Crossing-Symmetric Decomposition of the $n$-Point Veneziano Formula into Tree-Graph Integrals. I

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Given a cyclic ordering of external particles and an $n$-point tree Feynman graph $T$, the tree-graph integral $F_T$ is defined in such a way that $F_T$ has only the poles relevant to $T$, that there is a birational transformation by which $F_T$ is transformed into an integral identical with the $n$-point Veneziano formula apart from its integration domain, and that the crossing-symmetry property and Chan’s bootstrap condition are manifest. It is proved that the $n$-point Veneziano formula is written as a sum of $F_T$ over all tree graphs $T$ belonging to the given cyclic ordering of external particles.

§ 1. Introduction and summary

In previous work, we have shown for $n=4, 5, 6$ that the $n$-point Veneziano formula $F^{(n)}$ belonging to a cyclic ordering $(1, \cdots, n)$ of external particles can be decomposed into $n_T$ tree-graph integrals $F_T$:

$$F^{(n)} = \sum_{T \in T(n)} F_T,$$

where $T^{(n)}$ denotes the set of all the $n$-point tree Feynman graphs (three lines are incident with every vertex) belonging to the cyclic ordering $(1, \cdots, n)$, and $n_T$ is the number of the trees of $T^{(n)}$, that is:

$$n_T = (2n-4)!/(n-1)!(n-2)!.$$

The tree-graph integrals $F_T$ have a one-to-one correspondence to the $n$-point tree graphs $T$, and satisfy the following requirements:

[1] $F_T$ has only the singularities which can be existent in the Feynman amplitudes corresponding to $T$.

[2] There is a birational transformation of integration variables by which $F_T$ is transformed into an integral whose integrand is identical with that of the standard expression for $F^{(n)}$.

[3] If a tree graph $\tilde{T}$ is obtained from $T$ by a cyclic or anti-cyclic permutation of external particles, then $F_{\tilde{T}}$ is obtained from $F_T$ by the same permutation. In particular, if $T$ is invariant under certain cyclic or anti-cyclic permuta-

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Each tree-graph integral \( F_T \) satisfies Chan's bootstrap condition\(^{2} \), that is, if \( T \) is decomposed into two tree graphs \( T' \in T^{(n)} \) and \( T'' \in T^{(n-m+2)} \) by opening an internal line \( L_i \in T \), then the residue of \( F_T \) at \( \alpha_{P_i} = 0 \) is equal to \( F_{T'} F_{T''} \), where \( F_{T'} = 1 \) (or \( F_{T''} = 1 \)) if \( T' \in T^{(3)} \) (or \( T'' \in T^{(5)} \)) and \( \alpha_{P_i} \) is the Regge trajectory function corresponding to \( L_i \).

In the present paper, we extend our consideration to the case of \( n \) general. In § 2, we review a six-point symmetric tree-graph integral, and consider an extension to an eight-point symmetric tree-graph integral. In § 3, we explicitly construct a general birational transformation of integration variables by which \( F_T \) is transformed into an integral whose integrand is identical with that of \( F^{(5)} \) (see Requirement [2]). In § 4, after specifying the integration domain of \( F_T \), we show that the tree-graph integrals satisfy also all the other requirements stated above. Finally, the decomposition theorem (1·1) is proved for \( n \) general.

The unsolved problems are to prove the uniqueness of the definition of \( F_T \), to find an explicit, closed expression for the integrand of \( F_T \), and to investigate the asymptotic behavior of \( F_T \).

§ 2. Preliminaries and examples

As is well known, the \( n \)-point Veneziano formula belonging to a cyclic ordering \((1, \cdots, n)\) of external particles is written as\(^{3} \)

\[
\int_0^1 dv_1 \cdots \int_0^1 dv_{n-3} h \prod_{P} u_p^{-\alpha_P - 1}. \tag{2·1}
\]

Here \( P \) denotes a partition of the cyclic ordering \((1, \cdots, n)\) into two sets

\[
P' = (p, p+1, \cdots, q), \\
P'' = (q+1, \cdots, n, 1, 2, \cdots, p-1);
\]

\( \alpha_P = \alpha_P (s_P) \) stands for the Regge trajectory corresponding to a partition \( P \); \( u_p = u_{p-q} \) is the so-called "conjugate variable", which satisfies

\[
u_p = 1 - \prod_{P} u_p, \tag{2·3}
\]

where the product goes over all \( \overline{P} (= \overline{P'} + \overline{P''}) \) such that \( P' \cap \overline{P' \cap P''} \) is non-empty but equal to neither \( P' \) nor \( \overline{P'} \); \( \overline{P} \) is usually called a "dual" of \( P \), though this nomenclature is misleading mathematically.

The solution of (2·3) can be uniquely expressed in terms of \( n-3 \) independent variables \( v_1, \cdots, v_{n-3} \) if they are the \( n-3 \) conjugate variables corresponding to the partitions each of which is induced by opening an internal line of a particular tree graph \( T \in T^{(m)} \). The weight function \( h \) depends on the choice of the variables \( v_1, \cdots, v_{n-3} \). It is important to note that if and only if \( T \) has no internal vertices, that is, if at least one external line is incident with every ver-
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As is well known, it is most convenient to choose as $T$ a multiperipheral graph $M_r$ shown in Fig. 1. In particular, for the multiperipheral graph $M_r$, we have\(^5\)

$$u_{i-1+1}=v_i, \quad (i=1, \ldots, n-3) \quad (2.4)$$

$$u_p-1=\frac{(1-\prod_{i=p-2}^{q-2}v_i)(1-\prod_{i=p-1}^{q-1}v_i)}{(1-\prod_{i=p-2}^{q-2}v_i)(1-\prod_{i=p-1}^{q-1}v_i)} \quad \text{for } 2\leq p < q \leq n-1 \quad (2.5)$$

with $v_0=v_{n-3}=0$, and

$$h=\prod_{2\leq p \leq 2\leq n-1}u_p-q+1. \quad (2.6)$$

Hereafter, $v_1, \ldots, v_{n-3}$ always mean those variables defined by (2.4).

