Rayleigh's Principle in Finite Element Calculations of Seismic Wave Response

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Summary

Rayleigh's principle in the context of finite element modelling is shown to provide a powerful and convenient method for estimating the seismic eigenfrequencies of irregular Earth structures. It is not necessary to solve the eigenvalue problem completely, but instead to construct the elastic moduli and density matrices for the irregular structure, multiply them by an approximate eigenfunction vector, and form the Rayleigh quotient. The resulting error in frequency is of second order in the error of the eigenfunction. In order to conserve computer storage for large models the matrices need not be constructed and stored in their entirety, but multiplications can be accumulated one element at a time. Calculations for an inhomogeneous vibrating string and Rayleigh waves in a layered Earth model illustrate the technique.

1. Value of the variational method

Rayleigh's Principle is commonly invoked in the determination of periods of free oscillation. It states that when the mean potential and kinetic energies of the system in free oscillation are equated, the solution for the angular frequency takes a stationary value (see Moiseiwitsch 1966). Its use in seismology is not new. Meissner (1926) observed that it may be used to determine group velocity for Love waves without resorting to numerical differentiation. Jeffreys (1961) extended this work to Rayleigh waves, and further noted that perturbations in the eigenfrequency due to changes in the structure can be determined without repeating the entire calculation. The original form of the mode shape is retained, on the assumption that it approximates that for the modified structure. Because of the stationarity, the new eigenfrequency is obtained with an error of only second order. This observation led to the publication of Universal Dispersion Tables for Love and Rayleigh waves (Anderson 1964, and later papers) and for terrestrial eigenvibrations (Derr 1969) from which the effects of changing the structure on the period of surface waves or torsional and spheroidal oscillations with given wave number could be determined.
The purpose of this paper is to demonstrate how Rayleigh's principle may be fruitfully invoked for solving problems in which structures can be modelled by the finite element method. Two classes of problems suggest themselves. First, estimates of the seismic response of irregular Earth structures may be obtained from solutions of simpler problems without actually solving the more complicated eigenvalue problem. An approximate eigenfunction, calculated for a simple structure, is used to estimate the eigenfrequency for a more complex one, and this frequency is then accurate to second order. This procedure has application, for example, for realistic crustal models where either mathematical or numerical difficulties make work on wave propagation intractable. Secondly, from a set of computed solutions of defined models of geophysical interest (e.g. mountain roots, continental boundaries), the effects of small changes in the structure can be estimated by making corresponding changes in the finite element matrices. In both cases, the calculation is simple and direct.

Eigenvalue problems for perfectly elastic geophysical structures modelled by the finite element method may be expressed in the form (see Smith 1975)

\[-\omega^2 \mathbf{M} \mathbf{u} = \mathbf{F} - \mathbf{K} \mathbf{u}.\]  

(1)

\(\mathbf{K}\) is the elastic moduli (or stiffness) matrix and \(\mathbf{M}\) is the density (or mass) matrix. The examples given below will illustrate their form. \(\mathbf{K}\) and \(\mathbf{M}\) contain all the elastic and geometric properties of the discrete model of the geophysical structure. \(\mathbf{u}\) is a vector containing the displacements at all the nodes of the mesh, and thus represents the eigenfunction. \(\omega\) is the angular frequency. The eigenvalue problem is that of determining \(\omega\) and \(\mathbf{u}\) such that equation (1) is satisfied when \(\mathbf{F}\) is zero.

Rearrange equation (1) in the form

\[\omega^2 = \frac{\mathbf{u}^T \mathbf{K} \mathbf{u}}{\mathbf{u}^T \mathbf{M} \mathbf{u}}.\]  

(2)

This form at once suggests the application of Rayleigh's Principle because, on the right-hand side, the numerator is proportional to the potential energy and the denominator to the kinetic energy. An approximate form for the eigenfunction yields a close estimate of the eigenfrequency. Iterative schemes which successively improve both \(\omega\) and \(\mathbf{u}\) can be made to converge rapidly (e.g. Wilkinson 1965). But as far as we are aware, the value of invoking Rayleigh's Principle for variations in complicated geophysical structures using the matrix formulation (2) has not been widely appreciated. A new structure (different \(\mathbf{K}\) and/or \(\mathbf{M}\)) will have an eigenfunction slightly different from \(\mathbf{u}\). But if the original \(\mathbf{u}\) is retained the resultant error in \(\omega\) will be only of second order in the error of the eigenfunction. The Rayleigh quotient in equation (2) can be evaluated very easily by a digital computer, as it involves only the construction of the two matrices and their multiplication by vectors. The effect on the eigenfrequency of perturbations to even an extensive structure can thus be obtained at little cost.

