ditions, Equation [11], are listed below for Prandtl numbers of 0.1, 1, 10, and 100.

Pr	$\theta(0)$	$f''(0)$
0.1	−2.7507	1.6434
1	−1.3574	0.72196
10	−0.76746	0.30639
100	−0.46566	0.12620

Discussion

D. C. Hamilton.[7] This paper is most certainly a much-needed and useful contribution to the heat-transfer literature. The solution obtained, which the authors have called "exact solution," is more commonly called "numerical solution"; this is actually an abbreviation for the more precise definition "numerical approximation to the solution." In the discusser's opinion the authors' nomenclature will be misleading, particularly to the cursory reader.

A worth-while effort was expended by the authors in obtaining the solution of this same problem by the von Kármán method so that a comparison could be made. Since the von Kármán method is so useful in the more difficult boundary-layer problems that often occur in practice it is always gratifying to discover still another case where it gives a solution that is in satisfactory agreement with the more complete solution.

D. P. Timo.[8] The authors present an interesting, mathematically rigorous solution to the problem of the vertical heated plate with uniform surface heat flux. The term "mathematically rigorous" is perhaps preferable to the term "exact" in describing the solution, since the fluid properties other than the density were considered invariant. For this case of uniform surface heat flux, there is a longitudinal surface-temperature variation as well as a fluid-temperature variation normal to the plate. Thus, for large q, and hence large local values of $t_w - t_\infty$, the temperature at which to evaluate the fluid properties may present somewhat of a problem. However, for the laminar-flow region to which this analysis applies, the difficulty is relatively unimportant. The method of solution is analogous to that followed by Schmidt and Beckman,[9] in co-operation with Polhausen, in solving the problem of the vertical heated plate with uniform surface temperature.

The dimensionless group which the authors call the modified Grashof number is in reality the product of the conventional Grashof and Nusselt numbers. For example, the "modified Grashof number" based on x is

$$\mathrm{Gr}_x{}^* = \frac{g\beta x^4 q}{\nu^2 k} = \frac{g\beta x^4[h_x(t_w - t_\infty)]}{\nu^2 k} = \left[\frac{g\beta x^3(t_w - t_\infty)}{\nu^2}\right]\left[\frac{h_x x}{k}\right] = \mathrm{Gr}_x\mathrm{Nu}_x$$

If this expression is substituted into Equation [14*a*], there results

$$\frac{\mathrm{Nu}_x}{\mathrm{Gr}_x{}^{1/4}} = -\frac{1}{5^{1/4}\theta(0)^{5/4}}$$

Thus the Grashof number is seen to enter as the 1/4 power. This explains why transition from laminar to turbulent flow occurs at $\mathrm{Gr} \approx 10^9$ and $\mathrm{Gr}^* \approx 10^{11}$, as mentioned by the authors. The form of the relationship between Nu and Gr in this paper, after the foregoing substitution has been made, is more nearly similar to the form found by Schmidt and Beckman for the case of uniform surface temperature.

It is not clear what the "von Kármán-Pohlhausen method" referred to by the authors is. If the method referred to is the approximate momentum approach to the solution of the boundary-layer equations due to von Kármán,[10] it is not surprising that good agreement between this approximate method and the mathematically rigorous method of the authors exists. The former has been justified experimentally and by comparison with the rigorous solution of Schmidt and Beckman. Eckert[11] has made this comparison.

The fact that the uniform surface heat-flux case and the uniform surface-temperature case are not greatly different is illustrated by Fig. 3 of the paper. It is seen that the surface-temperature variation for the uniform heat-flux case is not great over the major portion of the plate height. Hence, in practice, it seems hardly necessary to differentiate between the two cases.

For those not familiar with the solution of convection problems, it may be well to elaborate slightly on the origin of Equations [1], [2], and [3] of the paper. The rigorous solution of the free-convection problem requires the simultaneous solution of the fluid-flow equations (Navier-Stokes equations and continuity equation) and the equation of heat conduction in a moving fluid (energy equation). Equations [1], [2], and [3] are the remainders of the continuity equation, the Navier-Stokes equations, and the equation of heat conduction in a moving fluid, respectively,[12] after irrelevant terms have been discarded.

Authors' Closure

The authors wish to thank the discussers for their interest and for their comments.

The phrase "exact solution" was used to distinguish between solutions of: (*a*) the partial-differential equations expressing conservation of mass, momentum, and energy for the boundary layer, and (*b*) approximate forms of these conservation laws. It is regretted that this phrase conveyed other impressions.

The modified Grashof number, $\frac{g\beta x^4 q}{\nu^2 k}$, includes quantities which would all be known by a designer at the beginning of a calculation. For this reason the modified Grashof number, rather than the conventional Grashof number, was used in presenting the results of the analysis. It may be noted that the conventional Grashof number, $\frac{g\beta x^3(t_w - t_\infty)}{\nu^2}$, contains a temperature difference which would not be known at the beginning of a calculation.

The von Kármán-Pohlhausen method consists of the rather often-used integral formulation of the conservation laws due to von Kármán, coupled with the use of polynomial approximations for the velocity and temperature profiles due to K. Pohlhausen.

A rather careful exposition of the origin of Equations [1], [2], and [3] of the paper is given in *Modern Developments in Fluid Dynamics—High-Speed Flow*,[13] vol. II, pp. 766–769 and p. 801.

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[9] "Das Temperatur- und Geschwindigkeitsfeld vor einer Wärme abgebenden senkrechten Platte bei natürlichen Konvektion," by E. Schmidt and W. Beckman, *Technische Mechanik und Thermodynamik*, vol. 1, 1930, pp. 1–24; the solution is also presented in "Heat Transfer," by M. Jakob, John Wiley & Sons, Inc., New York, N. Y., 1949, pp. 443–451.

[10] "Über laminare und turbulente Reibung," by T. von Kármán, *Zeitschrift für angewandte Mathematik und Mechanik*, vol. 1, 1921, pp. 235–252.

[11] "An Introduction to the Transfer of Heat and Mass," by E. R. G. Eckert, McGraw-Hill Book Company, Inc., New York, N. Y., 1950, pp. 158–164.

[12] See, for example, equations [3–14], [3–23], and [3–34] of "Heat Transfer," by M. Jakob, for these equations in their general form.

[13] L. Howarth, editor, Clarendon Press, Oxford, England, 1953.