Now, we consider the six-point function. In this case, we have two tree graphs which have an internal vertex. One of them, which is shown in Fig. 2, was called $C_1$. According to the result of the previous paper,\(^b\) with a simple transformation of variables, the tree-graph corresponding to $C_1$ is written as

$$F_{C_1}=\int \int d\rho dx dx dx(1-x_1x_2x_3)^{-1}\left[\frac{x_1(1-x_3x_3)}{1-x_1x_3x_3}\right]^{-a_{13}-1}\left[\frac{x_3(1-x_3x_3)}{1-x_1x_3x_3}\right]^{-a_{34}-1}\times\left[\frac{1-x_2}{1-x_2x_2x_3}\right]^{-a_{45}-1}\left[\frac{1-x_2}{1-x_2x_2x_3}\right]^{-a_{12}+1}\times\left[\frac{1-x_2x_2x_3}{1-x_2x_2x_3}\right]^{-a_{12}+1}\left[\frac{1-x_2}{1-x_2x_2x_3}\right]^{-a_{45}-1}\times\left[\frac{x_1(1-x_3x_3)}{1-x_1x_3x_3}\right]^{-a_{13}-1}\left[\frac{x_3(1-x_3x_3)}{1-x_1x_3x_3}\right]^{-a_{34}-1}\times\left[\frac{1-x_2}{1-x_2x_2x_3}\right]^{-a_{45}-1}\left[\frac{1-x_2}{1-x_2x_2x_3}\right]^{-a_{12}+1}\times\left[\frac{1-x_2x_2x_3}{1-x_2x_2x_3}\right]^{-a_{12}+1}\left[\frac{1-x_2}{1-x_2x_2x_3}\right]^{-a_{45}-1} \quad (2.7)$$

Here the integration domain $D_{C_1}$ is defined by

$$0 \leq x_i \leq 1 \quad (i=1, 2, 3) \quad (2.8)$$

and

$$\frac{1}{(1-x_i)(1-x_i)} \leq \frac{1}{(1-x_i)(1-x_i)} \quad (i=1, 2, 3) \quad (2.9)$$
where \((i, j, k)\) is a permutation of \((1, 2, 3)\). As is suggested by the conjugate variables corresponding to \(a_{13}, a_{125},\) and \(a_{56}\), the integrand of (2.7) is obtainable from that of the six-point Veneziano formula \(F^{(6)}\) by a birational transformation

\[
\begin{align*}
  v_1 &= \frac{x_1(1-x_2x_5)}{1-x_1x_2x_5}, \\
  v_2 &= \frac{(1-x_2)(1-x_1x_2x_5)}{(1-x_2x_5)(1-x_1x_5)}, \\
  v_3 &= \frac{x_3(1-x_1x_5)}{1-x_1x_2x_5},
\end{align*}
\]

that is,

\[
\begin{align*}
  x_1 &= \frac{v_1(1-v_2v_5)}{1-v_5}, \\
  x_2 &= \frac{1-v_1}{(1-v_1v_5)(1-v_5v_3)}, \\
  x_3 &= \frac{v_3(1-v_1v_5)}{1-v_1}.
\end{align*}
\]

The transformation (2.11) does not map the unit cube \(\{0 < v_i < 1, i = 1, 2, 3\}\) onto the unit cube \(\{0 < x_i < 1, i = 1, 2, 3\}\), but the important point is that the image of the latter by the transformation (2.10) entirely included in the former.

Next, we consider an eight-point tree graph, which we call \(E_1\), shown in Fig. 3. This graph is the simplest example which contains an internal line whose adjacent lines are all internal. The analysis of the tree-graph integral \(F_{E_1}\) is not only instructive but also important in the general consideration.

We first note that (2.10) is rewritten as

\[
\begin{align*}
  u_{12} &= \frac{x_1(1-x_2x_3)}{1-x_1x_2x_3}, \\
  u_{24} &= \frac{x_2(1-x_2x_1)}{1-x_1x_2x_3}, \\
  u_{34} &= \frac{x_3(1-x_1x_2)}{1-x_1x_2x_3}.
\end{align*}
\]

If we neglected the identity \(u_{1234} = u_{6075}\), in \(E_1\) we could adopt (2.12) and a similar parametrization for \(u_{36}, u_{7},\) and \(u_{476}.\) In order for \(u_{1234}\) to play a double
role, we remark the following identity:

\[
\frac{x_3'(1 - x_1 x_2)}{1 - x_1 x_2 x_3'} = \frac{x_3 (1 - x_2 x_3) (1 - x_4 x_5)}{1 - x_1 x_2 x_3 - x_2 x_3 x_4 + x_1 x_2 x_3 x_4 x_5}
\]

\[
= \frac{x_5'' (1 - x_4 x_5)}{1 - x_5'' x_4 x_5},
\]

(2.13)

where

\[
x_3' = \frac{x_3 (1 - x_2 x_3)}{1 - x_5 x_4 x_5},
\]

\[
x_3'' = \frac{x_3 (1 - x_1 x_2)}{1 - x_1 x_2 x_3}.
\]

(2.14)

In this way, the following parametrization is found to be appropriate:

\[
u_{12} = \frac{x_1 (1 - x_2 x_3')}{1 - x_1 x_2 x_3'},
\]

\[
u_{34} = \frac{x_3 (1 - x_3') x_1}{1 - x_1 x_2 x_3'},
\]

\[
u_{134} = \frac{x_3' (1 - x_2 x_3)}{1 - x_1 x_2 x_3'} = \frac{x_3'' (1 - x_4 x_5)}{1 - x_3'' x_4 x_5},
\]

\[
u_{56} = \frac{x_4 (1 - x_5 x_6 x_7)}{1 - x_5'' x_4 x_5},
\]

\[
u_{78} = \frac{x_8 (1 - x_7'' x_8)}{1 - x_7'' x_4 x_5}.
\]

(2.15)

