This study deals with a construction of the canonical formalism for fields associated with the unit-disk medium which is a continuous version of a planar Feynman graph. The variable which plays the role of the "time" is defined with the help of a conjugate field which is holomorphic in the domain outside the unit disk. Commutation relations for Taylor coefficients are obtained by means of canonical quantization. Equations of "motion", which are compatible with those derived by group-theoretical consideration, are consistently presented. Finally it is suggested that a quark line should always make a loop on a Riemann sphere if the new field holomorphic outside the unit disk is an analytic continuation of the original field which is holomorphic inside the unit disk.

§ 1. Introduction

The present study is an alternative, or a direct continuation, of the preceding work\(^3\) (to be referred to in what follows as [III]).\(^4\) In [II] we succeeded to some extent in associating a quantized field \(Z(\sigma, \tau)\) (and its complex conjugate) with the unit-disk medium on the basis of foregoing work\(^3\) (referred to as [II]). The set of configuration parameters \((\sigma, \tau)\) is a continuous version of the suffix attached to the four-position appearing in the Feynman propagators in position space. Accordingly, \(\sigma\) and \(\tau\) have nothing to do with "time", which is indispensable to a construction of the canonical formalism. We have thus been forced to employ in [II] the boundary conditions which played the role in [II] of quantum conditions on \(Z(\rho)\).

On the other hand, the formulation in terms of "proper-time" by Miyamoto,\(^3\) et al., is also promising. It is, therefore, still tempting to convert our framework to one compatible with such a canonical formalism.

In the previous work [II], we have taken into account only the \(Z(\rho)\) (and its complex conjugate) associated with \(U\). This is natural since, from results in [II], the medium \(\mathcal{G}\) (defined in [II]) should be distinct from the region outside \(\mathcal{G}\).

On the contrary, it will be shown in this paper that in order to formulate the canonical formalism we must introduce another operator-valued function \(Z^+(\rho)\) which is holomorphic outside \(\bar{U}\). Then one can consistently impose the canonical

\(^3\) When we cite equations in [II], the numbering system is by triples, e.g. as Eq. (II-2-2). We make constant use of notations in [II].
commutation relation regarding $-\log r$ ($r$ is the one used in [Π]) as the "time" and obtain equations of motion which agree with the ones obtainable by an argument based on homographic groups.

The operator $Z(\rho)$ previously considered is suggested to be the "time"-reflected quantity of $Z^+(\rho)$; and, furthermore, $Z(\rho)$ is suggested to be analytically continued to $Z^+(\rho)$ through the "world sheet" between two quark-lines. In fact, we see it natural for some medium to fill also outside $\mathcal{Q}$.

In this paper, we only consider the "boson" case, and as in [Π] we treat the one-component "free" $Z(\rho)$ (and $Z^+(\rho)$). No account for the interaction scheme is given.

§ 2. Canonical quantization and equations of "motion"

In [Π], we have made extensive use of Taylor-series development

$$Z(\rho) = \sum_{\nu=0}^{\infty} c_{\nu} \rho^{\nu} \quad \text{in } U$$

in the light of a unit-disk model of the conducting medium assumed in [Π]. In this paper, however, it will be convenient to map the unit-disk $U$ onto the domain $T^+(2\pi)$ in the $u(=s+it)$ plane by the conformal mapping

$$\rho = \exp iu .$$

Here $T^+(2\pi)$ is defined by

$$T^+(2\pi) = \{ s, t|0 \leq s < 2\pi, t > 0 \} .$$

We write hereafter $Z(u)$ (or $Z(s, t)$) for $Z(e^{iu})$ by abuse of notation. To be strict, $U$ is equivalent to $T^+(2\pi)$ only when the boundary of the latter is properly identified, that is, to the set of equivalence classes of the upper half-plane $T^+$ under the group generated by the translation $T: u \rightarrow u + 2\pi$. In this respect, $Z(u)$ is a modular form subject to

$$Z(T^nu) = Z(u) . \quad (n: \text{integer})$$

We sometimes prefer to represent $Z(u)$ as

$$Z(u) = \int_{0}^{\pi} dyc(\nu) e^{iu}$$

under the condition that $Z(u)$ obeys (2·4). The representation (2·5) is purely pro forma. If necessary the rhs should be replaced by the Fourier series. That form, however, is akin to the usual forms in conventional field theory, and it is rather clear in which domain $Z(u)$ is holomorphic.

