The validity of the Dyson formula for the $S$ matrix in non-linear field theories is examined by comparing it with the exact formula. It is concluded that this formula is valid in the tree approximation, at least in the lowest few orders. This justifies the tree approximation calculations in chiral dynamics.

§ 1. Introduction

In connection with the recent development of chiral dynamics, interest in non-linear field theories has revived, and it is a pressing problem to establish procedures of deriving the $S$ matrix from a given non-linear Lagrangian. In this article we shall study a general method of deriving the $S$ matrix in perturbation theory and shall then compare the results with those derived from the conventional Dyson formula generally valid in linear field theories.

For simplicity we shall consider only spinless fields in this paper. The distinction between linear and non-linear field theories is made according to the structure of the kinetic energy terms in given Lagrangians.

Suppose that a Lagrangian is given in the form

$$L = \frac{1}{2} \sum_{a,b} D_{ab} \partial_x \varphi_a \partial_x \varphi_b - V(\varphi_a), \quad (1.1)$$

where $\varphi_a (a=1,2,\cdots,N)$ denote Hermitian fields and $D_{ab}$ is a non-singular real symmetric matrix. The interaction term $V$ is a function of the field operators alone and does not depend on their derivatives.

When all the coefficients $D_{ab}$ are c-numbers the theory is called linear; otherwise we have a non-linear field theory. In either case the canonical commutation relations are given by

$$[\varphi_a(x), \varphi_b(y)] = iC_{ab} \delta^a(x-y) \quad \text{for} \quad x_0 = y_0, \quad (1.2)$$

where $C$ is the inverse matrix of $D$. Then, in linear field theories the r.h.s. of Eq. (1.2) represents a c-number, whereas in non-linear field theories it generally represents a q-number.

In this paper we find it convenient to introduce a further classification of non-linear theories. When a non-linear Lagrangian is transformed into a linear one by an appropriate point transformation

$$\varphi_a(x) = \chi_a(x) g(\chi(x)); \quad g(0) = 1, \quad (1.3)$$
the Lagrangian is called reducible. When it is impossible to find such a point transformation we have an irreducible non-linear Lagrangian.

In §2 we shall discuss a general method of deriving the $S$ matrix from a given Lagrangian of the type (1.1) with the help of a functional formalism. In §3 we shall introduce a solvable example of the reducible type and discuss some of the characteristic features of this model. In §4 we shall generalize the Feynman rules to reducible non-linear theories. In §5 we shall examine the validity of the Dyson formula for the $S$ matrix for the reducible model studied in §3, and then shall show that the application of the Dyson formula to the Weinberg Lagrangian leads to the correct results only in the tree approximation.

§ 2. Functional equations for the time-ordered Green functions

The $S$ matrix can be derived from the time-ordered Green functions with the help of the LSZ reduction formalism so that derivation of the $S$ matrix reduces to the evaluation of the time-ordered Green functions.

Without loss of generality we shall illustrate derivation of the equations for the time-ordered Green functions for the following Lagrangian:

$$L = -\frac{1}{2} \left(\frac{\partial_{\phi}^{2}}{1 + \phi f}\right) - V(\phi).$$  

(2.1)

The Euler-Lagrange equation and the canonical commutation relations are given, respectively, by

$$\square \phi (x) = j[\phi (x)]; \quad j[\phi (x)] = \frac{f}{2} \frac{(\partial_{\phi}^{2})}{1 + \phi f} + (1 + \phi f) V' (\phi)$$

(2.2)

and by

$$[\phi (x), \phi (y)] = i \delta^{4} (x - y) C[\phi (x)] \quad \text{for} \quad x_{0} = y_{0},$$

(2.3)

where

$$C[\phi (x)] = 1 + \phi f (x).$$

(2.4)

