On Love Waves in Laterally and Vertically Heterogeneous Layered Media

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(Received 1975 October 22; in original form 1975 June 2)

Summary

An analysis is made of the two-dimensional problem of Love wave propagation in layered media in which the rigidity and density vary in both horizontal (x) and vertical (z) directions. The ratio $\mu/\rho$ is assumed to be independent of $x$ but may be a function of $z$. Moreover, the horizontal variation in $\mu$ and $\rho$ is taken to be the same in all the layers. The Thomson-Haskell matrix method is used to derive the dispersion equation for a multilayered model. The dispersion equation for some particular cases is obtained explicitly and the conditions for the existence of its roots are investigated.

1. Introduction

Seismic surface wave dispersion is widely used to investigate the elastic properties and the structural details of the interior of the Earth. The dispersion of surface waves depends markedly upon the elastic properties of the medium of propagation. It is now well established that the density and the elastic properties of the Earth vary laterally as well as vertically. Therefore, it is important to study the dispersion of surface waves in media whose density and elastic parameters are functions of both $x$ and $z$.

Propagation of Love waves in vertically heterogeneous media has been studied by several authors. But the propagation of Love waves in laterally heterogeneous media has been considered by only a few investigators. Bhattacharya (1970b), Chatterjee (1972), Negi & Singh (1973a, b) and Singh (1974) applied the principle of constructive interference to derive the frequency equation of Love-type waves propagating in a laterally inhomogeneous layer overlying a homogeneous half-space. Bhattacharya (1970b) assumed that in the layer

$$\beta = \beta_0 (1 + bx), \quad \mu = \mu_0 (1 + mx)$$

and obtained the Love-wave dispersion curves for the first two modes. Chatterjee (1972) and Negi & Singh (1973a) studied the dispersion of Love waves when

$$\beta = \beta_0 e^{bx}, \quad \mu = \mu_0 e^{bx}.$$

These authors also obtained the Love-wave dispersion curves for the first two modes. Negi & Singh (1973b) discussed the exponential case all over again and, by a com-
parison of the theoretical Love-wave dispersion with the Delhi data of the Severnaya Zemlya earthquake of 1964 August 25, obtained estimates of the values of velocity, density and rigidity gradients as 1·14 × 10⁻³ s⁻¹, 0·42 × 10⁻³ g cm⁻³ km⁻¹ and 2·85 × 10⁻³ dynes cm⁻² km⁻¹ respectively. Singh (1974) assumed a transversely isotropic, laterally inhomogeneous layer lying over an isotropic, homogeneous half-space. He took the horizontal and vertical rigidities and density in the layer to be

\[ \mu_t = \mu_{t0}(1+ax)^3, \quad \mu_v = \mu_{v0}(1+ax), \quad \rho = \rho_0(1+ax), \]

and presented the dispersion curves for the first two modes.

Babich & Molotkov (1966) investigated the propagation of high-frequency Love waves in an elastic half-space which is inhomogeneous in the \( x \) and \( z \) directions. These authors used the parabolic equation method to study the dependence of Love waves on the co-ordinates and the frequency. De (1968) considered Love wave propagation in a two-layered medium assuming that in the layer

\[ \rho = \rho_1 e^{ex}, \quad \mu = \mu_1 e^{ex}, \]

and in the half-space

\[ \rho = \rho_2 e^{ex}, \quad \mu = \mu_2 e^{ex}. \]

He obtained the frequency equation by solving the equation of motion exactly and applying the appropriate boundary conditions.

We have considered the general two-dimensional problem of Love wave propagation in an \( n \)-layered medium in which \( \rho \) and \( \mu \) are functions of \( x \) and \( z \), but \( \mu/\rho \) is independent of \( x \). The Thomson–Haskell matrix method (Thomson 1950; Haskell 1953) is used to obtain the dispersion equation. The explicit form of the dispersion equation is derived for a half-space, for a layer over a rigid bottom and for a two-layered half-space. The conditions for the existence of the roots of the dispersion equation are discussed.

### 2. Equation of motion and its solution

Consider an isotropic elastic half-space occupying the region \( z \geq 0 \). If we specialize to a two-dimensional configuration in which the density \( \rho \), Lamé’s parameters \( \lambda, \mu \) and all the components of the displacement are independent of the co-ordinate \( y \), then the \( SH \) motion is decoupled from the \( P-SV \) motion (Kennett 1974). The equation of small motion of the \( SH \)-type is given by

\[ \frac{\partial}{\partial x} \left( \mu \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial z} \left( \mu \frac{\partial v}{\partial z} \right) = \rho \frac{\partial^2 v}{\partial t^2}, \tag{2.1} \]

where \( v \) is the \( y \)-component of the displacement. If we put

\[ v = W/\sqrt{\mu}, \tag{2.2} \]

equation (2.1) takes the form

\[ \mu \nabla^2 W + \frac{1}{2\mu} \left[ \mu_x^2 + \mu_z^2 \right] - \mu_{xx} - \mu_{zz} \right] W = \rho W_t, \tag{2.3} \]

where

\[ \mu_x = \frac{\partial \mu}{\partial x}, \quad \mu_{xx} = \frac{\partial^2 \mu}{\partial x^2}, \quad \text{etc.} \]