2. Vibrations of an elastic string

The oscillations of an inhomogeneous string provide a simple example of both finite element techniques and the use of the variational method (see Ramsey 1954, p. 327). A string of length \(L\) is shown in Fig. 1, modelled by eight elements of equal length. Displacements \(\mathbf{u}\) within an element are interpolated linearly from the values at the nodes. The first half of the string is taken to have density \(\rho\) and the second half \(\alpha \rho\), where \(\alpha\) is close to unity. The tension is \(P\), and the string is clamped at both ends.
Rayleigh’s principle in finite element calculations

The potential energy in an element of the string is

\[ V_e = \frac{P}{2} \int_0^h \left( \frac{\partial u}{\partial x} \right)^2 dx, \quad (3) \]

where \( h = L/8 \), the length of the element. If the two nodal displacements are collected in a vector \( u_e \), we can write the displacement within the element as

\[ u = \left[ 1 - \frac{x}{h}, \frac{x}{h} \right] u_e, \quad (4) \]

and the gradient as

\[ \frac{\partial u}{\partial x} = \frac{1}{h} \left[ -1, 1 \right] u_e. \quad (5) \]

So from (3), (4) and (5) we find

\[ V_e = \frac{P}{2h} u_e^T \left[ \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] u_e \]

\[ = \frac{1}{2} u_e^T K_e u_e, \quad (6) \]

where \( K_e \) is the element elastic modulus matrix. Adding together the contributions from all the elements, we find that the total potential energy is given by

\[ V = \frac{1}{2} u^T K u, \quad (7) \]

where \( u \) is a \( 9 \times 1 \) vector and \( K \) is given by

\[ K = \frac{8P}{L} \left[ \begin{array}{cccccc} 1 & -1 & -1 & 2 & -1 \\ -1 & 2 & -1 & -1 & 2 & -1 \\ -1 & 2 & -1 & -1 & 2 & -1 \\ -1 & 2 & -1 & -1 & 2 & -1 \\ -1 & 2 & -1 & -1 & 2 & -1 \\ -1 & 2 & -1 & -1 & 2 & -1 \\ -1 & 2 & -1 & -1 & 2 & -1 \\ -1 & 2 & -1 & -1 & 2 & -1 \\ -1 & 2 & -1 & -1 & 2 & -1 \end{array} \right] \quad (8) \]

The kinetic energy of an element is

\[ T_e = \frac{\rho}{2} \int_0^h \dot{u}^2 dx, \quad (9) \]
where a dot denotes differentiation with respect to time. Substituting from equation (4) we obtain

\[ T_e = \frac{\rho h}{12} \mathbf{u}_e^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{u}_e \]

\[ = \frac{1}{2} \mathbf{u}_e^T \mathbf{M}_e \mathbf{u}_e \]  

(10)

and \( \mathbf{M}_e \) is the element density matrix. The total kinetic energy is therefore given by

\[ T = \frac{1}{2} \mathbf{u}^T \mathbf{M} \mathbf{u}, \]  

(11)

where

\[ M = \frac{\rho L}{48} \begin{bmatrix} 2 & 1 \\ 1 & 4 & 1 \\ 1 & 4 & 1 \\ 1 & 4 & 1 \\ \end{bmatrix} \]

(12)

(Note that the element mass matrix is scaled by \( \alpha \) when the density of that particular element is \( \rho \alpha \).) For sinusoidal oscillation we can form the Rayleigh quotient of equation (2). This is the ratio of the mean potential and kinetic energies, averaged over one cycle.