By using the above equation and commutation relation we can easily derive a functional equation for the generating functional $\mathcal{D}$ of the time-ordered Green functions defined by

$$\mathcal{D} = \langle T^{*} \exp (-i \int \phi (x) J(x) \ dx) \rangle_{0},$$

(2.5)

where $J(x)$ is a c-number source and $\langle \rangle_{0}$ denotes the vacuum expectation value. Then, by making use of the formula

$$\square \langle T^{*}[\phi (x), AB\ldots] \rangle_{0},$$
The S Matrix in Non-Linear Field Theories

\[ = \langle T^* \square \varphi(x), AB \cdots \rangle \varnothing + i \langle T^* [C[\varphi(x)], \frac{\delta A}{\delta \varphi(x)}], B \cdots \rangle \varnothing + \cdots, \] (2.6)

we obtain

\[ \square x i \frac{\delta T^*}{\delta J(x)} = \square x \langle T^* [\varphi(x), \exp(-i \int \varphi(y) J(y) d^4 y)] \rangle \varnothing \]

\[ = \langle T^* [\varphi(x), J(y) \exp(-i \int \varphi(y) J(y) d^4 y)] \rangle \varnothing \]

\[ + J(x) \langle T^* [C[\varphi(x)], \exp(-i \int \varphi(y) J(y) d^4 y)] \rangle \varnothing. \] (2.7)

Thus the functional \( \mathcal{T} \) satisfies the following equation:

\[ \square x i \frac{\delta \mathcal{T}}{\delta J(x)} = \begin{pmatrix} \varphi(x) \\ J(x) \end{pmatrix} \mathcal{C} \begin{pmatrix} \varphi(x) \\ J(x) \end{pmatrix}, \] (2.8)

where we employed the following definition in conformity with the \( T^* \) product:

\[ \begin{pmatrix} \varphi(x) \\ J(x) \end{pmatrix} \mathcal{C} \begin{pmatrix} \varphi(x) \\ J(x) \end{pmatrix} = \frac{1}{2} \left( \begin{array}{c} \varphi(x) \\ J(x) \end{array} \right)^2 \] (2.9)

Rigorously speaking, the current \( j[\varphi(x)] \) must be defined as a normal product in order to avoid complications arising from the product of derivatives of field operators. The prescription of deriving a functional differential equation for \( \mathcal{T} \) can easily be generalized to cover a more general form (1.1). For instance, we shall consider the Weinberg Lagrangian of the form

\[ L = \frac{1}{2} \frac{1}{(1 + \lambda \varphi^a)^2} - V(\varphi^a), \] (2.10)

where the component index \( a \) is treated as a dummy index. Then we have

\[ \square \varphi_a(x) = j_a[\varphi(x)]; \] (2.11)

\[ j_a[\varphi^e(x)] = \frac{2 \lambda}{1 + \lambda \varphi^b} \begin{pmatrix} \partial_{\varphi_a} \varphi \partial_{\varphi_b} \varphi \partial_{\varphi_b} \varphi_a - (\partial_{\varphi_a} \varphi)^2 \varphi_a \\ -2 \varphi_a (1 + \lambda \varphi^b)^2 V'(\varphi^b) \end{pmatrix}, \] (2.12)

and

\[ [\varphi_a(x), \varphi_b(y)] = i \delta_{ab} C[\varphi(x)] \partial^8 (x - y) \text{ for } x = y \] (2.13)

with

\[ C[\varphi^e(x)] = (1 + \lambda \varphi^e)^2. \] (2.14)
Now introduce a generating functional $\mathcal{T}$ by
\[
\mathcal{T} = \langle T^* \exp (-i \int \varphi_a(x) J_a(x) d^4 x) \rangle_0. \tag{2.15}
\]
Then, $\mathcal{T}$ satisfies the following equation:
\[
\square_i \frac{\delta}{\delta J_a(x)} \mathcal{T} = J_a \left[ i \frac{\delta}{\delta J_c(x)} \right] \mathcal{T} + J_a(x) C \left[ i \frac{\delta}{\delta J_c(x)} \right] \mathcal{T}. \tag{2.16}
\]
By expanding the integrated form of Eq. (2.16) in powers of $J$ and using $LIF$ satisfying $\square LIF(x) = i\delta^4(x)$, we find
\[
\langle T^* [\varphi_a(x_1) \varphi_a(x_2) \cdots \varphi_a(x_n)] \rangle_0 = -i \int d^4 y \langle T^* [j_a(y), \varphi_a(x_2) \cdots \varphi_a(x_n)] \rangle_0
\]
\[
+ \sum_{i=2}^n \delta (a_i a_i) \langle T^* [\varphi_a(x_2) \cdots \varphi_a(x_n)] \rangle_0 \tag{2.17}
\]
We can solve the above set of equations successively in ascending powers of the coupling constant, and consequently we can determine the $S$ matrix in perturbation theory.