We now assume that

\[ W(x, z, t) = X(x) Z(z) e^{iot} \tag{2.4} \]

and

\[ \mu(x, z) = \mu_0 p(x) q(z), \quad \rho(x, z) = \rho_0 p(x) f(z), \tag{2.5} \]
where \( \omega \) is the angular frequency and \( \mu_0, \rho_0 \) are constants. From equations (2.3) to (2.5) we find that \( X(x) \) and \( Z(z) \) satisfy the following differential equations

\[
X_{xx} + \left[ r^2 + \frac{1}{4} \left( \frac{p_x}{p} \right)^2 - \frac{1}{2} \frac{p_{xx}}{p} \right] X = 0, \tag{2.6}
\]

\[
Z_{zz} + \left[ k_\beta^2 - r^2 + \frac{1}{4} \left( \frac{q_z}{q} \right)^2 - \frac{1}{2} \frac{q_{zz}}{q} \right] Z = 0, \tag{2.7}
\]

where \( r^2 \) is a separation constant and

\[
k_\beta^2 = \frac{\omega^2 f(z)}{\beta_0^2 q(z)}, \quad \beta_0^2 = \frac{\mu_0}{\rho_0}. \tag{2.8}
\]

We further assume that \( p(x) \) is of such a form that

\[
-\frac{1}{4} \left( \frac{p_x}{p} \right)^2 + \frac{1}{2} \frac{p_{xx}}{p} = a_0^2, \tag{2.9}
\]

where \( a_0^2 \) is a constant. In that case equation (2.6) reduces to

\[
X_{xx} + k^2 X = 0, \tag{2.10}
\]

where

\[
k^2 = r^2 - a_0^2. \tag{2.11}
\]

Therefore, \( X(x) = \exp(\pm ikx) \), and from equation (2.4), for waves propagating in the positive \( x \)-direction, we may take

\[
W(x, z, t) = Z(z) e^{(\omega t - kx)}. \tag{2.12}
\]

This shows that \( k \) is the wave number in the horizontal direction.

Equation (2.9) may be written as

\[
\frac{d}{dp} \left( \frac{p_{x}^2}{p} \right) = 4a_0^2. \tag{2.13}
\]

Integrating the last equation, we find

\[
\frac{dp}{dx} = \pm 2(a_0^2 p^2 + a_1 p)^{\frac{1}{2}}, \tag{2.14}
\]

where \( a_1 \) is a constant. Hence

\[
x + a_2 = \pm \frac{1}{2} \int (a_0^2 p^2 + a_1 p)^{-\frac{1}{2}} dp, \tag{2.15}
\]

where \( a_2 \) is another constant. We consider the following three cases:

**Case I:** \( a_1 = 0 \)

Equation (2.15) yields on integration

\[
p = e^{\pm 2a_0x}. \tag{2.16}
\]

taking \( a_2 = 0 \) which only amounts to dropping a constant factor from the right-hand side of equation (2.16).
Case II: $a_0 = 0$

Equation (2.15) now gives

$$p = (1 + x/a_2)^2,$$

where we have put $a_1 a_2^2 = 1$.

Case III: $a_0 \neq 0, a_1 \neq 0$

Equation (2.15) may be written as

$$x + a_2 = \pm \frac{1}{2} \int \left[ \left( a_0 p - \frac{a_1}{2a_0} \right)^2 - \frac{a_1^2}{4a_0^2} \right]^{-\frac{1}{2}} dp.$$

On integration, this yields

$$p = \sinh^2 \left( a_0 x + a_3 \right),$$

where $a_3 = a_0 a_2$ and we have taken $a_1 = a_0^2$.

Equation (2.18) gives the general solution of equation (2.13) when $a_0 \neq 0, a_1 \neq 0$. It may be noted that $a_0$ and $a_1$ cannot vanish simultaneously since, in that case, the medium reduces to a laterally homogeneous medium as is obvious from equation (2.14).

Using (2.11), equation (2.7) becomes

$$Z_{zz} + \left[ k \rho^2 - k^2 - a_0^2 + \frac{1}{4} \left( \frac{q_x}{q} \right)^2 - \frac{1}{2} \frac{q_{zz}}{q} \right] Z = 0.$$

This equation can be solved in terms of known functions such as Bessel and Whittaker functions for some particular forms of $q(z)$ and $f(z)$. Some examples together with the corresponding solutions of equation (2.19) are listed in the Appendix.