Henceforward, we treat the variable $0 \leq s < 2\pi$ as the "space"-component and
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t as the "time"-component. Relations of this set of variables to the previous ones are given by

\[
\begin{align*}
\begin{cases}
  s = \varphi , \\
  t = -\log r .
\end{cases}
\end{align*}
\]  (2.6)

We remark that we constantly use the Euclid metric \((s^2 + t^2 = |u|^2)\).

Let us here introduce \(c^+(v)\), an adjoint of \(c(v)\), whose support is also contained in \(v \geq 0\); and construct, by means of this, another operator-valued field

\[
Z^+(u) = \int_0^\infty d\nu c^+(\nu)e^{-ivu},
\]  (2.5)

which is assumed to be holomorphic in \(T^- (2\pi)\).

If we write

\[
\sqrt{2}^{-1}Z_1(s,t) / \sqrt{2}^{-1}Z_1(s,t)
\]

and

\[
\sqrt{2}^{-1}Z_1^+(s,t) / \sqrt{2}^{-1}Z_1^+(s,t)
\]

for the real/imaginary parts (hermitians) of \(Z(u)\) and \(Z^+(u)\), respectively, then they evidently satisfy

\[
\left( \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2} \right)Z_j(s,t) = 0 , \quad j = 1, 2
\]  (2.7)

in \(T^+ (2\pi)\) and

\[
\left( \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2} \right)Z_j^+(s,t) = 0 , \quad j = 1, 2
\]  (2.7+)

in \(T^- (2\pi)\). (The Laplace equation is invariant under any conformal mapping if the modulus of the mapping is finite.)

It is obvious that the following Lagrangian density is available to deliver (2.7) and (2.7+):

\[
\mathcal{L} = - \frac{1}{4\pi} \left[ \frac{\partial Z^+}{\partial s} \cdot \frac{\partial Z}{\partial t} + \frac{\partial Z^+}{\partial t} \cdot \frac{\partial Z}{\partial s} \right]
\]  (2.8)*

By this, one can define the conjugate momentum \(P(u)\) (resp. \(P^+(u)\)) of \(Z(u)\) (resp. \(Z^+(u)\)) as follows:

---

* When we consider \(Z^+\) and \(Z\) on the same footing as in the rhs of (2.8) (also in (2.11) below), we implicitly regard \(Z^+\) (or \(Z\)) as having an analytic extension to \(T^+ (2\pi)\) (or \(T^- (2\pi)\)). For example,

\[
(Z^+(u)) = \int_0^\infty du e^{-iu\varphi} = \frac{1}{iu} \quad \text{(Im } u < 0) \]

appears to have an analytic continuation in \(T^+\) (due to the rotational invariance of the integration path). Refer also to § 3 below.
and

\[ p^+(u) = -i \frac{\partial L}{\partial (\overrightarrow{Z}^+/\partial t)} = i \frac{\partial \overrightarrow{Z}}{\partial t} . \]  

(2.9)

We are now in a position to invoke canonical commutation relations. However, it should be noticed in advance that neither \( \overrightarrow{Z}(u) \) nor \( \overrightarrow{Z}^+(u) \) is the canonical variable. Thus we must bring them closer to conventional quantized field theory by replacing representations (2·5) and (2·5)* by

\[ \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} d\mu \frac{\mu}{2} e^{-\mu s^2 - \mu t} \delta (\nu^2 - \mu^2) \theta \mu \]  

(2·10)

and

\[ \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} d\mu e^{i \nu s + \mu t} \delta (\nu^2 - \mu^2) \theta \mu \]  

