = \psi^{\dagger}_e \psi^{\dagger}_i \), \( \Phi = \psi \phi \), \( \rho = -e \sum_e \phi_e^{\dagger} \phi_e \) and \( \psi_e^{\dagger}, \psi_e \) are the electron field operators. In Eq. (4), \( \mathbf{A} = (0, Hx, 0) \) is a static vector potential which gives rise to the equilibrium Abrikosov structure of vortices. \( U_i(r) \) is a potential due to impurities distributed at random. The BCS interaction is written as Eq. (5), \( g(\langle 0 \rangle) \) being a coupling constant. The approximation for \( \mathcal{H}_{\text{BCS}} \), done in Eq. (5') is equivalent to the Gorkov decoupling for Green's functions. The order parameter \( \Delta \) is defined self-consistently as \( \langle 0 \rangle = g(\langle \Phi^{\dagger} \phi \rangle) \), where \( \langle O \rangle \) is an ensemble average of the operator \( O \). \( \mathcal{H}_{\text{ext}} \) is an interaction due to the external electric field which is expressed by the scalar potential \( \phi = -Ex \).

In equilibrium \((\mathcal{H}_{\text{ext}} = 0)\), the Abrikosov state is realized, and \( \Delta(x) \) is given by

\[ \Delta^{\dagger}(r) = \sum_n C_n \exp \left[ -ikny - eH \left( x + \frac{kn_{\text{ext}}}{2eH} \right) \right], \]

which is time independent. In the non-equilibrium state \((\mathcal{H}_{\text{ext}} \neq 0)\), \( \Delta(x) \) varies with time. The response of \( \Delta \) to \( \mathcal{H}_{\text{BCS}} \) and \( \mathcal{H}_{\text{ext}} \) takes the form

\[ \Delta^{\dagger}(\mathbf{q}, \omega) = g \langle \Phi^{\dagger} \phi \rangle \Delta^{\dagger}(\mathbf{q}, \omega) + g \langle \Phi^{\dagger} \phi \rangle \Delta^{\dagger}(\mathbf{q}, \omega) \Delta^{\dagger}(\mathbf{q} + k, \omega), \]

where \( \mathbf{q}_1 + k = \mathbf{q} \). Each response function, expressed as \( \langle AA; BB, \cdots \rangle \), is defined as the ensemble average of a retarded commutator in the normal state without the electric field. They are related to the following thermal Green's functions:

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8) In the present paper the electronic change is written as \(-e\), and \( \hbar = c = k_B = 1 \). Note that in Eqs. (4) to (6) \( x \) means \((r, t)\), showing the explicit time dependence of the pair potential \( \Delta \).
\begin{equation}
\langle \mathcal{F}^\dagger; \mathcal{F} \rangle_{q, n} = \left\{ -\frac{1}{\beta} \int_0^\beta d\tau d\tau' \langle T, \mathcal{F}^\dagger(q, \tau) \mathcal{F}(-q, \tau') \rangle \times \exp \left[ i \omega_\nu (\tau - \tau') \right] \right\}_{\lambda_{\nu} = \nu + \frac{i}{\lambda}}. 
\end{equation}

\begin{equation}
\langle \mathcal{F}^\dagger; \mathcal{F}, \rho \rangle_{q, k, n, \nu_2} = \left\{ \frac{1}{\beta} \int_0^\beta d\tau d\tau' \int_0^\beta d\tau'' \langle T, \mathcal{F}^\dagger(q, \tau) \mathcal{F}(-q, \tau') \rho(-k, \tau'') \rangle \exp \left[ i \omega_\nu (\tau - \tau') + i \omega_\kappa (\tau - \tau'') \right] \right\}_{\lambda_{\nu} = \nu + \frac{i}{\lambda}}. 
\end{equation}

Here the \( \omega_\nu \)'s take the values \( 2\mu \pi T \) (\( \mu \): positive integers) and \( \beta = 1/T \).

With the aid of the static GL equation\textsuperscript{\textsuperscript{b}}

\begin{equation}
\Delta_\nu(r) = g \langle \mathcal{F}^\dagger \rangle_{q, n} \Delta_\nu(r),
\end{equation}

the deviation of the order parameter from its equilibrium value, \( \delta \mathcal{A}(x) = \mathcal{A}(x) - \Delta_\nu(x) \), is given by

\begin{equation}
\delta \mathcal{A}(q, \omega) = g \langle \mathcal{F}^\dagger; \mathcal{F} \rangle_{q, \nu} \delta \mathcal{A}(q, \omega) + g \langle \mathcal{F}^\dagger; \mathcal{F}, \rho \rangle_{q, k, \nu} \Delta_\nu(q) \phi(k, \omega). 
\end{equation}

Although this equation has been already discussed in CMI, it is worth reexamining, especially the second term of Eq. (12). The error in CMII is in the estimate of the corresponding term. We shall write it as \( P \). It is written as

\begin{equation}
P = 2g e T \sum \frac{dp}{(2\pi)^2} G(p, \epsilon_n) G(-p - q, -\epsilon_n) G(-p - q - k, -\epsilon_n - \omega_\nu) \times \eta(q, 0) \eta(q + k, \omega_\nu) \Gamma_0(k, \omega_\nu) \Delta_\nu(q) \phi(k, \omega_\nu). 
\end{equation}

Here, the one-particle thermal Green's function is given by

\begin{equation}
G(p, \epsilon_n) = \left( i \epsilon_n - \xi_p + \frac{i}{2\tau} \text{sign} \epsilon_n \right)^{-1},
\end{equation}

where \( \epsilon_n = (2n + 1) \pi T \), \( \xi_p = p^2/2m - \mu \), \( \mu \) is the chemical potential, and \( \tau \) is the relaxation time of electrons due to impurity scattering. The impurity correction factors \( \eta \) and \( \Gamma_0 \) are given by\textsuperscript{\textsuperscript{9},\textsuperscript{9},\textsuperscript{10}}

\begin{equation}
\eta(q, \omega_\nu) = \begin{cases} 
\frac{|2\epsilon_n + \omega_\nu| + 1/\tau}{|2\epsilon_n + \omega_\nu| + DQ^2}, & (\epsilon_n(\epsilon_n + \omega_\nu) > 0) \\
1, & (\epsilon_n(\epsilon_n + \omega_\nu) < 0)
\end{cases}
\end{equation}

and

\begin{equation}
\Gamma_0(k, \omega_\nu) = \begin{cases} 
1, & (\epsilon_n(\epsilon_n + \omega_\nu) > 0) \\
|\omega_\nu| + 1/\tau, & (\epsilon_n(\epsilon_n + \omega_\nu) < 0)
\end{cases}
\end{equation}

where \( Q = q - 2eA \). The treatments of the impurity effects are summarized in...
Appendix A. Notice that in Eq. (13), \( q_i \) is abbreviated to \( q \), and \( \omega_0 \) in Eq. (10) is replaced by zero because only the static \( \Delta_0^j \) is under consideration. The graphical representation of Eq. (13) is shown in Fig. 1.