Indeed, we obtain a birational transformation

\[
v_1 = \frac{x_1 (1 - x_2 x_3')}{1 - x_1 x_2 x_3'},
\]

\[
v_2 = \frac{(1 - x_3) (1 - x_2 x_3')}{(1 - x_1 x_2) (1 - x_3')},
\]

\[
v_3 = \frac{x_3' (1 - x_2 x_3)}{1 - x_1 x_2 x_3'} = \frac{x_3'' (1 - x_4 x_5)}{1 - x_3'' x_4 x_5},
\]

\[
v_4 = \frac{(1 - x_4) (1 - x_3'' x_4 x_5)}{(1 - x_3'' x_4) (1 - x_4 x_5)},
\]

\[
v_5 = \frac{x_5 (1 - x_5'' x_6 x_7)}{1 - x_3'' x_4 x_5},
\]

(2.16)

namely,

\[
x_1 = \frac{v_1 (1 - v_5 v_6)}{1 - v_5},
\]
The integrand of the tree-graph integral $F_{n1}$ is now well defined by (2.1), together with (2.4)~(2.6), and (2.16).

Finally, according to a rule suggested in the previous work,$^1$ the integration domain of $F_{n1}$ is expected to be given by

$$0 \leq x_i \leq 1 \quad (i = 1, \ldots, 5)$$

and

$$\left(\frac{x_1}{1 - x_1}\right)^2 \leq \frac{1}{(1 - x_2) (1 - x_3')} \ ,$$

$$\left(\frac{x_2}{1 - x_2}\right)^2 \leq \frac{1}{(1 - x_3') (1 - x_1)} \ ,$$

$$\left(\frac{x_3}{1 - x_3}\right)^2 \leq \frac{1}{(1 - x_1) (1 - x_2) (1 - x_4) (1 - x_5)} \ ,$$

$$\left(\frac{x_4}{1 - x_4}\right)^2 \leq \frac{1}{(1 - x_5) (1 - x_6')} \ ,$$

$$\left(\frac{x_5}{1 - x_5}\right)^2 \leq \frac{1}{(1 - x_5') (1 - x_6)} \ .$$

(2.19)

It should be noted that $x_3'$ and $x_6''$ are used instead of $x_3$ itself in the right-hand sides of (2.19) in conformity with (2.16). The appropriateness of (2.19) is verified in § 4.

§ 3. General birational transformation

In this section, given an arbitrary $n$-point tree graph $T \in T^{(n)}$, we find a birational transformation, which, together with (2.1), defines the integrand of the tree-graph integral $F_T$ corresponding to $T$.

Let $x_1, \ldots, x_{n-3}$ be the integration variables of $F_T$. They have a one-to-one correspondence to the internal lines $L_1, \ldots, L_{n-3}$ of $T$. By opening a line $L_i$, $T$ becomes disconnected. Let $P_i$ be the partition of the cyclic ordering $(1, \ldots, n)$
which is induced by opening $L_i$ in $T$. Then, according to Requirement [1] stated in § 1, $u_{P_i}$ should involve a factor $x_i$. We first propose a general rule for expressing $u_{P_i}$ in terms of $x_1, \ldots, x_{n-3}$, and then prove that this parametrization indeed induces a birational transformation.

First, we observe that any tree graph is 2-chromatic, that is, there is a mapping from the set of all its vertices to a set consisting of two colors $A$ and $B$ such that the two end vertices of any internal line correspond to two different colors. For any line $L_i$, we associate $x_i'$ with the end vertex of color $A$ and $x_i''$ with that of color $B$. The functions $x_i'$ and $x_i''$ are defined by the following recurrence relations:

$$x_i' = \frac{x_i' (1 - x_i'' x_m'')} {1 - x_i' x_i'' x_m''},$$

$$x_i'' = \frac{x_i (1 - x_i' x_k')} {1 - x_i' x_i' x_k'} \tag{3.1}$$

where $L_j$ and $L_k$ are the two lines adjacent to $L_i$ through the vertex of color $A$, and $L_i$ and $L_m$ are those through the vertex of color $B$ (see Fig. 4). If $L_i$ happens to be an external line, we set $x_i = 0$ so that $x_i' = x_i''$. The same rule also applies to $L_k, L_l$, and $L_m$. Therefore, if the color-$B$ (or color-$A$) end vertex of $L_i$ is external, then we have $x_i' = x_i$ (or $x_i'' = x_i$). It is easy to confirm that for any $L_i$, $x_i'$ and $x_i''$ are uniquely defined in terms of $x_1, \ldots, x_{n-3}$ by (3.1). Indeed, to calculate $x_i'$ (or $x_i''$) explicitly, we need only the information of the left-half (or right-half) structure of Fig. 4.

Now, we parametrize the conjugate variables $u_{P_i}$ by setting

$$u_{P_i} = \frac{x_i' (1 - x_i' x_i'')} {1 - x_i' x_i'' x_i''} = \frac{x_i (1 - x_i'' x_m'')} {1 - x_i' x_i'' x_m''} \tag{3.2}$$

The two expressions in (3.2) are equivalent to each other because of the identity (2.13). In particular, if the end vertex of color $A$ is external, (3.2) reduces to

$$u_{P_i} = x_i' = \frac{x_i (1 - x_i'' x_m'')} {1 - x_i' x_i'' x_m''}. \tag{3.3}$$

Furthermore, if both end vertices of $L_i$ are external, (3.2) of course reduces to

$$u_{P_i} = x_i. \tag{3.4}$$

Hence, if and only if $T$ has no internal vertices we have (3.4) for all internal
THEOREM 1. The transformation from \( \{x_1, \ldots, x_{n-3}\} \) to \( \{v_1, \ldots, v_{n-3}\} \) is *birational*, where \( v_i \) is defined by (2.4). The image of the unit hypercube
\[
\{0 \leq x_i \leq 1, \ i = 1, \ldots, n-3\}
\]
entirely lies in the unit hypercube
\[
\{0 \leq v_i \leq 1, \ i = 1, \ldots, n-3\}.
\]

**Proof:** Since it is well known\(^b\) that the transformation between any two multiperipheral graphs is birational and maps the unit hypercube of one onto that of the other, we may use \( w_1, \ldots, w_{n-3} \) instead of \( v_1, \ldots, v_{n-3} \), where \( w_1, \ldots, w_{n-3} \) are the integration variables (in an arbitrary order) of any multiperipheral graph \( M \).