(2·10)*

respectively. That is, we have altered the situation by introducing the light-like vector \((\nu, \mu)\) in place of \((\nu, \nu)\). It is then appropriate to specify the canonical position variable as a sum of (2·10) and (2·10)*, which we tentatively denote by \( \overrightarrow{Z}_c(s, t) \). It is then easy to see that

\[ \overrightarrow{Z}_c(s, t) = \overrightarrow{Z}(s, t) + \overrightarrow{Z}(-s, t) + \overrightarrow{Z}^+(s, t) + \overrightarrow{Z}^+(-s, t) . \]  

(2·11)

Hence, the canonical momenta are given by

\[ \overrightarrow{P}_c(s, t) = \overrightarrow{P}(s, t) + \overrightarrow{P}(-s, t) + \overrightarrow{P}^+(s, t) + \overrightarrow{P}^+(-s, t) . \]  

(2·12)

The canonical commutation relations then read

\[ [\overrightarrow{Z}_c(s, t), \overrightarrow{P}_c(s', t)] = i\delta (s-s'), \]  

(2·13)

\[ [\overrightarrow{Z}_c(s, t), \overrightarrow{Z}_c(s', t)] = 0 , \]  

(2·14)

\[ [\overrightarrow{P}_c(s, t), \overrightarrow{P}_c(s', t)] = 0 . \]  

(2·15)

First, it follows from (2·14) (or (2·15)) that

\[ 0 = \int dv dv' \cos \nu s \cos \nu' s' \{ [c(\nu), c(\nu')] \} e^{-(\nu + \nu')s} + [c^+(\nu), c(\nu')] e^{(\nu + \nu')s} \]

\[ + [c(\nu), c^+(\nu')] e^{-(\nu + \nu')s} + [c^+(\nu), c^+(\nu')] e^{(\nu + \nu')s} \} . \]  

(2·16)

Since \( \nu \) and \( \nu' \) are positive and (2·16) should hold at all the "time" \( t \), the following relations are readily derived

\[ [c(\nu), c(\nu')] = [c^+(\nu), c^+(\nu')] = 0 , \]

(2·17)

\[ [c(\nu), c^+(\nu')] = \nu \delta (\nu - \nu') . \]  

(2·18)
The quantity $\eta(v)$ is undetermined until (2·13) is used. Equation (2·13) can be rewritten with the use of (2·17) and (2·18) as follows:

$$\begin{align*}
\delta(s-s') &= \frac{1}{\pi} \int_0^\infty dv' dv' \cos v's' \cos vs \\
&\quad \times \{[c(v), c^+(v')] e^{-(s-s')v} - [c^+(v), c(v')] e^{(s-s')v}\} \\
&= \frac{2}{\pi} \int_0^\infty dv \eta(v) \cos vs \cos vs',
\end{align*}$$

whence we see that $\eta(v) = -v^{-1}$ (note that $s, s' > 0$).

Thus we are led to the following commutation relations between $c(v)$ and $c^+(v)$ on the basis of the Lagrangian formalism:

$$\begin{align*}
[c(v), c(v')] &= [c^+(v), c^+(v')] = 0, \\
[c(v), c^+(v')] &= \frac{1}{v} \delta(v-v'). (v \neq 0)
\end{align*}$$

Along this line of thought, we can further proceed to obtain equations of "motion" with the help of the canonical energy-momentum tensor. The Hamiltonian, which we expect to be the generator of displacement in the $t$ sense, is appropriately defined by

$$H = \int_0^{2\pi} ds \mathcal{H}(s, t),$$

where

$$\mathcal{H}(s, t) = iP(s, t) \frac{\partial Z(s, t)}{\partial t} + iP^+(s, t) \frac{\partial Z^+(s, t)}{\partial t} - \mathcal{L}(s, t)$$

$$= \frac{1}{4\pi} \left[ \frac{\partial Z^+}{\partial s} \frac{\partial Z}{\partial s} - \frac{\partial Z}{\partial s} \frac{\partial Z^+}{\partial s} \right].$$

One point to be remarked is that, because we define (2·21) by angular integration relevant to the unit-disk, we have to employ Fourier-series expansion for $Z(u)$, etc., in lieu of (2·5). It follows then that

$$H = \frac{1}{2} \sum_{\nu=0}^\infty \nu^2 \{c_\nu^+ c_\nu + c_\nu c_\nu^+\}. \tag{2·23}$$

On applying (2·20), we have

$$H = \sum_{\nu=0}^\infty \nu N_\nu \tag{2·24}\)
up to an additive (infinite) constant. In (2.24) we have set
\[ N_\nu = \nu c_\nu^+ c_\nu. \]

(See however 2° in § 3 below.)