Performing the integration with respect to \( p \), we obtain

\[
P = -ieN(0) g \cdot 4\pi T \left\{ \sum_{\varepsilon_n > -\omega_0} \frac{1}{(2\varepsilon_n + DQ^3)(2\varepsilon_n + \omega_0 + D(k + Q)^2)} \right. \\
+ \sum_{\varepsilon_n < -\omega_0} \frac{1}{(2\varepsilon_n - DQ^3)(\omega_0 + Dk^2)} \\
- \sum_{\varepsilon_n < -\omega_0} \frac{1}{(2\varepsilon_n - DQ^3)(2\varepsilon_n + \omega_0 - D(k + Q)^2)} \right\} \Delta^j_0(q) \phi(k, \omega_0),
\]

where \( N(0) \) is the electronic density of states at the Fermi energy. In Eq. (17), the operator \( D(k + Q)^2 \) operates both on \( \Delta^j_0 \) and \( \phi \), whereas \( DQ^3 \) does only on \( \Delta^j_0 \). Which term in Eq. (17) contributes mainly? This is connected with the proper order of operators discussed in § 1. We perform the \( n \)-summation, and then, replace \( i\omega_0 \) by \( \omega + i\eta \). Making use of \( DQ^3 \Delta^j_0 = \varepsilon_0 \Delta^j_0 \), \( \varepsilon_0 \) being \( 2D_0eH_0 \), we expand the result by \( \omega/\text{Max}(T, \varepsilon_0) \), because we are interested in the limit \( \omega \to 0 \). Then the main contribution comes from the second term of Eq. (17).

\[
P = 2ieN(0) g \frac{1}{4\pi T} \phi^{(0)}(\frac{1}{2} + DQ^3) \Delta^j_0(q) \phi(k, 0),
\]

which includes only the operator \( DQ^3 \). In deriving Eq. (18), we put \( ik\phi(k) = E \neq 0 \) and \( k^3\phi(k) = 0 \), because space charge is zero in the system so far as the zeroth order of \( |\Delta|^2 \) is concerned. Substituting Eq. (18) into Eq. (8) and combining it with Eq. (11), we obtain a linearized TDGL equation:

\[
\left\{ \phi\left(\frac{1}{2} + \frac{DQ^3 - i\omega}{4\pi T}\right) - \phi\left(\frac{1}{2}\right) + \log \frac{T}{T_c} + \frac{2ie\phi(r)}{4\pi T} \phi^{(0)}\left(\frac{1}{2} + \frac{DQ^3}{4\pi T}\right) \right\} \Delta^j(r, t) = 0.
\]

Equation (19) shows the proper order of operators; \( DQ^3 \) in \( \phi^{(0)} \) operates only on \( \Delta^j_0 \).

Now we will compare Eq. (19) with the corresponding TDGL equation obtained by CMII:

\[
\left\{ \phi\left(\frac{1}{2} + \frac{DQ^3 - i\omega + 2ie\phi(r)}{4\pi T}\right) - \phi\left(\frac{1}{2}\right) + \log \frac{T}{T_c} \right\} \Delta^j(r, t) = 0.
\]

Its linear order terms with respect to \( \phi \) become

Fig. 1. The graphical representation of \( P \), Eq. (14). The impurity corrections \( \tau \) and \( \Gamma_0 \) are expressed by the black and white circles, respectively.
This does not reduce to a correct result, the last term of Eq. (19), on account of the fact that \[ [Q^3, -E_x] = iEQ_x' \neq 0. \] Thus, Eq. (20) is incorrect even in the linear order term of \( \phi \). Exact calculations of the \( n \)-th order term of \( \phi \) lead to the result

\[
\frac{1}{n!} \left( \frac{-i\omega + 2i\psi}{4\pi T} \right)^n \phi^{(n)} \left( \frac{1}{2} + \frac{DQ^3}{4\pi T} \right) A^\dagger.
\]

From Eq. (22) it is clear that our result is gauge invariant in each order of \( \phi \) and that the expansion parameters are \( \omega / \text{Max}(T, \varepsilon_0) \) and \( 2\epsilon \psi / \text{Max}(T, \varepsilon_0) \).

The difference between Eqs. (19) and (20) is essential in the calculation of transport currents. As is already mentioned in §1, if Eq. (20) is assumed the operator \( A \) becomes fundamental and the procedure of CMII is appropriate. From Eq. (19), however, we have to regard \( DQ^3 \) instead of \( A \) as a fundamental operator and to treat \( i\omega - 2i\psi \) perturbationally. This procedure is that of CMI, and is a correct one for the analysis of the flux flow states.

For the proper order of operators, the result by FM is worth noting. They have introduced an electric field by a vector potential \( A' (t) = (-E_t, 0, 0) \). Their TDGL equation

\[
\left\{ \psi \left( \frac{1}{2} + \frac{DQ^3 - i\omega}{4\pi T} \right) - \psi \left( \frac{1}{2} \right) + \log \frac{T}{T_c} \right\} A'(r, t) = 0
\]

is obtained by replacement of \( A \) by \( A + A'(t) \) in Eq. (1). But detailed calculations of the \( A' \)-linear term of the TDGL equation, similar to those of the \( \phi \)-linear term shown just before, lead to the result

\[
\left\{ \psi \left( \frac{1}{2} + \frac{DQ^3 - i\omega}{4\pi T} \right) - \psi \left( \frac{1}{2} \right) + \log \frac{T}{T_c} \right\} A'(r, t)
\]

\[
-8\pi T \sum_{\varepsilon_n} \frac{1}{2\varepsilon_n - 2\varepsilon_0 + DQ^3} D\phi Q \left( \frac{1}{2\varepsilon_n - 2\varepsilon_0 + DQ^3} + \frac{1}{2\varepsilon_n + DQ^3} \right) \cdot A'(t)
\]

\[
\times A'(r, t) = 0.
\]

In this case the \( A' \)-linear term of Eq. (23) does not coincide with the fourth term of Eq. (24), because \( [A'(t), \partial / \partial t] = E \neq 0 \). So it can be concluded that both CMII and FM have made an error in the treatment of the ordering of operators.