We next assume that $f(z) = q(z)$ and $q(z)$ is such that

$$-\frac{1}{4} \left( \frac{q_x}{q} \right)^2 + \frac{1}{2} \frac{q_{zz}}{q} = b_0^2, \quad (2.20)$$

where $b_0^2$ is a constant. Equation (2.19) then becomes

$$Z_{zz} = \left[ k^2 + a_0^2 + b_0^2 - \omega^2 / \beta_0^2 \right] Z = 0.$$

Thus, we may take

$$Z = A e^{sz} + B e^{-sz}, \quad (2.22)$$

where $A, B$ are arbitrary constants and

$$s = \begin{cases} (a_0^2 + b_0^2 + k^2 - \omega^2 / \beta_0^2)^{\frac{1}{2}} & \text{for } \omega / \beta_0 < (a_0^2 + b_0^2 + k^2)^{\frac{1}{2}}, \\ -i (\omega / \beta_0^2 - a_0^2 - b_0^2 - k^2)^{\frac{1}{2}} & \text{for } \omega / \beta_0 > (a_0^2 + b_0^2 + k^2)^{\frac{1}{2}}. \end{cases} \quad (2.23)$$

From equations (2.2), (2.5), (2.12) and (2.22), we obtain

$$v(x, z, t) = \frac{1}{\sqrt{\left( \mu_0 p(x) q(z) \right)}} \left[ A e^{sx} + B e^{-sx} \right] e^{i(\omega t - kx)}, \quad (2.24)$$

$$p_{xz}(x, z, t) = \mu \frac{\partial v}{\partial z}$$

$$= \sqrt{\mu_0 p(x) q(z)} \left[ A \left( s - \frac{1}{2} \frac{q_x}{q} \right) e^{sx} - B \left( s + \frac{1}{2} \frac{q_x}{q} \right) e^{-sx} \right] e^{i(\omega t - kx)}. \quad (2.25)$$

Equations (2.24) and (2.25) hold when $\mu / \rho$ is independent of $x$ and $z$ and equations (2.9) and (2.20) are satisfied.
Equation (2.20) is similar to equation (2.9) and, therefore, can be integrated likewise. We thus get

\[
\left( \frac{dq}{dz} \right)^2 = 4(b_0^2 q^2 + b, q),
\]  

(2.26)

\[
q = e^{\pm 2b_0 z} \quad \text{for} \quad b_1 = 0,
\]  

(2.27)

\[
q = (1 + z/b_2)^2, \quad b_1 b_2^2 = 1, \quad \text{for} \quad b_0 = 0,
\]  

(2.28)

where \( b_1 \) is an arbitrary constant. When \( b_0 \) and \( b_1 \) are both different from zero, we have

\[
q = \sinh \left( b_0 z + b_3 \right),
\]  

(2.29)

where \( b_3 \) is another constant and we have taken \( b_0^2 = 1 \).

3. Love waves in a heterogeneous half-space

Since \( v \) should tend to zero as \( z \to \infty \), equation (2.24) shows that \( s \) must be real and \( A = 0 \). The boundary condition at the free surface gives

\[
p_{xx} = 0 \quad \text{at} \quad z = 0.
\]  

(3.1)

From equations (2.25) and (3.1), we obtain

\[
-\frac{1}{2} \left( \frac{1}{q(z)} \frac{dq(z)}{dz} \right)_{z=0} = s.
\]  

(3.2)

This is the frequency equation for Love waves in an inhomogeneous half-space. Equation (2.23) shows that \( s \) will be real if

\[
\omega^2 / \beta_0^2 - k^2 < a_0^2 + b_0^2.
\]  

(3.3)

Writing \( \omega = ck \), equation (3.3) reveals that the phase velocity \( c \) of Love waves is constrained by the condition

\[
c < \beta_0 \left[ 1 + (a_0^2 + b_0^2)/k^2 \right]^{1/2}.
\]  

(3.4)

Since \( s > 0 \), equation (3.2) will be satisfied only if

\[
\left[ \frac{1}{q(z)} \frac{dq(z)}{dz} \right]_{z=0} < 0.
\]  

(3.5)

This implies that in the neighbourhood of the free surface the density and rigidity decrease with depth. Therefore, in the case of a heterogeneous half-space in which

\[
\mu/\mu_0 = \rho/\rho_0 = p(x) q(z),
\]

where \( p(x) \) and \( q(z) \) satisfy equations (2.9) and (2.20), respectively, Love waves cannot exist if the rigidity increases with depth near the free surface.

Using equations (2.23) and (2.26), equation (3.2) can be transformed to

\[
\omega^2 / \beta_0^2 - k^2 = a_0^2 - b_1 q(0).
\]  

(3.6)

Thus the \( \omega - k \) graph is a hyperbola. The inequality (3.3) will be satisfied provided

\[
b_1 / q(0) > -b_0^2.
\]  

(3.7)

An examination of equations (2.27) to (2.29) reveals that the condition (3.7) can be
satisfied in all the three cases. The exact form of equation (3.6) is given in the following table.