Since the theorem is obviously true for \( n=4 \), we employ mathematical induction with respect to \( n \). Since the number of external lines is more than that of vertices in \( T \), there is at least one vertex with which two external lines are incident. For definiteness, let this vertex be of color \( B \). Let \( L_i \) be the unique internal line incident with this vertex, and \( L_j \) and \( L_k \) be the two lines adjacent to \( L_i \) at the other end vertex (of color \( A \)). Then \( T \) can be drawn on a plane as shown in Fig. 5. In a realization of \( T \) on a plane, we can define the set of

\[
\text{all the uppermost internal lines, which constitute a path } Q \text{ between two consecutive external particles.}
\]

We make a duality transformation\(^*\) for such an internal line \( L_i \) that it is incident with a vertex lying on \( Q \) but does not belong to \( Q \). Then we obtain a tree graph \( \tilde{T} \equiv T^{(n)} \), which has the uppermost path \( \tilde{Q} \) consisting of \( L_i \) and all the lines of \( Q \). In \( \tilde{T} \), we make a duality transformation for a line incident with a vertex on \( \tilde{Q} \) but not belonging to \( \tilde{Q} \). Performing this procedure successively, we finally reach a multiperipheral graph \( M \equiv T^{(n)} \), in which \( L_j \) and \( L_k \) are the right and left adjacent lines of \( L_i \), respectively. The details of \( M \) depend on the order of duality transformations performed, but this fact is unimportant.

Now, according to (3.2) together with (3.1), we can write

\(^*\) A duality transformation for an internal line is to change the incidence of its four adjacent lines with its two end vertices in such a way that the cyclic ordering of the four adjacent lines are left unchanged.
where \( \Phi, \mathcal{V}, \varphi_i \), and \( \psi_m \) are certain rational functions, \( V \) and \( W \) are the sets of internal lines as indicated in Fig. 5, and

\[
\begin{align*}
\mathcal{V} &= \{ x_i | L_1 \in V \}, \\
\mathcal{W} &= \{ x_m | L_m \in W \}.
\end{align*}
\]

From (3.1), we also have

\[
\begin{align*}
x_j' &= x_j \frac{1 - \Phi(x_\mathcal{V})}{1 - x_j \Phi(x_\mathcal{V})}, \\
x_k' &= x_k \frac{1 - \mathcal{V}(x_\mathcal{W})}{1 - x_k \mathcal{V}(x_\mathcal{W})}, \\
x_j'' &= x_j \frac{1 - x_i x_k'}{1 - x_j x_k' x_k''}, \\
x_k'' &= x_k \frac{1 - x_i x_j'}{1 - x_i x_j' x_k''}.
\end{align*}
\]

On the other hand, according to (2.4) and (2.5), the conjugate variables considered in (3.7) are expressed in terms of \( w_i, \ldots, w_{n-1} \) of \( M \) in the following way:

\[
\begin{align*}
u_{p_i} &= (1 - w_i) \frac{(1 - w_j w_k)}{(1 - w_i w_j w_k)}, \\
u_{p_j} &= w_j, \\
u_{p_k} &= w_k, \\
u_{p_i} &= f_i(w_j, w_k), \quad (L_1 \in V) \\
u_{p_m} &= g_m(w_k, w_w), \quad (L_m \in W)
\end{align*}
\]

where \( f_i \) and \( g_m \) are certain rational functions, and

\[
\begin{align*}
\mathcal{V} &= \{ w_i | L_i \in V \}, \\
\mathcal{W} &= \{ w_m | L_m \in W \}.
\end{align*}
\]

Therefore we obtain the following system of algebraic equations:
\[ \frac{(1 - w_i) (1 - w_j w_k)}{(1 - w_i w_j) (1 - w_i w_k)} = \frac{x_i (1 - x_j' x_k')}{1 - x_i x_j' x_k'}, \]

\[ w_j = \frac{x_j' (1 - x_i x_j')}{1 - x_i x_j' x_k'} = \varphi (x_j'', x_j'), \]

\[ w_k = \frac{x_k' (1 - x_i x_k')}{1 - x_i x_j' x_k'} = \psi (x_k'', x_w), \]

\[ f_i (w_j, w_k) = \varphi_i (x_j'', x_k'), \quad (L_i \in V) \]
\[ g_m (w_k, w_k) = \psi_m (x_k'', x_w), \quad (L_m \in W) \quad (3.12) \]

where

\[ \varphi (x, x_y) = \frac{x [1 - \Theta (x_y)]}{1 - x \Theta (x_y)} , \]

\[ \psi (x, x_y) = \frac{x [1 - \Phi (x_y)]}{1 - x \Phi (x_y)} . \quad (3.13) \]

We introduce two auxiliary graphs. Let \( T/L_i \) be the \((n-1)\)-point tree graph which is obtained from \( T \) by removing one of external lines adjacent to \( L_i \) and then contracting \( L_i \), and \( M/L_i \) be the \((n-1)\)-point multiperipheral graph which is obtained from \( M \) by same procedure. We denote the quantities of those auxiliary graphs by affixing a tilda to each of them. We have

\[ u_{P_i} = \bar{x}_j = \varphi (x_j, \bar{x}_y), \]
\[ u_{P_k} = \bar{x}_k = \psi (x_k, \bar{x}_w), \]
\[ u_{P_i} = \varphi_i (x_j, \bar{x}_y), \quad (L_i \in V) \]
\[ u_{P_m} = \psi_m (x_k, \bar{x}_w) \quad (L_m \in W) \quad (3.14) \]

from (3.2), and

\[ u_{P_j} = \bar{w}_j, \]
\[ u_{P_k} = \bar{w}_k, \]
\[ u_{P_i} = f_i (\bar{w}_j, \bar{w}_k), \quad (L_i \in V) \]
\[ u_{P_m} = g_m (\bar{w}_k, \bar{w}_k) \quad (L_m \in W) \quad (3.15) \]

from (2.4) and (2.5). Therefore

\[ \bar{w}_j = \varphi (x_j, \bar{x}_y), \]
\[ \bar{w}_k = \psi (x_k, \bar{x}_w), \]
\[ f_i (\bar{w}_j, \bar{w}_k) = \varphi_i (x_j, \bar{x}_y), \quad (L_i \in V) \]
\[ g_m (\bar{w}_k, \bar{w}_k) = \psi_m (x_k, \bar{x}_w) \quad (L_m \in W) \quad (3.16) \]