§ 3. A solvable example

In this section we shall consider a simple example defined by the following Lagrangian:
\[
L = -\frac{1}{2} \frac{(\partial_0 \varphi)^2}{1 + f\varphi}. \tag{3.1}
\]
This Lagrangian is non-linear but reducible as one can easily recognize as follows.
Introduce a new field $\chi$ defined by
\[
\partial_0 \chi = \frac{\partial_0 \varphi}{\sqrt{1 + f\varphi}} \tag{3.2}
\]
or
\[
\chi = \frac{2}{f} (\sqrt{1 + f\varphi} - 1). \tag{3.3}
\]
Then, in terms of the new field $\chi$ the Lagrangian reduces to
\[
L = -\frac{1}{2} (\partial_0 \chi)^2. \tag{3.4}
\]
Therefore, $\chi$ represents a free field so that we shall put
\[
\chi = \varphi^{in}. \tag{3.5}
\]
By solving Eq. (3·3) in favor of \( \varphi \) we find

\[
\varphi (x) = \chi (x) + \frac{f}{4} (\chi (x))^3
= \varphi^{in} (x) + \frac{f}{4} (\varphi^{in} (x))^3.
\] (3·6)

The \( S \) matrix in this model is equal to the unit matrix. We can use two alternative arguments to prove it. The first proof rests on the fact that the given Lagrangian (3·1) reduces to a free one by means of a point transformation of the type (1·3). Then it is clear that \( S = 1 \) from the arguments of Kamefuchi, O’Raifeartaigh and Salam and of Chisholm.\(^6\) The second proof is based on the observation that

\[
[\varphi (x), \varphi^{in} (y)] = 0 \quad \text{for} \quad (x-y)^2 > 0,
\] (3·7)

which is clear from Eq. (3·6). Then \( \varphi \) and \( \varphi^{in} \) belong to the same Borchers’ class,\(^6\) and we may replace \( \varphi \) by \( \varphi^{in} \) in the LSZ reduction formula for the \( S \) matrix as is clear from the HNZ theorem.\(^6\) Hence \( S = 1 \).

It is perhaps worth mentioning that the solution (3·6) satisfies the Yang-Feldman equation

\[
\varphi (x) = \varphi^{in} (x) - \frac{f}{2} \int d^n y \, d^p (x-y) (\partial_x \varphi (y))^3
\]
\[
1 + f \varphi (y).
\] (3·8)

This is interesting in the sense that the Yang-Feldman formalism\(^7\) leads to the correct quantization also in non-linear theories.

In this example the solution of the functional equation (2·8) with \( V = 0 \) can be obtained exactly. Although this is a trivial problem it is instructive when we consider its generalizations.

\[
\mathcal{S} = \langle T \exp \left( -i \int J (x) \varphi (x) d^4 x \right) \rangle, = \langle T \exp \left( -i \int J (x) (\varphi^{in} (x) + \frac{f}{4} (\varphi^{in} (x))^3) d^4 x \right) \rangle, = e^\mathcal{R}.
\] (3·9)

\( \mathcal{R} \) is the generating functional of the connected parts of the time-ordered Green functions. \( \mathcal{R} \) is given by

\[
\mathcal{R} = \mathcal{R}_{\text{closed}} + \mathcal{R}_{\text{open}},
\] (3·10)

where

\[
\mathcal{R}_{\text{closed}} = \sum_{n=3}^\infty \left( \frac{-if^n}{2^n n!} \right) \int d^4 x_1 \cdots d^4 x_n J (x_1) \cdots J (x_n)
\times \Delta_p (x_1 - x_2) \cdots \Delta_p (x_n - x_1),
\] (3·11)
Typical terms in (3·11) and (3·12) are represented, respectively, by Feynman diagrams in Figs. 1 and 2.

Fig. 1. The diagram corresponding to $\mathcal{R}_{\text{closed}}$.

Fig. 2. The diagram corresponding to $\mathcal{R}_{\text{open}}$.

From the above exact solution we can immediately read off the exact propagator

$$\Delta_{\phi^{'}}(x) = \Delta_{\phi}(x; 0) + \int_{0}^{\infty} dm^{2} \sigma(m^{2}) \Delta_{\phi}(x; m^{2}),$$

(3·13)

where

$$\sigma(m^{2}) = \frac{1}{2} \left( \frac{f}{2\pi} \right)^{2}.$$  

Since the coefficient of the free propagator $\Delta_{\phi}(x; 0)$ is unity, the $Z$ factor must be equal to unity. Also, there is no mass shift. These results seem to contradict Lehmann's formulas for the $Z$ factor and the self-energy when the spectral function $\sigma$ does not vanish. This is not a contradiction, however, since Lehmann's formulas are valid only in linear theories.