<table>
<thead>
<tr>
<th>( q(z) )</th>
<th>Frequency equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \exp(\pm 2b_0 z) )</td>
<td>( \omega^2/\beta_0^2 - k^2 = a_0^2 )</td>
</tr>
<tr>
<td>( (1+z/b_2)^2 )</td>
<td>( \omega^2/\beta_0^2 - k^2 = a_0^2 - 1/b_2^2 )</td>
</tr>
<tr>
<td>( \sinh^2 (b_0 z + b_3) )</td>
<td>( \omega^2/\beta_0^2 - k^2 = a_0^2 - b_0^2/\sinh^2 b_3 )</td>
</tr>
</tbody>
</table>

When \( \mu/\rho \) varies with depth, i.e. when \( f(z) \neq q(z) \), equation (3.2) is replaced by

\[
\frac{1}{2} \left[ \frac{1}{q(z)} \frac{dq(z)}{dz} \right]_{z=0} = \left[ \frac{1}{Z} \frac{dZ}{dz} \right]_{z=0},
\]

where \( Z \) is the solution of equation (2.19) which is bounded as \( z \to \infty \) (see appendix). Ewing, Jardetzky & Press (1957, pp. 347–349) discussed Love-wave propagation in a vertically heterogeneous half-space. Equation (3.8) is similar to equation (7–112) of Ewing et al. (1957).

4. Love waves in a heterogeneous layer overlying a rigid bottom

The boundary conditions are:

\[
p_{xx} = 0 \quad \text{at} \quad z = 0, \tag{4.1}
\]

\[
v = 0 \quad \text{at} \quad z = H, \tag{4.2}
\]

where \( H \) is the thickness of the layer. Equations (2.24), (2.25), (4.1) and (4.2) yield

\[
\int \left[ \frac{1}{q(z)} \frac{dq(z)}{dz} \right]_{z=0} + s \coth(sH) = 0. \tag{4.3}
\]

As \( H \to \infty \), equation (4.3) reduces to equation (3.2).

If

\[
\left[ \frac{1}{q(z)} \frac{dq(z)}{dz} \right]_{z=0} < 0, \tag{4.4}
\]

equation (4.3) can be satisfied for real as well as imaginary values of \( s \). However, if

\[
\left[ \frac{1}{q(z)} \frac{dq(z)}{dz} \right]_{z=0} > 0, \tag{4.5}
\]

equation (4.3) will have a solution only if \( s \) is imaginary. Equation (2.23) then yields

\[
\omega/\beta_0 > (a_0^2 + b_0^2 + k^2)^{1/2}. \tag{4.6}
\]

Putting \( s = -is' \), equation (4.3) becomes

\[
s' \cot(s'H) = \frac{1}{2} \left[ \frac{1}{q(z)} \frac{dq(z)}{dz} \right]_{z=0}. \tag{4.7}
\]

This equation will be satisfied provided \( \cot(s'H) < 0 \). Therefore, for the \( n \) th mode, we have

\[
(n+1/2)\pi < s'H < (n+1)\pi, \tag{4.8}
\]
On Love waves in heterogeneous layered media

which, on using equation (2.23), yields

\[(n + \frac{1}{2})^2 \pi^2 + (a_0^2 + b_0^2 + k^2) H^2 < \frac{\omega^2 H^2}{\beta_0^2} < (n + 1)^2 \pi^2 + (a_0^2 + b_0^2 + k^2) H^2.\]  (4.9)

In the case of a vertically homogeneous medium equation (4.3) reduces to

\[\coth(sH) = 0.\]  (4.10)

This yields

\[(\omega^2/\beta_0^2 - a_0^2 - k^2) H^2 = (n + \frac{1}{2})^2 \pi^2,\]  (4.11)

where \(n\) is any positive integer or zero and equation (2.23) with \(b_0 = 0\) has been used. If the medium is homogeneous, \(a_0 = 0\), and we have

\[(\omega^2/\beta_0^2 - k^2) H^2 = (n + \frac{1}{2})^2 \pi^2.\]  (4.12)

This is the well-known frequency equation for Love waves in a homogeneous layer of depth \(H\) lying over a rigid bottom (Hudson 1962, equation (2.3)).

Equations (4.11) and (4.12) are similar. There is no real solution of equation (4.12) for

\[\omega < \beta_0 \frac{(n + \frac{1}{2})}{H} \pi.\]  (4.13)

On the other hand, there is no real solution of equation (4.11) for

\[\omega < \beta_0 \left[ \frac{(n + \frac{1}{2})^2}{H^2} \pi^2 + a_0^2 \right]^\frac{1}{2}.\]  (4.14)

Thus the dimensionless cut-off frequency shifts from \((n + \frac{1}{2}) \pi\) for a homogeneous medium to

\[\left[ (n + \frac{1}{2}) \pi^2 + a_0^2 H^2 \right]^\frac{1}{2}\]

for a laterally heterogeneous medium. Assuming that \(a_0^2 H^2\) is small as compared to \((n + \frac{1}{2})^2 \pi^2\), the net shift is approximately equal to \(a_0^2 H^2/(2n+1)\pi\). Therefore the shift is maximum for the fundamental mode being equal to \(a_0^2 H^2 \pi^{-1}\) and vanishes for large \(n\).