It remains to define the eigenfunction. For a homogeneous string, it is of the form \( u = A \sin \pi x / L \). Provided that \( \alpha \) is not too different from unity, the fundamental eigenfunction for the inhomogeneous string will approximate this form. It may therefore be used in equation (2), with assurance from Rayleigh's principle that the resulting error in frequency will be of second order. Accordingly, the terms of the vector \( \mathbf{u} \) can be assigned the ordinates of the sinusoid, as shown in Fig. 1.

Define a dimensionless frequency \( \omega' = \omega L / (\rho / P) \). For \( \alpha = 1 \) (the homogeneous case), the evaluation of equation (2) yields \( \omega' = 3.16182 \). This is greater than the analytical value of \( \pi \) because of the discrete nature of the finite element modelling. For \( \alpha = 1.1 \), equation (2) yields \( \omega' = 3.08562 \), i.e. a decrease of 2.41 per cent. The true value of \( \omega' \) in this case is the solution of the transcendental equation

\[ \tan \omega' + \sqrt{\alpha} \tan \omega' \sqrt{\alpha} = 0, \]  

(13)

for which we obtained a solution \( \omega' = 3.06501 \) for \( \alpha = 1.1 \). This represents a decrease of 2.44 per cent from the homogeneous case \( (\omega' = \pi) \). It is clear that the Rayleigh quotient calculation gives a good approximation to the change in frequency when the homogeneous string is replaced by an inhomogeneous one.

3. Rayleigh waves in a layered-earth model

We examine as a further illustration of the method the estimation of changes in eigenfrequency for mantle Rayleigh waves. (A finite element formulation of Rayleigh wave propagation along a ridge has already been published by Burridge & Sabina (1972).) The mode shapes for defined Earth models are already computed for long-period Rayleigh waves (Dorman & Prentiss 1960; Bolt & Dorman 1961).
Let the displacements be \( u = (-i u_x, u_z) \exp \left( i(\omega t - kx) \right) \), where \( u_x, u_z \) are real. \( k \) is the wave number and \( \omega \) the angular frequency. The engineering strains are

\[
\begin{align*}
\varepsilon_{xx} &= -i(-ik) u_x = -k u_x \\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z} \\
\varepsilon_{xz} &= -i \frac{\partial u_x}{\partial z} - ik u_z.
\end{align*}
\]

The potential strain energy averaged over \( x \) and \( t \) is

\[
V = \frac{1}{2} \left( \lambda + 2\mu \right) \left[ k^2 u_x^2 + \left( \frac{\partial u_x}{\partial z} \right)^2 \right] - \lambda k u_x \frac{\partial u_x}{\partial z} + \frac{1}{2} \mu \left( \frac{\partial u_x}{\partial z} + k u_x \right)^2,
\]

where \( \lambda(z) \) and \( \mu(z) \) are the Lamé parameters. The mean kinetic energy is

\[
T = \frac{1}{2} \rho \omega^2 (u_x^2 + u_z^2).
\]

The Euler–Lagrange equation may be invoked to derive from (15) and (16) the differential equations

\[
\begin{align*}
-\rho \omega^2 u_x &= -(\lambda + 2\mu) k^2 u_x + (\lambda + \mu) k \frac{\partial u_x}{\partial z} + \mu \frac{\partial^2 u_x}{\partial z^2} \\
-\rho \omega^2 u_z &= (\lambda + 2\mu) \frac{\partial^2 u_x}{\partial z^2} - (\lambda + \mu) k \frac{\partial u_x}{\partial z} - \mu k^2 u_z.
\end{align*}
\]

For the finite element modelling we use the one-dimensional element shown in Fig. 2, aligned parallel to the \( z \)-axis and with two displacement components at each node. For

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*Fig. 2. Finite element used for the Rayleigh wave problem. There are two displacement components at each node. Propagation is in the \( x \) direction and \( z \) is vertically upwards.*
linear interpolation of displacements within an element we have

\[ \mathbf{u} = \begin{bmatrix} u_x \\ u_z \end{bmatrix} = \begin{bmatrix} 1 - \frac{z}{h} & 0 & \frac{z}{h} & 0 \\ 0 & 1 - \frac{z}{h} & 0 & \frac{z}{h} \end{bmatrix} \begin{bmatrix} u_{1x} \\ u_{1z} \\ u_{2x} \\ u_{2z} \end{bmatrix} \]