Owing to the induction assumption, the transformation defined by (3.16) is bira-
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tional. Comparing (3.12) with (3.16), therefore, we find that the transformation between \(\{w_j, w_k, w_r, w_w\}\) and \(\{x_j'', x_k'', x_r, x_w\}\) is birational. Furthermore, from (3.12) we have (cf. (2.10))

\[
w_i = \frac{(1-x_i)(1-x_i x'_j x'_k)}{(1-x_i x'_j)(1-x_i x'_k)}. \tag{3.17}
\]

Since \(x_j', x_k', x_j'', \text{ and } x_k''\) are rational functions of \(x_1, \ldots, x_{n-3}\), the variables \(w_1, \ldots, w_{n-3}\) are rational in \(x_1, \ldots, x_{n-3}\). Conversely, from (3.12) and (3.9) we have (cf. (2.11))

\[
x_i = \frac{1-w_i}{(1-w_i w_j)(1-w_i w_k)}, \tag{3.18}
\]

and

\[
x_j = \frac{x_j'}{1-\phi + x_j' \phi}, \tag{3.19}
\]

\[
x_k = \frac{x_k'}{1-\phi + x_k' \phi}.
\]

Therefore, \(x_1, \ldots, x_{n-3}\) are rational functions of \(w_1, \ldots, w_{n-3}\).

Finally, from (3.1) we note that in the hypercube (3.5) we have

\[
0 \leq x_q'' \leq x_q' \leq 1, \quad (q=1, \ldots, n-3)
\]

\[
0 \leq x_q'' \leq x_q \leq 1. \quad (q=1, \ldots, n-3) \tag{3.20}
\]

The induction assumption implies that if

\[
0 \leq x_{q}\leq 1, \quad (q \neq i) \tag{3.21}
\]

then

\[
0 \leq w_q \leq 1, \quad (q \neq i) \tag{3.22}
\]

that is, if

\[
0 \leq x_q'' \leq 1, \quad 0 \leq x_q'' \leq 1,
\]

\[
0 \leq x_i \leq 1, \quad (L_i \in V)
\]

\[
0 \leq x_m \leq 1, \quad (L_m \in W) \tag{3.23}
\]

then

\[
0 \leq w_q \leq 1, \quad (q \neq i) \tag{3.24}
\]
Hence (3·24) follows from (3·20). Furthermore, from (3·17) and (3·20) we have

\[ 0 \leq w_i \leq 1. \]

(3·25)

Thus the theorem is established. q.e.d.

It follows from the above theorem that in the hypercube (3·5) we have

\[ 0 \leq u_P \leq 1 \quad \text{for all } P, \]

(3·26)

as it should be.

§ 4. Proof of the decomposition theorem

We first specify the integration domain \( D_T \) of the tree-graph integral \( F_T \) corresponding to \( T \in T^{\infty} \). As is suggested by (2·18) and (2·19), we define \( D_T \) by

\[ 0 \leq x_i \leq 1 \quad (i=1, \cdots, n-3) \]

(4·1)

and

\[ \left( \frac{x_i}{1-x_i} \right)^2 \leq \frac{1}{(1-x'_f)(1-x'_e)(1-x''_e)(1-x''_m)}, \quad (i=1, \cdots, n-3) \]

(4·2)

where \( L_f, L_e, L_n, \) and \( L_m \) are the lines adjacent to \( L_t \) as shown in Fig. 4. The tree-graph integral is now defined completely.

Theorem 2. The tree-graph integrals satisfy the four requirements stated in § 1.

Proof: [1] The poles of \( F_T \) in \( s_P \) can arise only from

\[ u_P=0. \]

(4·3)

If \( P=P_t \), that is, \( P \) is the partition induced by opening an internal line \( L_t \in T \), then (4·3) is of course realized by \( x_i=0 \). In the following, we show that if

\[ P \neq P_t, \quad (i=1, \cdots, n-3) \]

(4·4)

then \( F_T \) has no poles corresponding to \( P \).

We assume that (4·3) is possible in \( D_T \) (otherwise the proof ends). Then, because of (2·3) and (3·26), (4·3) implies

\[ u_P=1 \quad \text{for any } P. \]

(4·5)

Since at least one of \( P_1, \cdots, P_{n-s} \) is dual to \( P \), (4·5) means that there is a line \( L_t \) such that

\[ u_{P_t}=1. \]

(4·6)

Hence (3·2) and (3·1) yield that

\[ x'_t=x'_t=x_t=1. \]

(4·7)
Let
\[ U = \{ L_q | x_q = 1 \}; \]  
(4.8)

(4.7) shows that \( U \) is non-empty.

**Lemma 2-1.** If (4.3) is realized at a point \( (x_1, \cdots, x_{n-3}) \) belonging to \( D_T \), then the set \( U \) contains at least three lines which are incident with a common vertex. \([\text{If} \ T \text{ has no internal vertices, this lemma implies that (4.3) is impossible in } D_T.\])

**Proof of Lemma 2-1.** Suppose that \( L_4 \in U \). Then (4.2) implies that at least one of its adjacent lines, say \( L_4 \), belongs to \( U \). By considering a product of the inequality (4.2) for \( L_i \) and that for \( L_4 \), we find that one of the lines adjacent to \( L_i \) but other than \( L_4 \) has to belong to \( U \). If this line is also adjacent to \( L_i \), then our lemma holds. Otherwise, we can proceed further by considering a product of three relevant inequalities. In this way, we finally find that *either* our lemma holds or \( U \) includes a path \( Q \). One end line of \( Q \) is \( L_4 \), but by construction its other end line should be external. Since, however, any external line cannot belong to \( U \) (because \( x = 0 \) for any external line), the latter alternative leads to a contradiction. Thus our lemma holds.