Let us next construct \([H, Z(u)]\). By virtue of the formula
\[ [c_\nu^+ c_\nu, c_\nu] = -\nu^{-1} \partial_{\nu\nu} c_\nu, \]
we can readily verify the equation of "motion":
\[ [H, Z(u)] = \frac{\delta Z(u)}{\delta t}. \]  

We can likewise proceed to the generator \(P\) of \(s\)-displacement. It may be defined by
\[ P = \int_0^{2\pi} ds \mathcal{P}(s, t), \]
where
\[ \mathcal{P}(s, t) = -p \frac{\partial Z}{\partial s} - p^+ \frac{\partial Z^+}{\partial s} + \frac{1}{4\pi i} \left[ \frac{\partial Z^+}{\partial t} \frac{\partial Z}{\partial s} + \frac{\partial Z}{\partial t} \frac{\partial Z^+}{\partial s} \right]. \]

In this case we obtain
\[ [P, Z(u)] = i \frac{\partial Z(u)}{\partial s}, \]
where
\[ P = \frac{1}{2} \sum_{\nu=0}^m \nu^2 \{c_\nu^+ c_\nu + c_\nu c_\nu^+\}. \]

It is to be recognized that
\[ P = H. \]

By further analogy to conventional field theory, we can define some other quantities. For example, the charge density may be given by
\[ \mathcal{J}(s, t) = \frac{1}{4\pi} \left[ \frac{\partial Z^+}{\partial t} Z - Z^+ \frac{\partial Z}{\partial t} \right], \]
so that the charge is proportional to
\[ \int_0^{2\pi} ds \mathcal{J}(s, t) = \sum_{\nu=1}^m N_\nu. \]

So far, we have dealt with the one-component operators \(Z(u)\) and \(Z^+(u)\). If, however, we extend them properly to become four-component operators, we may be able to establish such quantities as the spin-angular momentum tensor \( \mathcal{J}^{\mu\nu} \) to obtain the spin vector, and so on. We will present the result of such an attempt in the Appendix. (The spatial vector
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\[ M^{(k)} = \int_0^{2\pi} ds M^{(k)}(s, t) \]

presented in Appendix (9°) is the one in the framework of one-component fields; it may compete with the set of generators associated with a group of homographic transformations.

§ 3. Discussion and outlook

1° Most noteworthy is the role played by \( Z^+(\rho) \). Its presence has permitted us to perform the canonical formalism. In terms of \( \rho \), \( Z^+ \) is of the form

\[ Z^+(\rho) = \sum_{n=0}^{\infty} c_n \frac{1}{\rho^n}, \]

and \( T^-/T^n \) is mapped onto the domain \(|\rho|>1\), which we denote by \( U^+ \). The operator \( Z^+(\rho) \) is thus holomorphic in \( U^+ \); as also \( Z_{1+}^+(\rho, \bar{\rho}) \) and \( Z_{2+}^+(\rho, \bar{\rho}) \) are harmonic in \( U^+ \). The domain \( U^+ \) is homeomorphic to a disk around infinity, so that \( U \) and \( U^+ \) constitute, as it were, two 'distinct domains of the "day" and the "night" of a sphere [\( \partial U(=\partial U^+) \) constitutes a "twilight" line].

We have so far held the view that the medium \( \mathcal{G} \) is associated with \( U \), and the region outside \( U \) is empty. However, it now appears that another medium \( \mathcal{G}^+ \) may be symmetrically associated with \( U^+ \) where \( Z^+(\rho) \) plays the same role as \( Z(\rho) \) in \( U \), and further that there should exist a potential \( z^+(\rho) \) also in \( \mathcal{G}^+ \).

In this view, the external currents \( p_k \) which come in \( \mathcal{G} \) might be regarded as internal ones which go out of \( \mathcal{G}^+ \). Let there be several "craters" on the "twilight" line. Then light-beams will come discretely in \( \mathcal{G} \).