(18)

= \mathbf{N} \mathbf{u}_e

The element density matrix is given by

\[ \mathbf{M}_e = \int_0^h \mathbf{N}^T \rho \mathbf{N} \, dz. \]

(19)

The average strains are \( \mathbf{e} = \mathbf{B} \mathbf{u}_e \) and the stresses are \( \mathbf{p} = \mathbf{H} \mathbf{e} \), where

\[
\mathbf{B} = \begin{bmatrix}
-k \left(1 - \frac{z}{h}\right) & 0 & -k \frac{z}{h} & 0 \\
0 & -\frac{1}{h} & 0 & \frac{1}{h} \\
-\frac{1}{h} & k \left(1 - \frac{z}{h}\right) & \frac{1}{h} & k \frac{z}{h}
\end{bmatrix}
\]

(20)

and

\[
\mathbf{H} = \begin{bmatrix}
\lambda + 2\mu & \lambda & 0 \\
\lambda & \lambda + 2\mu & 0 \\
0 & 0 & \mu
\end{bmatrix}
\]

(21)

The element elastic moduli matrix is then

\[ \mathbf{K}_e = \int_0^h \mathbf{B}^T \mathbf{H} \mathbf{B} \, dz. \]

(22)

The global elastic moduli and density matrices \( \mathbf{K} \) and \( \mathbf{M} \) are accumulated in the usual way. For \( n \) elements in the model, each representing one layer, they will be \((2n + 2)\) square, but with a bandwidth of only 6. Construction of the vector \( \mathbf{u} \), containing all the \((2n + 2)\) displacement components, enables the evaluation of the energy terms \( \mathbf{u}^T \mathbf{K} \mathbf{u} \) and \( \mathbf{u}^T \mathbf{M} \mathbf{u} \) to be made. Their ratio is \( \omega^2 \). With the wave number

<table>
<thead>
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<th>Table 1</th>
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<td>Angular frequency, period and phase velocity for a wave number of 0.009909 km⁻¹</td>
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<table>
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<tr>
<th>Angular frequency (rad s⁻¹)</th>
<th>Period (s)</th>
<th>Phase velocity (km s⁻¹)</th>
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<tbody>
<tr>
<td>Dorman &amp; Prentiss</td>
<td>0.04192</td>
<td>149.9</td>
</tr>
<tr>
<td>Finite element</td>
<td>0.04203</td>
<td>149.5</td>
</tr>
<tr>
<td>Jeffreys model</td>
<td>0.04295</td>
<td>146.3</td>
</tr>
<tr>
<td>Bolt &amp; Dorman</td>
<td></td>
<td>146.3</td>
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</tbody>
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specified at the outset and the frequency now calculated, the phase velocity may be
determined.

Dorman & Prentiss (1960) published mode shapes for mantle Rayleigh waves for a
set of discrete frequencies in a horizontally layered Earth with a $P$ and $S$ velocity
distribution derived by Gutenberg and a density distribution known as Bullen Model
A. The period and phase velocities for one of these frequencies are reprinted in Table 1.
For a period of about 150 s the horizontal and vertical wave displacement in the
fundamental Rayleigh mode for a flat Earth model are perceptible to depths of
between 500 and 750 km (see Fig. 5, Dorman & Prentiss 1960; and Fig. 1, Bolt &
Dorman 1961). The mode shape in this case is approximated, as in the case of the
string, by linear elements. We use 19 elements, corresponding to layers of thickness
from 20 to 100 km, and modelling the Earth to a total depth of 800 km. Modelling
with a variety of element sizes presents no computational problem, because the global
matrices $K$ and $M$ (size $40 \times 40$) are built up element by element.