The contents of this section are all well known, but they are instructive when we consider a less trivial problem in the next section.

§ 4. Feynman rules for reducible non-linear theories

In order to write down the $S$ matrix we have to determine the generating functional $\mathcal{F}$ by solving Eq. (2·8) or (2·16). In some simple cases we can find a formal solution for $\mathcal{F}$. For simplicity we shall consider only the neutral scalar field and shall illustrate a solution of this problem by employing the Lagrangian (2·1).

Let us first consider the linear theory by putting $f=0$ in (2·1).

$$L = -\frac{i}{2} (\partial_{\mu} \phi)^{2} - V(\phi).$$  

(4·1)

In this case $\mathcal{F}$ satisfies

$$\Box_{x} i \frac{\delta \mathcal{F}}{\delta J(x)} = i \left[ i \frac{\delta}{\delta J(x)} \right] \mathcal{F} + J(x) \mathcal{F},$$  

(4·2)
The S Matrix in Non-Linear Field Theories

where

\[ j[\varphi(x)] = V[\varphi(x)]. \] (4.3)

The formal solution to Eq. (4.2) is given immediately by

\[ \mathcal{T} = \langle T^* \exp \left( -i \int d^4x \left( V[\varphi^\text{in}(x)] + \varphi^\text{in}(x) J(x) \right) \right) \rangle. \] (4.4)

The Dyson formula follows from Eq. (4.4) with the help of the LSZ reduction formalism and is given by

\[ S = T^* \exp \left( -i \int d^4x V[\varphi^\text{in}(x)] \right). \] (4.5)

Next we shall come back to the non-linear Lagrangian (2.1) and shall try to find a formal solution to Eq. (2.8). In guessing the correct formal expression for \( \mathcal{T} \) Eq. (3.9) is very helpful. The main difference between (3.9) and (4.4) consists in the term proportional to \( J(x) \) (\( \varphi^\text{in}(x) \))^2 in (3.9). The presence of such a term suggests that a possible generalization of (4.4) might be given by

\[ \mathcal{T} = \langle T^* \exp \left( -i \int d^4x \left( J[\varphi^\text{in}(x)] + F[\varphi^\text{in}(x) J(x)] \right) \right) \rangle. \] (4.6)

where

\[ \square \varphi^\text{in}(x) = 0, \]

\[ [\varphi^\text{in}(x), \varphi^\text{in}(y)] = i\delta^8(x - y) \quad \text{for} \quad x_0 = y_0. \]

The problem now is to determine \( J^\text{in} \) and \( F \) in such a way that the expression (4.6) satisfies Eq. (2.8).

\[ \square_x i \frac{\delta}{\delta J(x)} \mathcal{T} = \langle T^* \left[ \square_x F[\varphi^\text{in}(x)], \exp(\gamma) \right] \rangle_0 \]

\[ + \langle T^* \left[ \frac{\partial F}{\partial \varphi^\text{in}(x)} \left( \frac{\partial J}{\partial \varphi^\text{in}(x)} + J(x) \frac{\partial F}{\partial \varphi^\text{in}(x)} \right), \exp(\gamma) \right] \rangle_0 \]

\[ = \langle T^* \left[ \square_x F[\varphi^\text{in}(x)], \exp(\gamma) \right] \rangle_0 \]

\[ + \langle T^* \left[ \frac{\partial F}{\partial \varphi^\text{in}(x)} \frac{\partial J}{\partial \varphi^\text{in}(x)}, \exp(\gamma) \right] \rangle_0 \]

\[ + J(x) \langle T^* \left[ \frac{\partial F}{\partial \varphi^\text{in}(x)}, \exp(\gamma) \right] \rangle_0. \] (4.7)

By equating the coefficient of \( J(x) \) in Eq. (4.7) to the corresponding coefficient in Eq. (2.8) we find

\[ \langle T^* \left[ \frac{\partial F}{\partial \varphi^\text{in}(x)} \left( \frac{\partial F}{\partial \varphi^\text{in}(x)} \exp(\gamma) \right) \right] \rangle_0 = \left( 1 + i f \frac{\delta}{\delta J(x)} \right) \mathcal{T} \]
This equation implies
\[ \left( \frac{\partial F}{\partial \phi^\text{in}(x)} \right)^2 = 1 + fF. \] (4·8)