When \(p(x) = (1 + x/a_2)^2\), \(a_0^2 = 0\) and, therefore, the frequency equation (4.11) coincides with the frequency equation (4.12) for a homogeneous medium.

When \(\mu/\rho\) is a function of depth, the frequency equation (4.3) is replaced by

\[\frac{1}{2} \left[ \frac{1}{q(z)} \frac{dq(z)}{dz} \right]_o = \frac{Z_2(H) \left( \frac{dZ_1}{dz} \right)_o - Z_1(H) \left( \frac{dZ_2}{dz} \right)_o}{Z_1(0) Z_2(H) - Z_2(0) Z_1(H)} \]  (4.15)

where \(Z_1\) and \(Z_2\) are two independent solutions of (2.19).

5. Love waves in a multilayered heterogeneous half-space

Consider a semi-infinite medium made up of \(n - 1\) parallel, heterogeneous layers overlying a heterogeneous half-space. We number the layers serially, the layer at the top being layer 1. We place the origin of a Cartesian co-ordinate system \((x, y, z)\)
at the free surface. The $m$th layer is of thickness $d_m$ and is bounded by the interfaces $z = z_{m-1}$, $z = z_m$. Evidently, $z_0 = 0$ and $z_{n-1} = H$, where $H$ is the depth of the last interface. The rigidity and density of the $m$th layer are denoted by $\mu_m(x, z)$ and $\rho_m(x, z)$ respectively. We assume

$$\begin{align*}
\mu_m(x, z) &= \mu_{0m} p(x) q_m(z), \\
\rho_m(x, z) &= \rho_{0m} p(x) q_m(z),
\end{align*}$$

(5.1)

so that the lateral variation of $\mu$ and $\rho$ is the same in each layer,

$$\begin{align*}
\frac{1}{2} \frac{d^2 p}{dx^2} - \frac{1}{4} \left( \frac{1}{p} \frac{dp}{dx} \right)^2 &= a_0^2, \\
\frac{1}{2q_m} \frac{d^2 q_m}{dz^2} - \frac{1}{4} \left( \frac{1}{q_m} \frac{dq_m}{dz} \right)^2 &= b_m^2,
\end{align*}$$

where $a_0, b_m$ are constants. Further, we introduce the notation

$$\beta_{0m}^2 = \frac{\mu_{0m}}{\rho_{0m}},$$

(5.2)

$$s_m = \begin{cases} 
(a_0^2 + b_m^2 + k^2 - \alpha^2/\beta_{0m}^2)^+ & \text{for } \alpha/\beta_{0m} < (a_0^2 + b_m^2 + k^2)^+, \\
- i(\alpha^2/\beta_{0m}^2 - a_0^2 - b_m^2 + k^2)^+ & \text{for } \alpha/\beta_{0m} > (a_0^2 + b_m^2 + k^2)^+.
\end{cases}$$

For the $m$th layer, we may take as in equation (2.24)

$$v_m(x, z, t) = \frac{1}{\sqrt{\mu_{0m} p(x) q_m(z)}} \left[ A_m e^{sx} + B_m e^{-sx} \right] e^{i(\alpha t - kx)},$$

(5.3)

where $\alpha, \beta_{0m}$ are arbitrary constants. We thus have

$$\begin{align*}
v_m(x, z, t) &= \frac{1}{\sqrt{p(x)}} V_m(z) e^{i(\alpha t - kx)}, \\
(p_{yz})_m &= \mu_m \frac{\partial}{\partial z} v_m(x, z, t) \\
&= \sqrt{p(x)} P_m(z) e^{i(\alpha t - kx)},
\end{align*}$$

(5.4)

where

$$V_m(z) = \frac{1}{\sqrt{\mu_{0m} q_m(z)}} \left[ A_m e^{sx} + B_m e^{-sx} \right],$$

(5.5)

$$P_m(z) = \sqrt{\mu_{0m} q_m(z)} \left[ (s_m - \frac{1}{2q_m} \frac{d q_m}{dz}) e^{sx} A_m - \left( s_m + \frac{1}{2q_m} \frac{d q_m}{dz} \right) e^{-sx} B_m \right].$$

(5.6)

Equations (5.5) and (5.6) can be combined into a single matrix equation

$$\begin{bmatrix} V_m(z) \\ P_m(z) \end{bmatrix} = D_m(z) \begin{bmatrix} A_m \\ B_m \end{bmatrix},$$

(5.7)
where the elements of the $2 \times 2$ matrix $D_m$ are:

\[
\begin{align*}
D_{11} &= \frac{1}{\sqrt{(\mu_0 q_m(z))}} e^{s_m z}, \\
D_{12} &= \frac{1}{\sqrt{(\mu_0 q_m(z))}} e^{-s_m z}, \\
D_{21} &= \sqrt{(\mu_0 q_m(z))} \left(s_m - \frac{1}{2q_m} \frac{dq_m}{dz}\right) e^{s_m z}, \\
D_{22} &= -\sqrt{(\mu_0 q_m(z))} \left(s_m + \frac{1}{2q_m} \frac{dq_m}{dz}\right) e^{-s_m z}.
\end{align*}
\]