This lemma shows that there exists at least one line \( L_k \) such that \( P_k \) is not dual to \( P \) but we have \( x_k = 1 \), because any partition cannot be dual to all three partitions which correspond to three lines incident with a common vertex. \([\text{In the dual-graph language, any diagonal line of a polygon cannot intersect all three sides of a triangle formed by three diagonal lines.}\])

We transform \( F_T \) into an integral expressed in terms of the multiperipheral variables \( w_1, \cdots, w_{n-3} \) (see § 3). The mechanism by which the poles corresponding to \( P \) arise is well known in this form. Let \( P'_1, \cdots, P'_{n-3} \) be the partitions corresponding to \( w_1, \cdots, w_{n-3} \). Without loss of generality we may assume that \( P \) coincides with none of \( P'_1, \cdots, P'_{n-3} \). Then the poles corresponding to \( P \) arise from the points satisfying \( w_i = u_{P'_i} = 1 \) for all partitions \( P'_i \) dual to \( P \). It should be remarked that to yield the poles \( w_i \) has to be completely arbitrary for any \( P'_i \) non-dual to \( P \). In our case, however, there exists an extra condition \( x_k = 1 \), which is independent of (4.5). Hence, the volume element is not sufficient for yielding the poles corresponding to \( P \).

[3] Let \( \sigma \) be an arbitrary cyclic or anti-cyclic permutation operator of external particles, and suppose that
\[ \tilde{T} = \sigma T. \]  
(4.9)

Let \( x_1, \cdots, x_{n-3} \) and \( \tilde{x}_1, \cdots, \tilde{x}_{n-3} \) be the integration variables in \( F_T \) and those in \( F_{\tilde{T}} \), respectively. According to (3·2), if
\[ u_{P_i} = \varphi_i (x_1, \cdots, x_{n-3}) \]  
(4.10)
then we have
Furthermore, since (2.3) holds for $T$, we obtain
\[ u_{\sigma P} = 1 - \prod_{\sigma P} u_{\sigma P} \] (4.12)
for $\tilde{T}$. Therefore, the fact that for any $P$ (2.3) is uniquely solved as
\[ u_{P} = \psi(x_1, \ldots, x_{n-3}) \] (4.13)
in $T$ implies that we uniquely have
\[ u_{\sigma P} = \psi(\tilde{x}_1, \ldots, \tilde{x}_{n-3}) \] (4.14)
in $\tilde{T}$. If we identify $\tilde{x}_i$ with $x_i$ ($i=1, \ldots, n-3$), the domain $D_{\tilde{T}}$ of $F_{\tilde{T}}$ coincides with $D_P$ of $F_P$. Moreover, the Jacobians from the multiperipheral variables are the same for both $F_P$ and $F_{\tilde{T}}$. Thus $F_{\tilde{T}}$ is obtained from $F_T$ by merely replacing $\alpha_P$ by $\alpha_{\sigma P}$.

[4] The residue at $\alpha_P = 0$ is obtained by replacing $u_{\sigma P} = 0$ by $\delta(u_{\sigma P})$. Owing to [1], if and only if $P = P_i (1 \leq i \leq n-3)$, $\delta(u_{\sigma P})$ is nontrivial and proportional to $\delta(x_i)$ (the contribution from $x_j = x_k = 1$ is shown to be trivial by the same reasoning as in [1]). If we carry out the integration over $x_i$ in $F_T$, then the integrand of the resulting integral is obtained by setting $x_i = 0$ except for the factor $x_i$ in $u_{P_i}$. According to the general rule presented in §3 and the definition of $D_T$, to set $x_i = 0$ is realized by replacing $L_i$ by two external lines, namely, by opening $L_i$. The exact equality of the residue to $F_{\tilde{T}}$ can be confirmed by transforming the integration variables into the multiperipheral variables.

**Theorem 3.** The n-point Veneziano formula $F^{(n)}$ can be written as a sum of all tree-graph integrals $F_T$ corresponding to $T \in T^{(n)}$ (see (1·1)).

**Proof:** Since by definition $F_T$ is transformed into an integral whose integrand is identical with that of $F^{(n)}$, it remains only to prove that the unit hypercube (3·6) is exactly equal to the disjoint union of the images, $D(T)$, of the domains $D_T$ over all $T \in T^{(n)}$, where a disjoint union means a union such that any two of its constituents are disjoint (except for their boundaries). According to (4·1) and Theorem 1, all the images $D(T)$ are included in the unit hypercube (3·6).

We use the following two lemmas, which are proved afterwards.

**Lemma 3-1.** Let $L_i \in T \in T^{(n)}$, and let $T_i$ be the tree graph which is obtained from $T$ by making a duality transformation for $L_i$. Then $D(T)$ and $D(T_i)$ have a common boundary hypersurface, which we denote by $S(T, T_i)$, in the unit hypercube (3·6), and they lie in the opposite sides of $S(T, T_i)$.

**Lemma 3-2.** Let $\{T^1, T^2, \ldots, T^r\}$ be the set of the n-point tree graphs which are mutually obtainable by making duality transformations for two particular lines $L_i$ and $L_j$ alternately. Then $r = 4$ if $L_i$ and $L_j$ are not adjacent, and $r = 5$ if they are adjacent. Let
\[ C(T, T', T^{(n)}) = S(T, T') \cap S(T', T^{(n)}); \] (4·15)
then $r$ hypercurves

$$C(T^k, T^{k+1}, T^{k+2}) \quad (k=1, \cdots, r)$$

(4.16)

coincide with each other, where the superscripts should be considered in mod. $r$.