§3. Solution of the TDGL equation (flux flow state)

Using the expression for the equilibrium \( A^\dagger (r) \), Eq. (7), we obtain the solution for Eq. (19) as
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\[ A^i(r, t) = \left\{ 1 - i k u t + \frac{iu}{2D} \mathcal{P}(T) \left( x + \frac{k}{2eH} \right) \right\} A^i(r) \]

(25)

\[ \equiv \exp \left[ -ik(y + ut) - eH \left( x + \frac{k}{2eH} - \frac{i u}{4eH} \mathcal{P}(T) \right) \right] \]

(25')

where \( u = E/H \) is the velocity of the flux flow, and

\[ \mathcal{P}(T) = \frac{2 \phi(1/2 + \rho)}{\phi(1/2 + 3\rho) - \phi(1/2 + \rho)} \quad \rho = \frac{\epsilon_0}{4\pi T}. \]

(26)

Here and hereafter we will write only \( n=1 \) component of Eq. (7) for simplicity. It must be noted that Eqs. (25) and (25') are equivalent up to the first order of the expansion parameter given in § 2.

It can be easily checked that Eq. (25) is gauge covariant with the solution of Eq. (24), which corresponds to the equation given in CMI (Eq. (27)). Substituting \( A' = (-Ez, 0, 0) \) into Eq. (24), we obtain the solution

\[ A^i_1(r, t) = \left\{ 1 - 2ieE \left( t - \frac{\mathcal{P}(T)}{2\epsilon_0} \right) \left( x + \frac{k}{2eH} \right) \right\} A^i_1(r) \]

(27)

\[ \equiv \exp \left[ -iky - eH \left( x + \frac{k}{2eH} + iu \left( t - \frac{\mathcal{P}(T)}{2\epsilon_0} \right) \right) \right]. \]

(27')

Equation (27) is related with Eq. (25) by the gauge transformation

\[ A'(r, t) = e^{-Z\chi} A_0(r, t), \]

(28)

with \( \chi = -Ez \). As a matter of course, the gauge covariance of the TDGL equations (19) and (24) can be checked by the same gauge transformation. By the same arguments we can ascertain that Eq. (20), the TDGL equation in CMII, and Eq. (23), that in FM, are gauge covariant.* These relations are listed up in Table I.

Equation (25') shows explicitly the vortex motion. The vortices move with the velocity \((0, -u, 0)\). This is similar to the corresponding solution in CMII (Eq. (17)), but has a different polarization. It has an additional factor \( \mathcal{P}(T) \). The temperature dependence of \( \mathcal{P}(T) \) is shown in Fig. 2. It is clear that Eq. (25) itself shows the same vortex motion. Actually, \( (\partial / \partial t + u \partial / \partial y) \times A^i(r, t) \) is zero up to the linear order of \( u \), i.e. \( E \). (Though it might look as if Eq. (27') does not show the vortex motion, one can find that it really does, by summing up all the \( n \)-th component

* FM has insisted that their solution of Eq. (23) is equal to the corresponding solution of CMI. But they have neglected the terms proportional to \( i\omega A' = E \).
of Eq. (27') and using the characteristics of the theta function.)

Thus, we can conclude that flux flow state can be obtained by the usual linear response theory as CMI, so that the interpretations by FM, described in § 1, are not appropriate.

§ 4. Flux flow resistivity and microwave surface impedance

As mentioned in § 1, the modulation of the transport current by the order parameter can be divided into two parts, \( \mathcal{J}_n \) and \( \mathcal{J}_m \). They are given by the space averages of the current densities \( \mathcal{J}_n(r, \ell) \) and \( \mathcal{J}_m(r, \ell) \), where

\[
\mathcal{J}_n(r, \ell) = \left\langle \mathcal{J} \right\rangle_{\ell} = \left\langle j \right\rangle_{\ell, q_1, q_2} \delta \mathcal{A}(r, \ell) + \delta_j(r, \ell) \delta \mathcal{A}(r, \ell) \right|_{r=r_1=r_2},
\]

(29)

\[
\mathcal{J}_m(r, \ell) = \left\langle \mathcal{J} \right\rangle_{\ell} = \left\langle j \right\rangle_{\ell, q_1, q_2} \delta \mathcal{A}(r, \ell) \delta_j(r, \ell) \right|_{r=r_1=r_2}.
\]

(30)

Here, the response functions, \( \left\langle \mathcal{A} \right\rangle_{\ell} \)'s, are defined in § 2, and the \( q_1 \)'s and \( \omega_1 \)'s are the differential operators operating on the \( \mathcal{A} \)'s, and \( k \) on \( \phi \). The graphical representations of \( \left\langle \mathcal{A}; \mathcal{F}, \mathcal{F} \right\rangle \) and \( \left\langle \mathcal{A}; B, \mathcal{F}, \mathcal{F} \right\rangle \) are given in Figs. 3 and 4.\(^*\)

Calculation of the response function in Eq. (29) yields a result similar to Eq. (A3) in CMI:

\[
\mathcal{J}_n(q_1, \omega_1; \omega_2) = -\frac{eN}{2m} (Q_1 - Q_2) \left\{ \left( -i\omega_1 - i\omega_2 + DQ_1^2 - DQ_2^2 \right)^{-1} \right\}
\times \left[ \psi \left( \frac{1}{2} + \frac{DQ_2^2 - i\omega_1}{4\pi T} \right) - \psi \left( \frac{1}{2} + \frac{DQ_1^2 - i\omega_1}{4\pi T} \right) \right]
\times \left( -i\omega_1 - i\omega_2 + DQ_1^2 - DQ_2^2 \right)^{-1}
\times \left[ \psi \left( \frac{1}{2} + \frac{DQ_2^2 - i\omega_1}{4\pi T} \right) - \psi \left( \frac{1}{2} + \frac{DQ_1^2 - 2i\omega_1 - i\omega_1}{4\pi T} \right) \right]
\times \left[ \delta \mathcal{A}(q_1, \omega_1) \delta \mathcal{A}(q_2, \omega_2) + \delta \mathcal{A}(q_1, \omega_1) \delta \mathcal{A}(q_2, \omega_2) \right].
\]

(31)

Here, \( N \) is the number density of electrons, and \( \omega \) does not mean \( \omega - 2e\phi \). Substituting Eq. (25) and its conjugate into Eq. (31), and performing the space average of \( \mathcal{J}_n(r, \ell) \), we obtain

\[
\mathcal{J}_n = -4eM \frac{1}{8\pi T}\rho \mathcal{E}.
\]

(32)

Fig. 3. The graphical representation of \( \left\langle \mathcal{A}; \mathcal{F}, \mathcal{F} \right\rangle \). \( \delta \mathcal{A} \) or \( \delta \mathcal{A} \) is the linear modulation of the order parameter due to the perturbational field.