2° If we think of each mode-state of \( Z(\rho) \) as representing a state of quantum "moving" forwards in time with positive frequency, then the quantum associated with the mode-state of \( Z^+(\rho) \) can be considered to move \textit{backwards} in time. Consequently \( Z^+(\rho) \) is an image (in a mirror) of \( Z(\rho) \). It will be appropriate to define the number operator \( N_\nu \) by \( N_\nu = \nu c_\nu c_\nu^+ \) in \( \mathcal{G}^+ \) and also \( H \) by \( \sum_{\nu=0}^{\infty} \hat{\nu}^2 c_\nu c_\nu^+ \).

3° The operator \( \overline{Z(\rho)} \) used extensively in [III] is identical at the limit \( t \to 0^+ \) with \( \lim_{t \to 0^-} Z^+(\rho) \), if we identify \( \bar{\nu} \) with \( c_\nu^+ \). In effect, we can regard \( \overline{Z(\rho)} \) as the quantity reflected in time of \( Z^+(\rho) \), that is

\[ Z^+(s, t) = \overline{Z(s, -t)}. \]

4° As noted in (2·6), we have the relation

\[ -\log r = t, \quad \varphi = s. \]
Thus \(-\log r\) is the "time"-component in our formulation. When we quantized \(Z(\rho)\) in \(\Pi\) by means of the boundary condition, the conjugate of \(Z_f(\rho, \bar{\rho})\) necessarily took the form

\[
 r \cdot \frac{\partial}{\partial r} Z_f(\rho, \bar{\rho}). \tag{3.3}
\]

(See (II·2·9) and (II·2·15).) By virtue of (3·2), the quantity (3·3) is identical with

\[
-\frac{\delta}{\delta t} Z_f(\rho, \bar{\rho}),
\]

so that (3·3) is said to be such a conjugate momentum as defined in (2·9) (cf. 10° in \(\Pi\)).

As noted in 16° in \(\Pi\), \(\varphi (= s)\) appears also to be the "time" because the generator \(P\) is identical with \(H\) (see (2·31)).

5° The complex potential obtained in \(\Pi\) can be written with the use of \(Z^+(\rho)\) as follows:

\[
z(\rho) = \frac{\alpha i}{\pi} \sum_{k=1}^{n} p_k [Z(\rho), Z^+(\rho_k)], \quad |\rho| < 1 \tag{3·4}
\]

up to the additive "free" term. In (3·4) it is, of course, assumed that the following limit uniformly exists

\[
\lim_{\rho \to \rho_k} Z^+(\rho) = Z^+(\rho_k). \tag{3·5}
\]

Incidentally, the propagator

\[
[Z(\rho), Z^+(\rho')] = -\log \left(1 - \frac{\rho}{\rho'}\right), \quad |\rho| < 1, \quad |\rho'| > 1 \tag{3·6}
\]

is obviously invariant under the dilations

\[
\rho \to m \rho, \quad (m: \text{complex})
\]

\[
\rho' \to m \rho'. \tag{3·7}
\]

(Cf. the discussion in 8° in \(\Pi\).) If use is made of \(u = s + it\), the transformations (3·7) imply translations both in \(s\) and \(t\). It is evident that the generators which give rise to such translations are the \(P\) and \(H\) previously obtained.

Let us now construct the complex potential \(z^+(\rho)\) in \(U^+\) by symmetrical analogy with \(z(\rho)\):

\[
z^+(\rho) = -\frac{\alpha i}{\pi} \sum_{k=1}^{n} p_k [Z^+(\rho), Z(\rho_k)], \quad |\rho| > 1, \tag{3·8}
\]

where \(Z(\rho_k)\) is the distribution assumed by
As speculated in 1°, it seems natural to suppose the potential $z^+(\rho)$ as the one associated with the external medium $\mathcal{G}^+$. When we draw a Feynman graph, we are not concerned with the place where external momenta come in or go out. Incoming external lines should link at infinity with outgoing lines since momenta are conserved. The situation, then, would be the same if we imagine the network at infinity as an image of the Feynman graph. If we translate the network to a continuous medium as we did in [II], then the above potential $z^+(\rho)$ would be produced by “outgoing” external momenta.

