Discussion

CHIHIRO HAYASHI. Subharmonic oscillations of order 1/2 may occur in nonlinear systems and also in linear systems when system parameters change periodically with time. The device shown in Fig. 9 of the paper has a nonlinear characteristic provided by the guiding support of the pendulum. It also has a time-varying characteristic given by the rotating mass m around the motor. It is rather difficult to see which one of these characteristics (nonlinear or time-varying) gives rise to the subharmonic response. If the guiding surface of the support is removed and still subharmonic oscillations of order 1/2 could be obtained, one could conclude that the subharmonic response is due to the time-varying characteristic of the pendulum.

Generally speaking, subharmonic oscillations of order 1/2 in nonlinear systems are apt to occur when $f(x)$ in Equation [6] is a nonodd function of $x$, for example, when $f(x)$ is given by

$$f(x) = c_1x + c_2x^2 + c_3x^3$$

However, even when the nonlinearity is symmetrical, i.e., when $f(x)$ is an odd function of $x$, subharmonic oscillations of order 1/2 may occur. In this case the nonlinearity is symmetrical, but the oscillation itself becomes unsymmetrical and self-biased. In other words, the oscillation contains a unidirectional component. Further, subharmonic oscillations of order 1/2 with different amplitudes and wave forms may occur in the same system under the action of an equal periodic force. This is evidently due to the nonlinearity of the system and cannot be explained by the time-varying characteristic.

K. KLOTTER. The author has given a survey of a number of nonlinear phenomena and, with his often displayed skill, has demonstrated some of them by well-performed experiments. Among the experiments shown at the Conference (not recorded, however, in the preprints of the paper) was the pendulum with oscillating support. As the author has mentioned, it is well known that such a pendulum may oscillate at a frequency which is half the frequency of the support, and it is equally well known that the phenomenon can be described by a Mathieu differential equation of form

$$\ddot{x} + \omega^2 - \frac{U}{i} \cos x = 0$$

where $U$ and $\Omega$ denote amplitude and frequency of the support, $i$ is the equivalent length of the pendulum, and $x = \Omega t$. Equation [9] is a linear differential equation with nonconstant (sinoidally varying) coefficients.

The author then went on to demonstrate very clearly subharmonics of order 1/2, 1/3, 1/5, 1/9, etc., by means of the device shown in Fig. 9 of the paper. The subharmonics in this experiment were all attributed to the nonlinearity in the restoring force of that device.

The writer has some doubts as to the correctness of this explanation in regard to the subharmonics of order 1/2 (or of any order 1/2m). Certainly, the experiment, as described and performed leaves room for other explanations: Because the exciting force is provided by the inertia force of a single rotating mass, it has not only a horizontal component (as used for explaining the results of the experiment) but also a vertical one. This sinoidally varying vertical component of the inertia force produces a periodically varying effective stiffness of the spring; an equation of motion of the Mathieu type will follow and hence the situation is quite analogous to the one of the pendulum with oscillating support. The subharmonics observed may be solutions of a linear Mathieu differential equation, not solutions of a nonlinear differential equation.

The writer attributes quite a bit of significance to this situation, because in so far as he is aware, the existence of subharmonics of order 1/2 (or of order 1/2m generally) in systems having nonlinear restoring forces which are point-symmetrical (represented by purely odd functions of the displacement) has not yet been proved or disproved beyond doubt. Hence it would seem highly desirable to refine the experiments shown by the author in two different ways:

1. The experiment of Fig. 9 should be carried out with a linear spring (omitting the constraining walls and using very small displacements) in order to prove or disprove the existence of subharmonics of order 1/2.

2. The device shown in Fig. 9 should be modified so as to replace the linear restoring force by one with a highly nonlinear restoring force, such as a restoring force of the form

$$f(x) = Kx + cx^2$$

where $K$ and $c$ are constants. This modification would make it possible to observe the existence of subharmonics of order 1/2 and also to study the effects of the nonlinear restoring force on the behavior of the system.

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placements). If the subharmonics of order 1/2 still readily appear (as this writer surmises) the explanation by means of the linear Mathieu equation is the only one applicable.

2 In the nonlinear device of Fig. 9 an excitation should be produced by means of a purely horizontal force (sinusoidally varying). This could be accomplished by the use of two masses rotating clockwise and counterclockwise, respectively, thus destroying their vertical components. Depending on whether or not a subharmonic of order 1/2 will appear in this case one would have an indication about the existence of subharmonics of order 1/2 in systems having odd restoring forces.

The writer would like to urge the author strongly to devote his experimental skills to finding clear-cut answers to this still rather beclouded problem of subharmonics of order 1/2.

AUTHOR'S CLOSURE

Drs. Hayashi and Klotter both indicate in their discussions that the subharmonic of order one half may arise in systems which have nonlinearities, in systems which have time-varying characteristics, and in systems which have both. Since the system consisting of a rotating unbalance mounted on top of a vertical cantilever has both characteristics, it is difficult to decide which gives rise to the subharmonic.

Both writers suggest an experiment in which the nonlinearity is removed, but not the time-varying characteristic; and Dr. Klotter suggests also the possibility of removing the time-varying characteristics while retaining the nonlinearity. The author is in full agreement with these suggestions, and as soon as time permits he will perform these experiments and any others which suggest themselves in order to locate uniquely the characteristic which gives rise to subharmonic oscillations of order one half.

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