* We have not explained in detail the treatments of these response functions and their graphical representations. Readers can find them in references 2) and 4) (\( \left\langle \mathcal{A}; \mathcal{F}, \mathcal{F} \right\rangle \)), and in references 3) and 11) (\( \left\langle \mathcal{A}; B, \mathcal{F}, \mathcal{F} \right\rangle \)).
Fig. 4. The graphical representations of $\langle A; B, \mathbf{F}^n, \mathbf{F}^s \rangle \mathcal{A}_n \mathcal{A}_s$. The broken lines between Green's functions, antiparallel and parallel, are the impurity corrections $I_1$ and $I_2$, respectively. They are defined in Appendix A. In the diagrams, $p^\pm = p \pm k/2$ and $\varepsilon_n^- = \varepsilon_n - \omega_0$.

$M$ is the magnetization of the system given by

$$M = J_0 \mathcal{A}_n \mathcal{A}_s,$$  

where $J_0$ is the normal conductivity and $\mathcal{A}_n \mathcal{A}_s$ is the space average of $|\mathcal{A}_n \mathcal{A}_s|^2$. The straightforward calculation of Eq. (33) leads to the result

$$\mathcal{J}^s = 4eM \frac{1}{8\pi T \rho} \left[ 1 + \frac{\rho \phi(1/2 + \rho)}{\phi(1/2 + \rho)} \right] E.$$

The total current, which is proportional to $\langle |\mathcal{A}_n \mathcal{A}_s|^2 \rangle$, is given by

$$\mathcal{J}^s = \mathcal{J}_1^s + \mathcal{J}_2^s = -4eM \frac{L_D(T)}{8\pi T \rho} E,$$

where $L_D(T)$ is defined, as in CMII, by

$$L_D(T) = 2 + \frac{\rho \phi(1/2 + \rho)}{\phi(1/2 + \rho)}.$$

Then the variation of the flux flow resistivity with $H$ at $H_{c2}$ is given by

$$\alpha = \frac{H}{R_n} \frac{\partial R}{\partial H} \bigg|_{H = H_{c2}} = \frac{4\kappa_2^2(0)}{2\kappa_2^2(T) - 1} \frac{L_D(T)}{1.16},$$

where $\kappa_1$ and $\kappa_2$ are the Ginzburg-Landau parameters. The temperature dependence of $\alpha$ is shown in Fig. 5. The obtained $\mathcal{J}^s$ is identical with that of Thompson but not with the one in CMII. The latter is given by Eq. (35), if $L_D(T)$ is replaced by unity. Note that $\mathcal{J}^s$ itself is gauge independent but $\mathcal{J}^{st}$
and $\mathcal{J}^{\text{el}}$ are dependent. This is easily seen if our results are compared with those by Thompson. (His results are quoted in Table II in Appendix B.)

The frequency dependent electromagnetic response function $\mathcal{Q}(\omega)$, defined by $\mathcal{J}_{z} \equiv -\mathcal{Q}(\omega) A_z$, has been obtained by Thompson. In the low frequency limit, $\omega/\text{Max}(T, \varepsilon_0) \ll 1$, it is given by

$$\mathcal{Q}(\omega) = \mathcal{Q}^M(\omega) + \mathcal{Q}^T(\omega) + \mathcal{Q}^F(\omega),$$  

(38)

where

$$\mathcal{Q}^M(\omega) = -4\varepsilon M \left\{ 1 - \frac{i\omega}{2(\varepsilon_0 - i\omega)} \right\}$$

$$- \frac{i\omega}{8\pi T} \left\{ 3 - \frac{i\omega}{\varepsilon_0 - i\omega} \right\} \frac{\psi^{(3)}(1/2 + \rho)}{\psi^{(1)}(1/2 + \rho)},$$  

(39)

$$\mathcal{Q}^T(\omega) = 4eM \left\{ \frac{2\varepsilon_0^2}{4\varepsilon_0^2 + \omega^2} - \frac{i\omega}{4\pi T} \frac{\varepsilon_0}{2\varepsilon_0 + i\omega} \frac{\psi^{(3)}(1/2 + \rho)}{\psi^{(1)}(1/2 + \rho)} \right\},$$

(40)

$$\mathcal{Q}^F(\omega) = 4eM \frac{2\pi T\varepsilon_0}{X(\rho, \omega)} \left\{ X(\rho, \omega) - \psi^{(1)}(1/2 + \rho) \right\} \left( 2\varepsilon_0 - i\omega \right)$$

$$- \frac{i\omega}{2\pi T} \left\{ 2\varepsilon_0 - i\omega \right\} \frac{\psi^{(3)}(1/2 + \rho)}{\psi^{(1)}(1/2 + \rho)}.$$  

(41)

Here, the functions $X(\rho, \omega)$ and $Y(\rho, \omega)$ are

$$X(\rho, \omega) = \frac{1}{\psi^{(1)}(1/2 + \rho)} \left\{ \psi \left( \frac{1}{2} + 3\rho \right) - \psi \left( \frac{1}{2} + \rho \right) \right\}$$

$$- \frac{i\omega}{4\pi T} \psi^{(1)}(1/2 + 3\rho) + \frac{i\omega}{4\pi T} \frac{\psi^{(3)}(1/2 + 3\rho)}{\psi^{(1)}(1/2 + \rho)}.$$  

(42)

$$Y(\rho, \omega) = X(\rho, \omega) + \frac{i\omega}{2\pi T} \left\{ \psi \left( \frac{1}{2} + 3\rho \right) - \psi \left( \frac{1}{2} + \rho \right) \right\}$$

$$- \frac{i\omega}{2\pi T} \psi^{(1)}(1/2 + \rho) + \frac{i\omega}{2\pi T} \frac{\psi^{(3)}(1/2 + \rho)}{\psi^{(1)}(1/2 + \rho)}.$$  

(43)

$\mathcal{Q}^M$ is the contributions from the diagrams in Fig. 4A, calculated by Maki.\(^{13}\) In its calculation, the spatial variation of the order parameter is not necessary so that it is also correct for other gapless superconductors, such as the one with magnetic impurities, if an appropriate pair-breaking parameter is taken for $\varepsilon_0$\(^{14}\). $\mathcal{Q}^T$ comes from diagrams in Fig. 4B calculated by Thompson. This term is indispensable, if the order parameter varies spatially. $\mathcal{Q}^F$, which is given in CMI, gives rise to the current $\mathcal{J}^{\text{el}}$. In the static limit $\omega/\varepsilon_0 \ll 1$, the real part of $\mathcal{Q}$ vanishes and the imaginary part gives a stationary current written as Eq. (35). In the high frequency limit $\varepsilon_0 \ll \omega/4T$, which is realized only near $T_c$, $\mathcal{Q}$ is determined by $\mathcal{Q}^M$ and is written as
**Linear Response Theory in the Vortex State**

\[ Q(\omega) \equiv -4eM \left( \frac{3}{2} + O \left( \frac{\omega}{T_c}, \frac{\varepsilon_0}{T_c} \right) \right). \]  

(44)

In this case the external field changes so quickly that the spatial variation of \( \Delta_0^+ (Q^+) \) and the motion of the vortices (\( Q^F \)) are smeared out. The response becomes the same as that of the ordinary gapless superconductors.