6° In light of (3·6), let us conjecture that the following commutation relation holds

$$[Z(\rho_1), Z^+(\rho_2)] = [Z(\rho_1), Z^+(\rho_2)] \text{ for } \rho_1, \rho_2 \in \partial \mathcal{U} \tag{3·10}$$

modulo a sum of some “periods” (consequences of the presence of “craters”). On the other hand, we have

$$[Z(\rho_1), Z(\rho_2)] = [Z(\rho_1), Z(\rho_2)] \text{ for } \rho_1, \rho_2 \in \partial \mathcal{U}. \tag{3·11}$$

Hence, we can consistently conjecture that

$$Z^+(\rho) = Z(\rho) \text{ for } \rho_k \in \partial \mathcal{U} \tag{3·12}$$

in the sense of distribution. It is tempting then to suggest that (3·12) can be further converted to a form such that the relation

$$\lim_{t \to 0^-} z^+(s, t) = \lim_{t \to 0^+} Z(s, t) \tag{3·13}$$

is satisfied continuously for a finite neighbourhood of $s_k = \varphi_k$, e.g. for

$$s_k - \frac{l_0}{2} < s < s_k + \frac{l_0}{2}. \tag{3·14}$$

(See Fig. 3.) If so, we can readily apply Painlevé's edge-of-the-wedge theorem to obtain $Z^+(\rho)$ as the analytic continuation of $Z(\rho)$. It should, of course, be borne in mind that this holds only locally since the master function is no longer single-valued. Such a conjecture cannot, however, be applied to $z(\rho)$ or $z^+(\rho)$, at least at $\rho = \rho_k$, because $z(\rho)$ and $z^+(\rho)$ are singular at the source points.

It should be noted that the above plot trivially resembles the one in which the positive-frequency Wightman function is analytically continued to a negative-frequency Wightman function by local commutativity. Recall that the holomorphy envelope of the forward tube, plus the backward tube, plus a complex neighbour-
hood of the space-like region, is identical with the «extended tube» which is obtained by sole use of complex Lorentz transformations \( L_n(C) \). In view of this, one might anticipate the same result if we start with full homographic transformations. Concerning this, it should be pointed out that in the work by Fubini and Veneziano, the expression

\[
b^{(-)}(x) = \sum_{n=0}^{\infty} b_n^{(-)} x^{-n}
\]

shows up (see (3.11) in Ref. 5)). Refer also to Eq. (102) in Susskind's paper. (In our case, however, we have no Laurent series since we are concerned with a no-loop model.)

7° To sum up, the discussions in 1° and 6° seem to suggest a topological viewpoint. Let us consider a punctured Riemann sphere, on which \( Z(\rho) \) is specified as a free potential (necessarily, multiple-valued). Let there be a number of “craters” or “holes” in such a way that each boundary of holes consists of a quark-line having an orientation (necessarily, a circuit). If we draw a “twilight” line across all the craters, one side of the sphere belongs to the “day” and the other to the “night”. Suppose that we are at midday. Then the “twilight” line stands far away, and another half of each quark-loop is below the “horizon” so that external mesons appear to come from the horizon in the shape of the so-called “world-sheets” (of Susskind⁶), each sandwiched by two quark-lines. (It is probable that the lower limit of the width of a world sheet is around \( l_\alpha \) used in (3.14)). This is the topological view for an \( n \)-point graph without

---

**Fig. 4.** The graphs in the lower are the consequences of “hooks” represented by the dotted segments; the “twilight line” also plays the role of a hook.
the "loop". (See Fig. 4.)

If we keep the "twilight" line away from some craters, or if we suppose some craters inside the meridian region, then the situation becomes the same as the one for multiple-loop graphs. In this case also, we must imagine similar craters in the night region by an analogy with «electrical images».

Suppose finally that two quark-loops with distinct orientations are identical. Then they should disappear so that the punctured Riemann sphere becomes a punctured torus (or a double torus since we should take into account the image). If both quark-loops are of the same orientation, then we would obtain a punctured Klein's bottle (or its double).