(5.8)

From equation (5.7), we have

\[
\begin{bmatrix}
A_m \\
B_m
\end{bmatrix} = \left[D_m(z)\right]^{-1} \begin{bmatrix}
V_m(z) \\
P_m(z)
\end{bmatrix},
\]

(5.9)

where the inverse matrix $[D_m]^{-1}$ is given by

\[
[D_m]^{-1} = \frac{1}{2s_m} \begin{bmatrix}
-D_{22} & D_{12} \\
D_{21} & -D_{11}
\end{bmatrix}.
\]

(5.10)

Equations (5.7) and (5.9) yield

\[
\begin{bmatrix}
V_m(z_m) \\
P_m(z_m)
\end{bmatrix} = G_m \begin{bmatrix}
V_m(z_{m-1}) \\
P_m(z_{m-1})
\end{bmatrix},
\]

(5.11)

where $G_m$ is given by

\[
G_m = [D_m(z_m)] [D_m(z_{m-1})]^{-1}.
\]

(5.12)

Since $v$ and $p_{yz}$ are continuous across each interface, equation (5.4) shows that $V_m(z)$ and $P_n(z)$ must be continuous across each interface. Therefore

\[
V_m(z_{m-1}) = V_{m-1}(z_{m-1}), \quad P_m(z_{m-1}) = P_{m-1}(z_{m-1})
\]

(5.13)

and equation (5.11) may be written as:

\[
\begin{bmatrix}
V_m(z_m) \\
P_m(z_m)
\end{bmatrix} = G_m \begin{bmatrix}
V_{m-1}(z_{m-1}) \\
P_{m-1}(z_{m-1})
\end{bmatrix}.
\]

(5.14)

Using equation (5.14) repeatedly, we obtain, with the help of equation (5.11) and (5.13),

\[
\begin{bmatrix}
V_n(z_{n-1}) \\
P_n(z_{n-1})
\end{bmatrix} = J \begin{bmatrix}
V_1(z_0) \\
P_1(z_0)
\end{bmatrix},
\]

(5.15)

where

\[
J = G_{n-1} G_{n-2} \ldots G_1.
\]

Since the free surface is stress free, $P_1(z_0) = 0$. Hence we get

\[
\frac{V_n(H)}{P_n(H)} = \frac{J_{11}}{J_{21}},
\]

(5.16)

where $H = z_{n-1}$. The radiation condition demands that $s_n$ be real and $A_n = 0$. From equation (5.2) $s_n$ will be real if

\[
\omega / \beta_n < (a_0^2 + b_n^2 + k^2)^{\frac{1}{2}}.
\]
Further, if $A_n = 0$, equations (5.5) and (5.6) yield

$$\frac{P_n(H)}{V_n(H)} = -\mu_{on} q_n(H) \left[ s_n + \frac{1}{2q_n(z)} \frac{dq_n(z)}{dz} \right]_{z = h}.$$ 

Hence, from equation (5.16), we find

$$\mu_{on} q_n(H) \left[ s_n + \frac{1}{2q_n(z)} \frac{dq_n(z)}{dz} \right]_{z = h} + \frac{J_{21}}{J_{11}} = 0. \quad (5.17)$$

This is the desired frequency equation for the propagation of Love waves in a multilayered heterogeneous half-space.

Two-layered half-space

In this case $n = 2$, $J = G_1$ and

\[
\begin{align*}
J_{11} &= \frac{1}{s_1} \left[ \frac{q_1(0)}{q_1(H)} \right]^{1/2} \left[ s_1 \cosh s_1 H + \frac{1}{2} \left( \frac{1}{q_1} \frac{dq_1}{dz} \right)_0 \sinh s_1 H \right], \\
J_{21} &= \frac{\mu_{o1}}{s_1} \left( q_1(0) q_1(H) \right) \left[ s_1^2 \sinh s_1 H - \frac{1}{s_1} \left( \frac{1}{q_1} \frac{dq_1}{dz} \right)_0 \cosh s_1 H \\
&\quad + \frac{1}{2} s_1 \left( \frac{1}{q_1} \frac{dq_1}{dz} \right)_0 \cosh s_1 H \\
&\quad - \frac{1}{2} \left( \frac{1}{q_1} \frac{dq_1}{dz} \right)_0 \left( \frac{1}{q_1} \frac{dq_1}{dz} \right)_H \sinh s_1 H \right]. \quad (5.18)
\end{align*}
\]