The surface impedance \( Z(\omega) \) in the case of the specular reflection at the surface is easily calculated from these expressions for \( Q(\omega) \). The variation of the surface resistance at \( H_{c2} \) becomes

\[ S_\perp (\omega, T) = \frac{H'}{Z} \frac{\partial \text{Re} Z}{\partial H} \bigg|_{n=n_{c2}} = - \frac{4\kappa_1^2(0)}{1.16(2\kappa_2^3(T) - 1)} \mathcal{S}(\omega, T), \]  

(45)

where

\[ \mathcal{S}(\omega, T) = \frac{\varepsilon_0}{\omega} (\text{Im} \tilde{Q}(\omega) - \text{Re} \tilde{Q}(\omega)), \]  

(46)

with \( \tilde{Q} = -Q/4eM \). Note that the variation of the flux flow resistivity \( \alpha \), Eq. (37), is given from Eq. (45) as \( \lim_{\omega \to 0} S_\perp (\omega, T) = \frac{1}{2} \alpha(T) \). A difference between the theoretical results by FM or CMII and ours appears in the expressions for \( \mathcal{S}(\omega, T) \). They are common to all type-II superconductors, whereas the other factor in Eq. (45) strongly depends on a material.

The numerical results of \( S_\perp (\omega, T) \) for the \( \text{Pb}^{83} \text{In}^{17} \) alloy at the frequencies 55 and 23GHz are shown in Fig. 6 with the experimental points given by FM. The frequency dependence of \( S_\perp (\omega, T) \) cannot be neglected even in the low temperature range, because \( \omega/\varepsilon_0(T = 0^\circ \text{K}) \) is about 0.2 for \( \omega = 55 \text{GHz} \).

The fact that near \( T_c \) the dissipative part cannot completely follow the rapid change of the applied field appears at temperatures even below \( T_{\omega} \), which is determined by \( \omega = \varepsilon_0(T_{\omega}) \). (\( T_{\omega}/T_c \) is 0.85 and 0.94 for \( \omega = 55 \) and 23GHz, respectively.) But the present theory cannot be applied at the immediate vicinity to \( T_c \), where the high frequency effect in small \( \omega \) appears, because in this range the thermal fluctuations of the higher Abrikosov modes appear.15

Now let us examine the dif-
ferences between our analysis and that of FM for \( S_1(\omega, T) \), excluding the high frequency effects mentioned above. In ours, the temperature dependence of \( S_1(\omega, T) \) originates from \( \kappa_1(T) \) and \( \mathcal{F}(\omega, T) \). For the latter we have used \( Q(\omega) \) given in the expressions from Eq. (39) to (43). For the factor including \( \kappa_1(T) \) we have used the expression in the dirty limit for \( \kappa_1(T)/\kappa_1(0) \), and the value 1.2 for \( \kappa_1(0)/\kappa_1(0) \). This value seems to be explained by the purity dependence of \( \kappa_1 \) and \( \kappa_2 \). On the other hand, FM have analysed their data putting \( \mathcal{F}(\omega, T) = \frac{1}{2} \) (their Eq. (20)), and have taken the strongly temperature dependent \( \kappa_1(T)/\kappa_1(0) \). Then, \( \kappa_2(0)/\kappa_1(0) \) becomes about 2. They have attributed this large value to the strong coupling effect.

Concerned with the flux flow resistivity, the result by CMII agrees better with experiment\(^{10}\) than ours. In reference 18), \( \kappa_1(T) \) and \( \kappa_2(T) \) have been determined independently from the magnetization measurement. In the following section we will further examine this discrepancy.

§ 5. Discussion

In the preceding sections, it has been shown that the TDGL equation, under an applied electric field, is not given by Eq. (20) or (23) but by Eq. (19) or (24). The transport currents, in the vortex state, can be calculated by the linear response theory, as CMI, which intrinsically includes the effect of the vortex motion. The difference between the present discussions and CMII is important not only quantitatively but also qualitatively in order to understand the characteristics of the flux flow state in the regime of the TDGL equation.

First, let us examine the obtained TDGL equation (19). As shown in § 2, the form of the \( \phi \)-linear term in the TDGL equation is determined by careful treatment of the impurity effect. We had to take account of the fact that the impurity correction factor \( I_\gamma \) is singular in the restricted range \( 0 > \varepsilon_\gamma > -\omega_p \). This was also important when Gorkov and Eliashberg\(^{13}\) derived the \( \phi \)-linear term in their TDGL equation for a superconductor with paramagnetic impurities.\(^{8}\) As far as a superconductor in the gapless region is concerned, we can at once derive their final equation, Eq. (22), using the discussions in § 2 of the present paper. We only need to replace \( \varepsilon_\gamma \) by \( 2/\tau_s \), \( \tau_s \) being the spin-flip relaxation time of electrons due to paramagnetic impurities. It is worth noticing that the gapless condition ensures the validity of the expansions of various quantities with respect to \( \Delta \) and \( \omega/\text{Max}(T, \varepsilon_\delta) \) and, accordingly, the existence of the TDGL equation of a diffusion type.\(^{13}\)

The field-free TDGL equation, Eq. (1), is equivalent to

\[
- \frac{\partial}{\partial t} \mathcal{A}(r, t) = (DQ^2 - \epsilon_\delta) \mathcal{A}(r, t).
\]

\(^{8}\) Note that the singularity of their \( I(\omega, k) \), Eq. (10), is the same as that of \( I_\gamma(k, \omega) \) in the present paper. Their condition \( \rho=0, \rho \) being a charge density, is represented by \( \epsilon_\phi = 0 \) in our work.
The $n$-th eigenmode, $|n\rangle$, has the pure imaginary eigenfrequency $2in\omega$, $\omega$ being $2DcH_{c2}$. The lowest eigenmode $|0\rangle$ is the Abrikosov state given by Eq. (7). The scalar potential $\phi = -Ex$ induces the higher diffusive mode $|1\rangle$, $\delta\tilde{A}$ of Eq. (25), because $\langle 1|\phi|0\rangle \neq 0$. This mode rapidly decays accompanied with the dissipation of energy. This is the theoretical interpretation of the dissipation mechanism associated with the flux flow.