Apart from the boundaries of the unit hypercube (3.6), each $D(T)$ is bordered by $n-3$ algebraic hypersurfaces because of (4.2). Since the number of the tree graphs which are obtainable from $T$ by making a duality transformation for only one line of $T$ is exactly $n-3$, $D(T)$ is bordered by $S(T, T_i), \ldots, S(T, T_{n-3})$, apart from the boundaries of (3.6), according to Lemma 3-1. Each of those boundary hypersurfaces of $D(T)$ coincides with a boundary hypersurface of one of the domains adjacent to $D(T)$.

The true boundary of $D(T)$ on $S(T, T_i)$ is a hypersurface segment bordered by $n-4$ hypercurves $C(T_j, T, T_i)$ ($j \neq i$) apart from the boundaries of (3.6). Let $T_{ij}$ be the tree graph which is obtained from $T_i$ by making a duality transformation for $L_j$. Then according to Lemma 3-2, we have

$$C(T_j, T, T_i) = C(T, T_i, T_{ij}). \quad (j \neq i)$$

(4.17)

This relation shows that the bounding hypersurface segment of $D(T)$ on $S(T, T_i)$ exactly coincides with that of $D(T_i)$, that is, there are no gap and no overlapping. Thus (3.6) is the disjoint union of $D(T)$ over all $T \in T^{\infty}$.

(Proof of Lemma 3-1). Suppose that $T$ is drawn as in Fig. 4. According to (3.2), we have

$$u_{pj} = \frac{x_j'(1-x_j'x_k')}{1-x_j'x_j'x_k'},$$

$$u_{pk} = \frac{x_k'(1-x_j'x_k')}{1-x_j'x_j'x_k'},$$

$$u_{pt} = \frac{x_i'(1-x_j'x_k')}{{1-x_j'x_j'x_k'}},$$

$$u_{pt} = \frac{x_j'(1-x_j'x_i')}{1-x_j'x_j'x_i'},$$

$$u_{pt} = \frac{x_i'(1-x_i'x_i')}{1-x_i'x_i'x_i'},$$

(4.18)

We denote the variables of $F_{T_i}$ by $y_l, y_j, \ldots$ Let $P_i$ be the partition induced by opening the line $L_i$ in $T_i$. Then, supposing that the vertex with which $L_i, L_m,$ and $L_j$ are incident is of color $A$, we have

$$u_{pm} = \frac{y_m'(1-y_l'y_j')}{1-y_l'y_m'y_j'},$$

$$u_{pj} = \frac{y_j'(1-y_j'y_m')}{1-y_j'y_m'y_j'},$$
It is important to note that the fact of the coincidence of two hypersurfaces is a geometrical property, that is, we may prove it by expressing their equations in any system of \( n - 3 \) independent variables. Thus to prove Lemma 3-1 is equivalent to prove
\[
f(x_i; x'_i, x'_i, x''_i, x''_i) = [f(y_i; y'_m, y'_k, y''_k, y''_i)]^{-1},
\]
where
\[
f(x; a, b, c, d) = \frac{(1-x)^2}{x^2(1-a)(1-b)(1-c)(1-d)}.
\]
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\[ u_{P_1} = \frac{(1 - w_2)(1 - w_1 w_2 v_3)}{(1 - w_1 w_2)(1 - w_2 v_3)} , \]

\[ u_{P_1} = w_3 , \]

\[ u_{P_2} = \frac{(1 - w_1)(1 - w_2 w_3 v_4)}{(1 - w_2 w_3 v_4)(1 - w_1 v_4)} , \]

\[ u_{P_3} = w_5 . \]  

(4·24)

It should be noted that the relation between \( \{v_1, \ldots, v_5\} \) and \( \{w_1, \ldots, w_5\} \) is exactly the transformation between the variables for \( M_1 \) and those for \( M_7 \), that is, we have

\[ w_1 = v_5 , \]

\[ w_2 = \frac{1 - v_1 v_3 v_4 v_5}{1 - v_1 v_2 v_3 v_4 v_5} , \]

\[ w_3 = \frac{(1 - v_1 v_2 v_3)(1 - v_1 v_2 v_3 v_4)}{(1 - v_1 v_2 v_3 v_4)(1 - v_1 v_2 v_3 v_4 v_5)} , \]

\[ w_4 = \frac{(1 - v_5 v_3 v_4 v_5)}{(1 - v_5 v_3 v_4)(1 - v_5 v_3 v_4 v_5)} , \]

\[ w_5 = \frac{(1 - v_5)(1 - v_5 v_4 v_5)}{(1 - v_5 v_4)(1 - v_5 v_4 v_5)} , \]  

(4·25)

as is directly confirmed by (4·22), (4·23), and (4·24). As in § 2, \( \{x'_j, x'_k, \ldots\} \) and \( \{y'_m, y'_n, \ldots\} \) can be expressed in terms of \( \{v_1, \ldots, v_5\} \) and \( \{w_1, \ldots, w_5\} \), respectively, that is,

\[ x'_j = \frac{v_1(1 - v_2 v_3)}{1 - v_3} , \]

\[ x'_k = \frac{1 - v_3}{(1 - v_1 v_3)(1 - v_2 v_3)} , \]

\[ x'_l = \frac{v_3 (1 - v_1 v_2)}{1 - v_1} , \]

\[ x''_1 = \frac{v_3 (1 - v_4 v_5)}{1 - v_5} , \]

\[ x''_k = \frac{1 - v_4}{(1 - v_3 v_4)(1 - v_4 v_5)} , \]

\[ x''_l = \frac{v_5 (1 - v_4 v_5)}{1 - v_5} , \]  

(4·26)

and

\[ y'_m = \frac{w_1 (1 - w_2 v_3 v_4)}{1 - w_3} . \]
Furthermore, from (3·1) we have

\[
(1 - x_i')/x_i = (1 - x_i'^{''} x_m'') (1 - x_i')/x_i' \\
= (1 - x_j x_i') (1 - x_i'')/x_i'', \tag{4·28}
\]

and similar expressions for \((1 - y_i)/y_i\). Therefore, a simple calculation yields

\[
f(x_i; x_j', x_k', x_i'', x_m'') = \frac{(1 - v_5 v_3) (1 - v_4 v_4)}{v_5 v_3 v_4 (1 - v_5 v_3) (1 - v_4 v_4)},
\]

\[
f(y_i; y_m', y_j', y_k'', y_i'') = \frac{(1 - w_5 w_3) (1 - w_4 w_4)}{w_5 w_3 w_4 (1 - w_5 w_3) (1 - w_4 w_4)}. \tag{4·29}
\]

It is now straightforward to check (4·20) by using (4·25). Thus Lemma 3-1 is established.