Also shown in Table 1 is the corresponding angular frequency. The wave number,
calculated to be 0.009909 km$^{-1}$, is used in the finite element expression (2) with the
published mode shape and the same Gutenberg model. Table 1 lists the resulting
values for angular frequency, period and phase velocity. Agreement is very close; the
small discrepancies of 0.25 per cent between these estimates and those of Dorman &
Prentiss probably arise from the linear finite element modelling. Additional precision
should result from smaller element size or higher-order spline interpolation with, of
course, additional computing time required. These numerical problems are outside
the aim of the present paper.

To illustrate the ease with which the effect of changes in the model may be
estimated, and the accuracy of doing so, we now calculate the period and phase
velocity for a Rayleigh wave of the same wave number, but propagating in a mantle
model with $P$ and $S$ velocity distributions derived by Jeffreys. This variation in the
model requires, among other changes, the removal of the low velocity zone of the
Gutenberg model between 50 and 200 km depth. Elements in the elastic moduli
matrix (21) require changes in value of up to 10 per cent. The original mode shape of
Dorman & Prentiss is retained as an approximate eigenfunction for the new model.
The $40 \times 40$ matrices are constructed as before and the products $u^T Ku$ and $u^T Mu$
accumulated. The resulting values of angular frequency, period and phase velocity
are shown in Table 1. Also shown are the relevant figures for the same wave number
interpolated from Table 2 of Bolt & Dorman (1961). Their model was of a spherical,
gravitating Earth with the Jeffreys velocity distribution and the Bullen A densities,
but they indicate that the sphericity and gravity affect the phase velocity by 0.09 km s$^{-1}$
at a period of 146 s. This correction for the flat-layered case has been made in the phase
velocity listed in Table 1. The increase in the phase velocity of 2.2 per cent as calculated
by the finite element modelling and the variational principle compares well with the
accurate value of 1.9 per cent when it is remembered that a perturbation of up to
10 per cent in the model elastic parameters has been made. This calculation, therefore,
gives some indication of the size of error obtainable in the variational approach.

4. Discussion

We have shown that Rayleigh's principle provides a powerful tool for the
approximate determination of eigenfrequencies for realistic geophysical structures
modelled by the finite element method. When the eigenfunction for a regular structure
is known, the effect on the eigenfrequency of irregularities introduced into the
structure can be estimated easily. The original eigenfunction is retained, and
Rayleigh's principle guarantees that the resulting error in frequency will be of second
order. The actual magnitude of the error will depend on the particular problem in
hand.
The oscillating string exemplifies all free oscillation problems. Irregularities may be introduced very easily and the new eigenfrequency estimated. It is not necessary to solve an eigenvalue problem, but only to construct the matrices and perform multiplications. The Rayleigh surface wave problem illustrates further how eigenfunctions chosen on the basis of known seismological solutions can give useful results for irregular Earth structures. The modelling of the free oscillations of the Earth is another obvious application, where the perturbations due to lateral inhomogeneities could be estimated. The finite element method is well suited to the modelling of inhomogeneous structures with or without damping.

The computer storage required for three-dimensional models normally limits severely the scope of such calculations. For an element with eight nodes, and three degrees of freedom at each node, the element stiffness and mass matrices are of size \(24 \times 24\). Accumulating these into global matrices escalates rapidly the storage requirements, even for quite small assemblages of elements. Such large amounts of storage are not necessary with the present technique, because the products \(Ku\) and \(Mu\) can be accumulated one element at a time. The element stiffness matrix \(K_e\) can be multiplied by the appropriate terms of the eigenvector \(u\), and accumulated into the vector \(Ku\). The energy term \(u^T Ku\) is then obtained as the dot product of the two vectors. The saving in space, therefore, is effectively that required for the two global matrices \(K\) and \(M\), and these constitute probably 90 per cent of the storage for any finite element problem.

It should also be appreciated that, storage problems aside, the size of a tractable problem is normally limited by computer roundoff error. For this reason eigenvalue solutions for problems with many thousands of elements are beyond the capability of present-day computers. But the Rayleigh's principle approximation is not subject to such limitations, so very large problems may be treated. When the eigenvalue problem is tractable the computer time required is much greater than for the present method, by an order of magnitude. This is apparent when it is realized that the Rayleigh Quotient calculation is but one small part of each iteration in the eigenvalue solution procedure.

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References