Secondly, we must pay attention to the calculations of $J^t$ in the vortex state. $J^t$ is not gauge independent by itself as shown in § 4. The induced currents have always another term, $J^n$, in a system with an order parameter which can be evaluated by the mean field approximations. The total current $J^t = J^u + J^n$ is shown to be gauge independent in Appendix A, in the case of gapless superconductors. It is worth pointing out that the arguments can be at once applied to the problem of the paraconductivity above $T_c$, where the superconducting fluctuations are described by the Aslamazov-Larkin’s theory. It is shown that the AL result for the paraconductivity is not gauge independent, but the sum with the terms considered by Maki becomes invariant.

The quantitative difference of the transport coefficients obtained by the present work from those by CMII or FM has already been discussed in § 4. Here we would like to make further comments. The one is about the analysis of the experimental data. The observed $R_f - H$ curve is rounded off at the immediate vicinity of $H_{c2}$ (18). Muto et al. have obtained their $\alpha$ values from the linear portion of the curve appearing in the narrow range close to $H_{c2}$ (18). On the other hand, Axt et al. (20) have analysed experimental data using the extrapolation of $R_f (H, T)$, at the magnetic field not close to $H_{c2}$ to the limit $H \to H_{c2}$. They have obtained the stronger temperature dependence of $\alpha$’s (the chain line in Fig. 5) than those of CMII. We believe that this extrapolation method is the better. Since the rounding off does not appear in the magnetization versus magnetic field curve (18), that of the $R_f (H, T)$ curve seems to indicate the deviation of the actual system from the ideal vortex state. In order to clarify this point, further investigations are necessary.

The second comment is about the electromagnetism in the vortex state. If an applied electric field is modulated in the superconductors by the order of $\langle |d_{\phi}|^2 \rangle$, it makes a contribution to the flux flow resistivity through the normal conductivity, i.e. $J' = \sigma (E_{\text{int}} - E_{\text{ext}})$. This possibility arises, if the vortex pattern is distorted by the field and the system is polarized so that $\rho (r) \propto E \cdot v |d_{\phi} (r)|^2$. This is now under consideration.

Now, as it has been established how to calculate transport currents in the vortex state, we shall reexamine other transport coefficients given in CMII. In order to treat the thermal effect microscopically, we assume that the linear response of a physical quantity $X$ with respect to the thermal gradient is given by

$$\langle X_\kappa \rangle = -\lim_{\omega \to 0} \text{Re} \left\{ \frac{1}{i\omega} \mathcal{J}_x (k, \omega + i0) \right\} \left( \frac{\mathcal{V}^T}{T} \right)_\kappa, \quad (47)$$
where the average of the retarded commutator $\mathcal{F}_X$ is given by

$$
\mathcal{F}_X(k, \omega + i0) = \frac{1}{\beta} \int_0^\beta \int_0^\beta \exp(-i\omega (\tau - \tau')) \langle T_r, X(k, \tau) j_h(-k, \tau') \rangle \left|_{\omega_{-n-1/2}} \right.,
$$

and $j_h$ is the heat current operator. We can easily check that if we take $j_h$ itself for $X$, Eqs. (47) and (48) give the well-known Kubo formula for the thermal conductivity. However, the definition of the heat current is rather ambiguous. In this work, we take the electronic heat current operator $j_{ho}$ given by

$$
j_{ho}(x) = -\sum_x \frac{1}{2m} \left[ \nabla' \frac{\partial}{\partial t} + \nabla \frac{\partial}{\partial t'} \right] \phi^\dagger(x') \phi(x) \Big|_{x' \rightarrow x}. \tag{49}
$$

This was taken also by CMII. Then, the calculation is straightforward. The details are given in Appendix B.

Putting $X = g T^l$ in Eqs. (47) and (48), we obtain the TDGL equation under the thermal gradient. Its solution does not show the motion of the vortex structure, but only its distortion. The resulting transport coefficients agree with those of CMII and satisfy the Onsager's reciprocal theorem, though their derivations are quite different. (In CMII, the thermal response was not calculated explicitly.) But this result leads to the fact that the entropy, carried by a vortex with its motion, diverges at $T \rightarrow 0^\circ K$. To get rid of this difficulty, Maki has taken account of the energy flow due to the change of the magnetization. But, then, the Onsager's reciprocal theorem is violated unless we add to $J_y$ some additional electrical current, the nature of which is quite unknown at present. This problem is directly connected with the definition of the heat current operator and, accordingly, with the exact microscopic treatment of the temperature gradient.

Finally, we would like to make a remark on the Hall effect. The Hall effect in the vortex state cannot be explained by the present theory. It is determined by the off-diagonal component of the conductivity tensor, $\sigma_{xy}$, which is defined by

$$
J_y = -\sigma_{xy} E_x. \tag{50}
$$

Experimentally, it shows a large increase as the magnetic field is lowered through $H_c2$, but calculation of $J_y^{\text{H}}$ with Eq. (31) yields a vanishing $\sigma_{xy}$. Maki has introduced the term of the kinetic energy of a Cooper pair into the TDGL equation through the chemical potential of electrons. Following his idea but using our TDGL equation derived in § 2, we obtain

$$
\sigma_{xy} = \sigma_{xy}^{\text{H}} - \sigma \frac{4k_F^3(0)}{1.16 (2k_F^3(T) - 1)} \frac{H_c2 - H}{4mD} \mathcal{P}_1(T), \tag{51}
$$

where

$$
\mathcal{P}_1(T) = \frac{2\rho \psi^{(1)}(1/2 + 3\rho)}{\psi(1/2 + 3\rho) - \psi(1/2 + \rho)}. \tag{52}
$$
The value of $\sigma_{xy}$ in the normal state, written as $\sigma^{n}_{xy}$, is negative for electrons. The result obtained by Maki did not have the factor $P_1(T)$, which is shown in Fig. 2. In these calculations, the quantities with the order of $\tau e H_{\text{ext}}/m$, the Hall angle in the normal state, are neglected compared with $(4mD)^{-1}$ in the dirty limit $\tau e \ll 1$. The same approximation has been applied to $J_y^\text{nt}$.

Agreement of Maki's result with experiment, in reference 26), is good in the order of magnitude with a value of $(4mD)^{-1}$ adjusted reasonably, but not good in the temperature dependence. Our result, Eq. (51), does not improve this dependence. This would suggest that the other mechanisms are important for understanding the Hall effect correctly.