(Proof of Lemma 3-2). First, we consider the case in which two lines \(L_i\) and \(L_j\) are not adjacent. Then the two duality transformations for them are commutative. Thus we obtain four tree graphs \(T^1, T^2, T^3,\) and \(T^4\). It is evident that

\[
S(T^1, T^2) = S(T^3, T^4),
\]

\[
S(T^2, T^3) = S(T^4, T^1). \tag{4·30}
\]

Hence the hypercurves (4·16) are the same.

In the case in which \(L_i\) and \(L_j\) are adjacent, we obtain five tree graphs \(T^1, \ldots, T^5\) because in the dual-graph representation they correspond to five different triangulations of a pentagon. It is sufficient to show, for example,

\[
C(T^1, T^3, T^5) = C(T^1, T^3, T^5), \tag{4·31}
\]

that is, to prove that the equation of \(S(T^3, T^5)\) coincides with that of \(S(T^3, T^5)\) on the hypersurface \(S(T^3, T^5)\).

As above, it is convenient to introduce the variables corresponding to the lines of ten-point multi peripheral graphs \(M_1\) and \(M_7\) (see Fig. 1). [We need also \(M_8, M_9, M_6\) if we consider the equalities other than (4·31).] Let
{v₁, ⋯, v₇} and {w₁, ⋯, w₇} be the variables in M₁ and those in M₂, respectively, in the formal sense. Then, as shown above (see (4·2), (4·20) and (4·29)), we have

\[ S(T^0, T^1) : \frac{v₁v₅v₇(1 - v₅v₄)(1 - v₅v₇)}{(1 - v₅v₄)(1 - v₅v₇)} = 1, \]  

(4·32)

\[ S(T^1, T^1) : \frac{(1 - v₅v₄)(1 - v₅v₇)}{v₅v₄v₅(1 - v₅v₄)(1 - v₅v₇)} = \frac{(1 - w₅w₄)(1 - w₅w₇)}{(1 - w₅w₄)(1 - w₅w₇)} = 1, \]  

(4·33)

\[ S(T^1, T^0) : \frac{(1 - w₅w₄)(1 - w₅w₇)}{w₅w₄w₅(1 - w₅w₄)(1 - w₅w₇)} = 1. \]  

(4·34)

The relation between \{v₁, ⋯, v₇\} and \{w₁, ⋯, w₇\} is as follows (cf. (2·5)):

\[ w₁ = \frac{(1 - v₅)(1 - v₅v₅v₇)}{(1 - v₅v₄)(1 - v₅v₇)}, \]

\[ w₂ = \frac{1 - v₅v₇}{1 - v₅v₄v₇}, \]

\[ w₃ = v₅, \]

\[ w₄ = \frac{1 - v₅v₅v₁v₄}{1 - v₅v₅v₄v₅}, \]

\[ w₅ = \frac{(1 - v₅v₅v₄)(1 - v₅v₅v₅v₇)}{(1 - v₅v₅v₄)(1 - v₅v₅v₅v₇)}, \]

\[ w₆ = \frac{(1 - v₅v₅)(1 - v₅v₅v₅v₇)}{(1 - v₅v₅v₄)(1 - v₅v₅v₅v₇)}, \]

\[ w₇ = \frac{(1 - v₅)(1 - v₅v₅v₅)}{(1 - v₅v₄)(1 - v₅v₇)}. \]  

(4·35)

On substituting (4·35) in the left-hand side of (4·34), we find that it equals

\[ \frac{(1 - v₅v₅)(1 - v₅v₇)}{v₅v₄v₅(1 - v₅v₄)(1 - v₅v₇)} \cdot \frac{(1 - v₅v₄)(1 - v₅v₅v₄v₇)}{v₅v₄v₅(1 - v₅v₄)(1 - v₅v₅v₄v₇)}. \]  

(4·36)

The first factor of (4·36) is identical with the inverse of the left-hand side of (4·32), and the second factor is equal to unity on S(T¹, T¹), because from (4·33) we have

\[ v₁v₅ = 1 - \frac{(1 - v₅v₅)(1 - v₅v₇)}{v₅v₄v₅(1 - v₅v₄)}. \]  

(4·37)

Thus Lemma 3–2 has been proved. q.e.d.

The above two lemmas can be summarized diagrammatically by introducing
a homogeneous graph $G^{(n)}$. Each vertex $v(T)$ of $G$ represents a tree graph $T \in T^{(n)}$. Each line between two vertices $v(T)$ and $v(T')$ correspond to $S(T, T')$. Therefore, any vertex is of degree $n - 3$, that is, $n - 3$ lines are incident with every vertex. The graph $G^{(n)}$ is connected, because for any vertex $v(T)$ there exists a path between $v(T)$ and $v(M_l)$, whose length is at most $n - 3$ (see §3). Each of certain circuits, whose length is four or five, corresponds to a bounding hypercurve defined in Lemma 3-2. It is expected that any other circuit has a length more than five.

References


Note added in proof: The author has found the Koba-Nielsen representation of the tree-graph integral $F_T$, which is very elegant in form. Detailed accounts will appear in a succeeding paper.

*) $G^{(4)}$ consists of only one line, $G^{(5)}$ is a pentagon, and $G^{(6)}$ is a planar graph. The author is very grateful to Dr. Y. Shimamoto for his interesting comments on homogeneous graphs.