By taking account of the boundary condition (1·3), namely,
\[ \frac{F[\phi^\text{in}]}{\phi^\text{in}} \to 1 \quad \text{for} \quad \phi^\text{in} \to 0, \]
we find the following solution
\[ F[\phi^\text{in}(x)] = \phi^\text{in}(x) + \frac{f}{2} (\phi^\text{in}(x))^2. \] (4·9)

We have completely determined \( F \), and the next problem is the determination of \( \mathcal{H} \).

Since we have
\[ \Box \phi F[\phi^\text{in}(x)] = f \left( \frac{\partial}{\partial x} \phi^\text{in}(x) \right)^2 = \frac{f}{2} \frac{(\partial_x F)^2}{1 + fF}, \]
the remaining part of Eq. (2·8) may be written as
\[ j \left[ i \frac{\partial}{\partial J(x)} \right] = \left\langle T^*_x \left[ f \left( \frac{\partial}{\partial \phi^\text{in}(x)} \right)^2 + \frac{\partial F}{\partial \phi^\text{in}(x)} \cdot \frac{\partial \mathcal{H}}{\partial \phi^\text{in}(x)} \right. \right. \exp \left( \cdot \right) \right\rangle_0 \]
\[ \text{or} \]
\[ (1 + fF) V'[F] = \frac{\partial F}{\partial \phi^\text{in}(x)} \cdot \frac{\partial \mathcal{H}}{\partial \phi^\text{in}(x)}. \] (4·10)

The second equation may also be written as
\[ \left( \frac{\partial F}{\partial \phi^\text{in}(x)} \right)^2 \frac{\partial V}{\partial F} = \frac{\partial F}{\partial \phi^\text{in}(x)} \cdot \frac{\partial \mathcal{H}}{\partial \phi^\text{in}(x)}. \]

A further simplified equation reads
\[ \frac{\partial V}{\partial \phi^\text{in}(x)} = \frac{\partial \mathcal{H}}{\partial \phi^\text{in}(x)}, \]
and hence \( \mathcal{H} \) is given by
\[ \mathcal{H} = V[F] = V \left[ \phi^\text{in}(x) + \frac{f}{4} (\phi^\text{in}(x))^2 \right]. \] (4·11)

Thus we have fixed the formal solution (4·6).

In the above treatment we have expressed \( \mathcal{I} \) in terms of the massless field \( \phi^\text{in}(x) \), but it is also possible to express it in terms of a massive field of mass \( \mu \). Let \( \phi^\text{in} \) satisfy


then we have the same $F$ as (4.9), but $\mathcal{H}$ is modified as

$$\frac{\partial V}{\partial \varphi^{\text{in}}(x)} = \frac{\partial \mathcal{H}}{\partial \varphi^{\text{in}}(x)} + \mu^2 \varphi^{\text{in}},$$

and consequently we obtain

$$\mathcal{H} = V(F) - \frac{1}{2} \mu^2 (\varphi^{\text{in}}(x))^2$$

$$= V(\varphi^{\text{in}}(x) + \frac{f}{4} (\varphi^{\text{in}}(x))^2) - \frac{1}{2} \mu^2 (\varphi^{\text{in}}(x))^2. \quad (4.12)$$

Application of the LSZ reduction technique to (4.6) then leads to the following $S$ matrix formula:

$$S = T^* \exp \left( -i \int d^4 x \left( V \left( \varphi^{\text{in}}(x) + \frac{f}{4} (\varphi^{\text{in}}(x))^2 \right) - \frac{1}{2} \mu^2 (\varphi^{\text{in}}(x))^2 \right) \right). \quad (4.13)$$

It is important to recognize that the second term $f/4 \cdot (\varphi^{\text{in}})^2$ on the r.h.s. of Eq. (4.9) gives no contributions to the $S$ matrix, although it gives non-vanishing contributions to the time-ordered Green functions as we have seen in the preceding section. Equation (4.13) instructs us how to modify the Feynman rules for evaluating the $S$ matrix in this case.