Substituting in equation (5.17) we get, after some simplification,

\[
\begin{align*}
tanh (s_1 H) \left[ \mu_{o1} q_1(H) \left\{ s_1^2 + \frac{1}{2} \left( \frac{1}{q_1} \frac{dq_1}{dz} \right)_0 \left( \frac{1}{q_1} \frac{dq_1}{dz} \right)_H \right\} \\
+ \frac{1}{2} \mu_{o2} q_2(H) \left( \frac{1}{q_1} \frac{dq_1}{dz} \right)_0 \left\{ s_2 + \frac{1}{2} \left( \frac{1}{q_2} \frac{dq_2}{dz} \right)_H \right\} \right] \\
+ \frac{s_1}{2} \mu_{o1} q_1(H) \left[ \left( \frac{1}{q_1} \frac{dq_1}{dz} \right)_0 - \left( \frac{1}{q_1} \frac{dq_1}{dz} \right)_H \right] \\
+ s_1 \mu_{o2} q_2(H) \left[ s_2 + \frac{1}{2} \left( \frac{1}{q_2} \frac{dq_2}{dz} \right)_H \right] &= 0. \quad (5.19)
\end{align*}
\]

The effect of the lateral heterogeneity is to introduce an additional term $a_0^2$ in the expressions for the radicals $s_1$ and $s_2$ (equation (5.2)). If $\mu$, $\rho \propto (1 + x/a_2)^2$, then $a_0 = 0$ and these radicals for a laterally heterogeneous model are the same as for a laterally homogeneous model. Therefore, the frequency equation (5.19) for a model in which $\mu$, $\rho \propto (1 + x/a_2)^2$ is identical with the corresponding frequency equation for a laterally homogeneous model. The same applies to the frequency equation (5.17) for Love waves in a multilayered half-space.
As \( \mu_0 \to \infty \), equation (5.19) becomes

\[
\frac{1}{q_1} \left( \frac{dq_1}{dz} \right)_0 \tanh (s_1 H) + s_1 = 0,
\]

which coincides with the dispersion equation (4.3) for Love waves in a heterogeneous layer over a rigid bottom.

If \( q_1 = 1 \), equation (5.19) reduces to

\[
\mu_0 \: s_1 \tanh (s_1 H) + \mu_0 q_2 (H) \left[ s_2 + \frac{1}{q_2} \left( \frac{dq_2}{dz} \right)_H \right] = 0,
\] (5.20)

where now

\[
s_1 = (a_0^2 + k^2 - \omega^2/\beta_{01}^2)^{\dagger} \text{ for } \omega/\beta_{01} < (a_0^2 + k^2)^{\dagger},
\]

\[
= -i(\omega^2/\beta_{01}^2 - a_0^2 - k^2)^{\dagger} \text{ for } \omega/\beta_{01} > (a_0^2 + k^2)^{\dagger}.
\] (5.21)

Equation (5.20) is the frequency equation for Love waves in a laterally heterogeneous two-layered model in which the layer is vertically homogeneous but the substratum is vertically inhomogeneous.

If \( q_1 = q_2 = 1 \), equation (5.20) becomes

\[
\mu_0 \: s_1 \tanh (s_1 H) + \mu_0 \: s_2 = 0,
\] (5.22)

where

\[
s_2 = (a_0^2 + k^2 - \omega^2/\beta_{02}^2)^{\dagger}.
\] (5.23)

Equation (5.22) represents the frequency equation for Love waves in a two-layered laterally inhomogeneous half-space. If we take \( p(x) = e^{\alpha x} \), equation (5.22) coincides with the frequency equation derived by De (1968). Lastly, if the layer as well as the half-space are homogeneous, equation (5.22) reduces to the standard dispersion equation for Love waves in a two-layered homogeneous half-space. In fact, equation (5.22) can be interpreted in terms of the corresponding equation for a homogeneous model if we replace \( k^2 \) by \( k^2 + a_0^2 \).

For the existence of surface waves it is necessary that \( s_2 \) be real. Consequently, from equation (5.23) we find that

\[
\omega < \beta_{02}(a_0^2 + k^2)^{\dagger}. \tag{5.24}
\]

With \( s_2 \) real and positive, equation (5.22) cannot be satisfied for real values of \( s_1 \). Hence, from equation (5.21), we have

\[
\omega > \beta_{01}(a_0^2 + k^2)^{\dagger} \tag{5.25}
\]

and \( s_1 = -is_1' \), where \( s_1' \) is real and positive. From (5.24) and (5.25), we find

\[
\beta_{02}(a_0^2 + k^2)^{\dagger} > \omega > \beta_{01}(a_0^2 + k^2)^{\dagger}. \tag{5.26}
\]

Therefore \( \beta_{02} > \beta_{01} \), i.e. for all \( x \), the shear wave velocity in the underlying half-space is greater than the shear wave velocity in the layer.