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**Appendix A**

In any approximate calculations, the electric current $J_\mu$ should be invariant with respect to the gauge transformation

$$A_\nu(k) \rightarrow A_\nu(k) + ik \chi. \quad (A\cdot1)$$

With the definition of the electromagnetic response function, $Q_{\mu\nu}$, by

$$J_\mu(k) = \sum_{\nu=0}^{3} Q_{\mu\nu} A_\nu,$$

this requirement turns out to be the condition

$$\sum_{\nu=0}^{3} k_\nu Q_{\mu\nu} = 0. \quad (A\cdot2)$$

The contributions to $Q_{\mu\nu}$, in the present calculation, come from the terms which are represented by diagrams in Figs. 3 and 4. $\delta\mathcal{A}$ should be determined by Eq. (12), represented graphically in Fig. 1. The total $Q_{\mu\nu}$ will be shown to satisfy Eq. (A·2). This treatment is based on the arguments by Nambu, and is very similar to that in the theory of Hall effect by one of the present authors and others.

First, let us summarize the various impurity corrections: the electron self-energy part $\Sigma$, the vertex corrections $\gamma$ and $\Gamma_\mu$, and the corrections between two Green's functions $I_1$ and $I_2$. For example, $\gamma$, $I_1$ and $I_2$ are determined by the following equations:

$$\gamma(q, \varepsilon_n, \omega_n) = 1 + \frac{1}{2\pi i N(0)} \int \frac{dp}{(2\pi)^3} \mathcal{G}(p-q, \varepsilon_n-\omega_n) \mathcal{G}(-p, -\varepsilon_n) \gamma(q, \varepsilon_n, \omega_n), \quad (A\cdot3)$$
$$I_{1,2}(q, \varepsilon_n, \omega_n) = \frac{1}{2\pi^2 N(0)} \left[ 1 + \int \frac{dp}{(2\pi)^d} G(p-q, \varepsilon_n-\omega_n) G(\pm p, \pm \varepsilon_n) I_{1,2}(q, \varepsilon_n, \omega_n) \right].$$

(A·4)

Their graphical representations are given in Fig. 7. Equations for $\Sigma$ and $\Gamma_\mu$ are given only graphically also in Fig. 7. Although we easily obtain solutions for them as Eqs. (14), (15) and (16), etc., we use only these equations in the arguments below.

These impurity corrections are called the ladder-type approximation. They are consistent in the sense that they satisfy the following "generalized Ward's identity" (GWI) for the external electromagnetic vertex $\Gamma_\mu$:  

$$\sum_{\mu=0}^3 k_\mu \Gamma_\mu (p^-, \varepsilon_n - \omega_n; p^+, \varepsilon_n) = G^{-1}(p^-, \varepsilon_n - \omega_n) - G^{-1}(p^+, \varepsilon_n).$$

(A·5)

Equation (A·5) multiplied by $G(p^-, \varepsilon_n - \omega_n) G(p^+, \varepsilon_n)$ is graphically represented in Fig. 8. Note that Eq. (A·5) and the arguments below are formally valid also for systems with two-body interactions. We have only to replace the impurity correction line $(2\pi^2 N(0))^{-1}$ by the effective two-body interactions, $V(p-p', \varepsilon_n - \varepsilon_n')$. Here $p-p'$ and $\varepsilon_n - \varepsilon_n'$ are the momentum and energy transfers, respectively.

Contributions to $Q_\mu$ from the terms represented by the diagrams in Fig. 4 are $Q_\mu^0$ and $Q_\mu^1$. The vertices $A$ and $B$ are $-e\Gamma_\mu$ and $-e\Gamma_\mu$, respectively. The fluctuation part $Q_\mu^f$ is expressed as follows. We rewrite Eq. (12) as

$$\delta A'(q, \omega_n) = -\mathcal{D}(q, \omega_n) \langle \mathcal{W}^1; \mathcal{V}, J_\mu \rangle_{q,k,\varepsilon_n} \mathcal{A}^d(q_\mu) \Lambda(k, \omega_n).$$

(A·6)

The fluctuation propagator $\mathcal{D}(q, \omega_n)$ satisfies the equation

$$\mathcal{D}(q, \omega_n) = \frac{1}{q^2 + \omega_n^2 - \varepsilon_n^2}.$$
Substituting Eq. (A·6) for $\delta A'$ in Eq. (29), we obtain an expression for $Q_{\mu\nu}^\prime$ graphically shown in Fig. 9. The factor 2 comes from the spin factor. The graphical representations of Eq. (A·7) and of the static GL equation (11) are given in Fig. 10.

![Graphical representation of $Q_{\mu\nu}^\prime$](image)

**Fig. 9.** The graphical representations of $Q_{\mu\nu}^\prime$. The wavy line denotes the propagator of the fluctuation, $\phi$.

Using the GWI and the various equations given above, we now carry out the operation $\sum k_s$ on each contribution to $Q_{\mu\nu}$ and show that the various terms cancel out each other. Carrying out the operation, for example, on $Q_{\phi,\phi}^\prime$, the diagram (f) in Fig. 4B) and using the GWI, we obtain

$$\sum k_s Q_{\phi,\phi}^\prime = T \sum_{\epsilon_n} \phi \int \frac{dp_1}{(2\pi)^3} I_1(q_z + k, \epsilon_n, \omega_n) \eta(q_z, \epsilon_n, 0)$$

$$\times \mathcal{G}(-p_1^+ + q_1, -\epsilon_n) \left[ \mathcal{G}(p_2^+ - q_1 - q_2, \epsilon_n) \mathcal{G}(p_3^+ - q_1 - q_2 - \epsilon_n) \mathcal{G}(-p_2^- + q_1, -\epsilon_n) \right],$$

where

$$\phi = -2 \int \frac{dp_1}{(2\pi)^3} \Gamma_\mu \mathcal{G}(p_1^+, \epsilon_n) \mathcal{G}(p_1^- - q_1 - q_2, \epsilon_n) \mathcal{G}(-p_2^+ + q_1, -\epsilon_n)$$

$$\times \eta(q_1, \epsilon_n, 0) J_{\phi}(q_1) J_{\phi}^+(q_2).$$

With the aid of Eqs. (A·3) and (A·4), Eq. (A·8) becomes

$$\sum k_s Q_{\phi,\phi}^\prime = T \sum_{\epsilon_n} \phi \int \frac{dp_1}{(2\pi)^3} \left[ \left[ 1 + \int \frac{dp_3}{(2\pi)^3} \mathcal{G}(p_3^- - q_1 - q_2, \epsilon_n) \mathcal{G}(-p_3^+ + q_1, -\epsilon_n) I_2 \right] \right]$$

$$\times \frac{1}{2\pi N(0)} \mathcal{G}(p_1^- - q_1 - q_2, \epsilon_n) \mathcal{G}(-p_2^+ + q_1, -\epsilon_n) \eta$$

$$- I_3 \mathcal{G}(p_2^+ - q_1 - q_2, \epsilon_n) \mathcal{G}(-p_3^+ + q_1, -\epsilon_n) \eta$$

$$\times \left[ 1 + \frac{1}{2\pi N(0)} \int \frac{dp_3}{(2\pi)^3} \mathcal{G}(p_3^+ - q_1 - q_2, \epsilon_n) \mathcal{G}(-p_3^+ + q_1, -\epsilon_n) \eta \right].$$