We have also tried to find an alternative expression for the formal solution in the form

$$\mathcal{T} = \langle T^* \exp \left( -i \int d^4 x \left( \mathcal{H}[\varphi^{\text{in}}(x)] + G[\varphi^{\text{in}}(x)] J(x) \left( 1 + i f \frac{\delta}{\partial J(x)} \right) \right) \rangle \rangle. \quad (4.14)$$

Substitution of this representation into the functional equation (2.7) leads to the following solution:

$$G[\varphi^{\text{in}}(x)] = \frac{2}{f} \ln \left( 1 + \frac{f}{2} \varphi^{\text{in}}(x) \right),$$

$$\mathcal{H}[\varphi^{\text{in}}(x)] = V \left( \varphi^{\text{in}}(x) + \frac{f}{4} (\varphi^{\text{in}}(x))^2 \right) - \frac{1}{2} \mu^2 (\varphi^{\text{in}}(x))^2. \quad (4.15)$$

From this solution we can infer that the $S$ matrix is given again by (4.13).

We conclude that for a reducible non-linear theory we can always find a formal expression for the $S$ matrix which is a slight modification of the Dyson formula. The situation is completely different, however, with irreducible non-linear theories.\(^9\) We have tried to express the formal solution for $\mathcal{T}$ in the form (4.6) or (4.14) for a simple irreducible non-linear theory, but we reached the conclusion that the functional equation for $\mathcal{T}$ is integrable in neither form. We
suspect, therefore, that we might not be able to find a formal solution in a simple form in the irreducible cases. It is a very challenging problem to find a formal solution in this case, since otherwise we do not have Feynman rules for irreducible non-linear theories in the sense that the method of construction of the $S$ matrix depends critically on the number of external lines.

§ 5. The validity of the Dyson formula in non-linear field theories

In a linear field theory the total Lagrangian is split into two parts:

$$L_{\text{total}} = L_{\text{free}} + L_{\text{int}},$$

where $L_{\text{free}}$ is given, for scalar fields, by

$$L_{\text{free}} = -\frac{1}{2}(\partial_{\mu}\varphi)^2,$$

where the component index $a$ is treated as a dummy index. Then the $S$ matrix is given by the Dyson formula

$$S = T^{*} \exp \left( i \int d^4x L_{\text{int}}[\varphi_a^a(x)] \right).$$

As we have mentioned in the preceding section we do not know of any corresponding formula for irreducible non-linear theories. In handling such a theory, however, the validity of the Dyson formula is often tacitly assumed. So we shall examine what the Dyson formula leads us to in non-linear field theories.

As the first example we shall consider the Lagrangian (3.1) for which we know that $S=1$. Then we have

$$L_{\text{int}} = \frac{1}{2} f \varphi (\partial_{\mu}\varphi)^2.$$  

(5.4)

We have evaluated the lowest $S$-matrix elements for the 4- and 5-leg tree diagrams represented by Figs. 3 and 4, respectively. By making use of the definition of the $T^{*}$ product we can easily verify that all these contributions vanish. Thus the Dyson formula gives the correct $S$-matrix elements in the tree approximation, at least for the lowest few orders in the coupling constant.

![Fig. 3. The 4-leg tree diagrams in the order $f^2$ of the Lagrangian (3.1).](image)

![Fig. 4. The 5-leg tree diagrams in the order $f^3$ of the Lagrangian (3.1).](image)
We then tried to evaluate the contributions of the 4-leg diagrams in the order $f^4$. The result is divergent, and it seems there is no accidental cancellation among the divergences. Similarly, we have evaluated the modified propagator and have found that the final expression deviates from the exact result (3.13). Thus we may conclude that the Dyson formula leads to the correct $S$-matrix elements for tree diagrams, but it leads to incorrect results when closed loops take part in. We have not yet understood why the Dyson formula works well in the tree approximation.

As the second example we have considered a less trivial example represented by Eq. (2.10) with $V=0$.

We have studied the 4- and 6-point Green functions by two methods. First, we have determined them by solving Eq. (2.17). Then we have evaluated them by using the Dyson formula for the generating functional:

$$\mathcal{T}' = \langle T^*(i \int d^4x (L_{\text{int}}(\phi_i^a)(x)) - \phi_i^a(x) J_\phi(x)) \rangle_{\phi}.$$  (5.5)

In both methods the Green functions have been determined to the lowest order in the coupling constant $\lambda$. The result is rather remarkable in that both methods have yielded identical expressions.