Using equations (5.21), (5.23) and (5.26), the frequency equation (5.22) may be written as

\[
\tan [kH(c^2/\beta_{01}^2 - 1 - a_0^2/k^2)^{\dagger}] = \frac{\mu_{02}(1 + a_0^2/k^2 - c^2/\beta_{02}^2)^{\dagger}}{\mu_{01}(c^2/\beta_{01}^2 - 1 - a_0^2/k^2)^{\dagger}}. \tag{5.27}
\]

6. Conclusions

We have made two simplifying assumptions. Firstly, it has been assumed that the lateral variation at all depths is the same. Secondly, it is supposed that the shear
wave velocity does not vary laterally. This assumption, though not physically justified, has been made for the sake of mathematical tractability of the problem. Thus our analysis gives the effect of the lateral variation of $\mu$ and $\rho$ when $\mu/\rho$ is independent of $x$. Though, we have taken $\mu/\rho$ as constant in each layer, the case when $\mu/\rho$ varies with $z$ can be discussed on parallel lines with the help of the results given in the Appendix.

Acknowledgments

This research was financed by the Department of Atomic Energy, Government of India under project No. BRNS/Maths/8/74. The authors are thankful to the referee for his helpful comments which improved the presentation of the paper.

References


Appendix

We list below the solutions of equation (2.12) for some particular forms of $q(z)$ and $f(z)$ (see also Avtar 1967; Bhattacharya 1970a).
(i) \( q(z) = (1 + \delta z)^2, \quad f(z) = (1 + e z), \quad Z = A W_{i,m}(\xi) + B W_{-i,m}(-\xi). \)

Here \( A \) and \( B \) are arbitrary constants and \( W_{\pm i,m}(\pm \xi) \) are Whittaker functions. Moreover,

\[ \xi = 2 \sqrt{(k^2 + a_0^2)(z + 1/\delta)}, \]

\[ l = \frac{k_0^2}{2 \delta^2 \sqrt{(k^2 + a_0^2)}}, \]

\[ m^2 = \frac{k_0^2}{\delta^2} (e - \delta) + \frac{1}{4}. \]

(ii) \( q(z) = (1 + \delta z)^2, \quad f(z) = (1 + \delta z)^2 (1 + e z)^{-2}, \quad Z = \xi^\pm [A K_m(\theta) + B I_m(\theta)]. \)

Here \( I_m(\theta) \) and \( K_m(\theta) \) are modified Bessel functions. Moreover,

\[ \theta = \sqrt{(k^2 + a_0^2)(z + 1/\delta)}, \]

\[ m^2 = \frac{1}{4} - \frac{k_0^2}{\epsilon^2}, \quad \xi = 1 + e z. \]

(iii) \( q(z) = (1 + \delta z)^2, \quad f(z) = (1 + \delta z)^2 (1 + \gamma e^z)^{-2}, \quad Z = \xi^\pm [A K_m(\theta) + B I_m(\theta)]. \)

Here \( H_m^{(1)}(l e^z) \) and \( H_m^{(2)}(l e^z) \) are Hankel functions and

\[ \xi = \frac{1}{2} \epsilon z, \quad l = \frac{2 k_0}{\epsilon} \sqrt{\gamma}, \]

\[ m^2 = \frac{4}{\epsilon^2} (k^2 + a_0^2 - k_0^2). \]

(iv) \( q(z) = e^{\delta z}, \quad f(z) = e^{\delta z}/(1 + e z), \quad Z = A W_{i,m}(\xi) + B W_{-i,m}(-\xi). \)

Here

\[ \xi = \frac{1}{\epsilon} (4k^2 + 4a_0^2 + \delta^2)^\frac{3}{4}(1 + e z), \]

\[ l = \frac{k_0^2}{\epsilon} (4k^2 + 4a_0^2 + \delta^2)^{-\frac{1}{4}}, \quad m = \frac{1}{4}. \]

(v) \( q(z) = (1 + \delta z)^p, \quad f(z) = (1 + \delta z)^{p-2}, \quad Z = \xi^\pm [A K_m(\theta) + B I_m(\theta)]. \)

Here

\[ \xi = 1 + \delta z, \]

\[ m^2 = \frac{1}{4} (p - 1)^2 - k_0^2/\delta^2, \]

\[ \theta = \sqrt{(k^2 + a_0^2)(z + 1/\delta)}. \]

(vi) \( q(z) = f(z) = (1 + \delta z)^p, \quad Z = \xi^\pm [A K_m(l \xi) + B I_m(l \xi)]. \)

Here

\[ \xi = 1 + \delta z, \quad l = \frac{1}{\delta} (k^2 + a_0^2 - k_0^2)^\frac{3}{4}, \quad m = \frac{1}{4}(e - 1). \]
(vii) \( q(z) = e^{\xi z} \), \( f(z) = e^{\xi z} \), \( Z = AH_m^{(1)}(l e^\xi) + BH_m^{(2)}(l e^\xi) \).

In this case
\[
\xi = \frac{1}{2}(\varepsilon - \delta) z, \\
l = \frac{2k_\theta}{\varepsilon - \delta}, \\
m^2 = (4k^2 + 4a_0^2 + \delta^2)(\varepsilon - \delta)^{-2}.
\]