(A·9)
The equation of $\vartheta$ is used at the $A_0$ vertex and at the right-hand vertex of the propagator $\vartheta$. We obtain the last equation, applying to (k. 1) and (k. 2) the equations of $A_0$ and $\vartheta$, respectively, which are shown graphically in Fig. 10.
The remaining contributions in Fig. 12 are put together to give

\[-k_nT \sum_n \int \frac{dp}{(2\pi)^3} \delta G (p, q_1, q_2, \varepsilon_n),\]  

(A-10)

which is proportional to the modulation of the particle number due to the static order parameter. This term, which is usually neglected, cancels formally the result of the operation \( \sum k_nQ_{\mu} \) on the diamagnetic current term. Thus all the terms of \( \sum k_nQ_{\mu} \) vanish. In the above proof, it is important that the cancellation occurs between the terms from \( Q^\mu + \Omega^\rho \) and from \( Q^\rho \), for example, between (1) in Fig. 12 and (k.1) + (k.2) in Fig. 13. It shows that \( J^{(1)} \) or \( J^{(2)} \) alone is not gauge invariant.

The same arguments can be applied to the calculation of the excess conductivity above \( T_c \). The diagrams for \( Q_{\mu \nu} \) in this case correspond to those in Figs. 4 and 9 one by one. We only need to connect, by a wavy line, between the vertices of \( \Delta_{\theta} \) and of \( \Delta_{\theta}' \) in each diagram, and to replace \( \Delta_{\theta}' \Delta_{\theta} \) by \( \delta \langle q, 0 \rangle \). The arguments of the gauge invariance given above provide the necessity condition that all diagrams corresponding to those in Figs. 4 and 9 must be summed up. But, it is another problem which terms actually give the largest contribution to the excess conductivity. In the classical range where the mean field theory of the fluctuations is valid, they are the Maki term, which corresponds to (c) in Fig. 4, and the Aslamazov-Larkin term, which corresponds to (c) in Fig. 9.

Appendix B

The self-consistent response of the order parameter with respect to the temperature gradient is written schematically as

\[ \delta A = g \int \frac{d^3q}{i\omega} \langle T^+ j_x \rangle \frac{F}{T} + g \langle T^+ T^\rho \rangle \delta A. \]  

(B-1)

Straightforward calculation of Eq. (B1) leads to the TDGL equation

\[ \{ \frac{1}{2} + \frac{DQ^\rho - i\omega}{4\pi T} \} \phi \left( \frac{1}{2} \right) + \log T_\infty - \frac{D\Psi(T)}{T} \nabla T \cdot Q \} A'(r, t) = 0, \]  

(B-2)

where

\[ \Psi(T) = -\frac{1}{\epsilon} \left\{ \frac{3}{2} \left[ \phi \left( \frac{1}{2} + 3\rho \right) - \phi \left( \frac{1}{2} + \rho \right) \right] - \rho \phi^{(n)} \left( \frac{1}{2} + \rho \right) \right\}. \]  

(B-3)

When the temperature gradient is in \( x \) direction, the solution of Eq. (B2) is given by

\[ A'(r, t) = A'(r) = \left\{ 1 - 2eH \frac{\Psi}{T} \frac{F_x T}{T} \right\} A'(r) \]  

(B-4)

\[ \simeq \exp \left[ -iky - eH \left( x + \frac{k}{2eH} + \frac{\Psi}{T} \frac{F_x T}{T} \right) \right]. \]  

(B-4')
which is time independent and does not show the vortex motion. Only the vortex pattern is distorted. The polarization factor $P_3(T)$ is

$$P_3(T) = \frac{3}{2} \left[ \frac{\rho \phi^{(1)}(1/2 + \rho) - \phi(1/2 + \rho)}{\phi(1/2 + 3\rho) - \phi(1/2 + \rho)} \right]. \quad (B.5)$$

The transport heat current $\mathcal{J}_h$ is also calculated by dividing it into two parts, $\mathcal{J}_{h\text{st}}$ and $\mathcal{J}_{h\text{r}}$. An explicit expression for $\mathcal{J}_{h\text{st}}$ is given by Eq. (A7) of CMIL. It vanishes because $\mathcal{A}(r, t)$, Eq. (134), is time independent. $\mathcal{J}_{h\text{st}}$ is obtained from the response function $\mathcal{R}(\mathcal{J}_h; \mathcal{J}_h, T')$, first calculated by Caroli and Cyro. The results are listed up on Table II. Note that Onsager’s reciprocal theorem is satisfied, as expected, by the independent calculations of $(\mathcal{J}_{h\text{st}})_y$ due to $F_x T/T$ and $(\mathcal{J}_{h\text{r}})_y$ due to $E_x$.

Table II. The transport currents induced by an electric field and a thermal gradient. In the table, $M$ is the magnetization given by Eq. (33), $A=\mathcal{B}^{-1}=2e/\epsilon_0$, and the $L(T)$’s are defined by

$$L_1(T) = L_{B}(T) - 1, \quad L_2(T) = 2 - \frac{\phi^{(1)}(1/2 + 3\rho)}{\phi^{(1)}(1/2 + \rho)} - \frac{\phi(1/2 + 3\rho)}{\phi(1/2 + \rho)},$$

where $L_{B}(T)$ is given by Eq. (36). We omit the $J^{(1)}$, which are given by $J^x - J^y$. Here, the responses to $E_x$ are calculated by a vector potential $A'$. In this case, $\lim_{a \to 0} \text{Im}(J_x^y) \neq 0$, though $\lim_{a \to 0} \text{Im}(J_x^x) = 0$. In the table only the real part of the current is written.

<table>
<thead>
<tr>
<th>Applied Field</th>
<th>Transport Current</th>
<th>$J^x$</th>
<th>$J^y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_x$</td>
<td>$(\mathcal{J}_h)_y$</td>
<td>$-AML_3(T)$</td>
<td>$-AML_3(T)$</td>
</tr>
<tr>
<td>$T$</td>
<td>$(\mathcal{J}_h)_x$</td>
<td>$ML_0(T)$</td>
<td>$ML_0(T)$</td>
</tr>
</tbody>
</table>

References

   JETP 8 (1959), 1060].
   Chap. 18.