For the 4-point Green function corresponding to the diagram in Fig. 5 we find

$$G(1234) = 4i\lambda \delta(a_1 a_2) \delta(a_3 a_4)$$

$$\times \int d^4x \{ \partial_\mu A_\mu(x_1 - y) \partial_\mu A_\mu(x_2 - y) A_\mu(x_3 - y) A_\mu(x_4 - y)$$

$$+ A_\mu(x_1 - y) A_\mu(x_2 - y) \partial_\mu A_\mu(x_3 - y) \partial_\mu A_\mu(x_4 - y) \}$$

$$+ (2\leftrightarrow 3) + (2\leftrightarrow 4),$$  (5.6)

where $\partial_\mu = \partial/\partial y_\mu$. The 6-point function is represented by the diagrams in Fig. 6.

![Fig. 5. The 4-leg tree diagram in the order $\lambda$.](https://example.com/fig5)

![Fig. 6. The 6-leg tree diagrams in the order $\lambda^2$.](https://example.com/fig6)

6 and is given by

$$G(123456) = -16\lambda^3 \int d^4y \int d^2x \delta(a_1 a_2) \delta(a_3 a_4) \delta(a_5 a_6)$$

$$\times [ \partial_\mu A_\mu(x_1 - y) \partial_\mu A_\mu(x_2 - y) A_\mu(x_3 - y)$$

$$+ A_\mu(x_1 - y) A_\mu(x_2 - y) \partial_\mu A_\mu(x_3 - y) \partial_\mu A_\mu(x_4 - y) \}$$

$$+ (2\leftrightarrow 3) + (2\leftrightarrow 4).$$  (5.6)
K. Nishijima and T. Watanabe

\[ x \left\{ \partial_\mu A^\nu (y-z) \partial_\nu A^\mu (x_4-z) A^\nu (x_5-z) \right\} \\
+ A^\nu (y-z) A^\mu (x_4-z) \partial_\mu A^\nu (x_5-z) \partial_\nu A^\mu (x_6-z) \right\} \\
+ A^\nu (x_1-y) A^\mu (x_2-y) \partial_\mu A^\nu (x_3-y) \\
x \left\{ \partial_\mu A^\nu (y-z) \partial_\nu A^\mu (x_4-z) A^\nu (x_5-z) \right\} \\
+ \partial_\mu A^\nu (y-z) A^\mu (x_4-z) \partial_\nu A^\mu (x_5-z) \partial_\nu A^\mu (x_6-z) \right\} \\
\right] \\
+ (3 \leftrightarrow 4) + (3 \leftrightarrow 5, 4 \leftrightarrow 6) + (3 \leftrightarrow 6, 4 \leftrightarrow 5) + (1 \leftrightarrow 3, 2 \leftrightarrow 4) \\
+ (1 \rightarrow 5 \rightarrow 3 \rightarrow 1, 2 \rightarrow 6 \rightarrow 4 \rightarrow 2) \right] \\
- 24 i^2 \int d^2 y \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \frac{\partial}{\partial x_4} \right\} \\
\times A^\nu (x_1-y) A^\mu (x_4-z) A^\nu (x_5-y) A^\mu (x_6-y) \\
+ \text{all possible ways of dividing 6 variables into 3 pairs,} \quad (5.7)

where \( \partial_\mu = \partial/\partial y_\mu \) and \( \partial_\nu = \partial/\partial z_\nu \). Contrary to the previous example, the contributions from these tree diagrams do not vanish because of the presence of the isotopic spin factors.

We may conclude that the Dyson formula gives the correct \( S \) matrix in the tree approximation, \emph{at least in the lowest few orders}, even in irreducible non-linear theories. This justifies the calculations hitherto carried out in chiral dynamics.

It still remains to be clarified why the Dyson formula works in the tree approximation even in non-linear theories. It is also a challenging problem to find a modified Dyson formula for an irreducible non-linear theories.

References
2) K. Nishijima, \textit{Fields and Particles} (W. A. Benjamin, New York, 1967), p. 269. Equation (2.6) is a generalization of the formula in this reference.
3) S. Weinberg, Phys. Rev. 166 (1968), 1568.
4) S. Kamefuchi, L. O’Raifeartaigh and A. Salam, Nucl. Phys. 28 (1961), 529.
13) In most papers on chiral dynamics the Dyson formula is used to write down the \( S \) matrix. See, for instance, S. Gasiorowicz and D. A. Geffen, Rev. Mod. Phys. 41 (1969), 531.