Topology associated with $z(\rho)$ or $z^+(\rho)$ should be somewhat different from the one for $Z(\rho)$ or $Z^+(\rho)$, and this problem must be connected with the interaction scheme whose riddle lies in the twilight zone.

Appendix

We here try to define the spin-angular momentum operator by extending $Z(\rho)$ to a four-component field. Let us denote the spatial-components of $Z(\rho)$ and $Z^+(\rho)$ by $Z^{(k)}(\rho)$ and $Z^{+(k)}(\rho)$ $(k=1,2,3)$, respectively. We start by assuming the commutation relations

$$
[c^{(k)}_\nu, c^{(l)}_\mu] = [c^{+(k)}_\nu, c^{+(l)}_\mu] = 0 ,
$$

$$
[c^{(k)}_\nu, c^{+(l)}_\mu] = \frac{1}{v} \delta_{\nu\mu} \delta_{kl} ,
$$

where $c^{(k)}_\nu$ (or $c^{+(k)}_\nu$) stands for the expansion coefficient of $Z^{(k)}(\rho)$ (or $Z^{+(k)}(\rho)$).

We define the spatial spin-tensor by

$$
S^{kl} = \int_0^{2\pi} ds J^{kl}(s,t) ,
$$

where

$$
J^{kl} = \frac{i}{2\pi} \left( Z^{+(k)} \frac{\partial Z^{(l)}}{\partial t} + \frac{\partial Z^{+(l)}}{\partial t} Z^{(k)} \right) .
$$

Then we readily obtain

$$
S^{kl} = -i \sum_{m=8}^{10} \left( c^{+(k)}_\nu c^{(m)}_\mu - c^{+(m)}_\nu c^{(k)}_\mu \right) .
$$

It is easy to prove that

$$
[S^{(k)}, S^{(l)}] = i\delta^{kl} S^{(m)} ,
$$

where $S^{(k)}$ denotes a component of the spin-vector defined by
We have, for example,

\[ [S^{(b)}, Z^{(a)}] = iZ^{(c)}, \]

\[ [S^{(b)}, Z^{(b)}] = -iZ^{(b)}, \]

while

\[ [S^{(b)}, Z^{(b)}] = 0. \]

9° We show that it is possible in the framework of the one-component \( Z(p) \) to construct a three-vector

\[ M^{(b)} = \int_0^{2\pi} ds M^{(b)}(s, t), \]  

(A6)

which satisfies the same algebra as (A5).

Let us define the spatial densities by

\[ \mathcal{M}^{(1)} = \frac{1}{2\pi} \left( \cos \theta \sin \theta, 1 \right) \frac{\partial Z^+}{\partial t} \cdot \frac{\partial Z}{\partial t}, \]  

(A7)

where \( \theta \) is the one used in the text. It is, however, convenient to work with

\[ \mathcal{M}^\pm = M^{(1)} \pm iM^{(2)}, \]  

(A8)

and so with

\[ M^\pm = M^{(1)} \pm iM^{(2)}. \]  

(A9)

We then readily obtain

\[ M^+ = -\sum_{\nu=0}^{\infty} \nu(\nu + 1) c_{\nu+1}^+ c_{\nu}, \]  

(A10)

\[ M^- = -\sum_{\nu=0}^{\infty} \nu(\nu + 1) c_{\nu}^+ c_{\nu+1}, \]  

(A11)

\[ M^{(b)} = -\sum_{\nu=0}^{\infty} \nu^2 c_{\nu}^+ c_{\nu}, \]  

(A12)

where \( c_{\nu}^+ \) and \( c_{\nu} \) are assumed to satisfy (2.20). It is easy to prove that

\[ [M^+, M^-] = 2M^{(b)}, \]  

(A13)

\[ [M^{(b)}, M^\pm] = \pm M^+, \]  

(A14)

so that

\[ [M^{(b)}, M^{(b)}] = i\varepsilon_{2\nu \mu} M^{(\mu)}. \]  

(A15)

It is apparent from (A10) that we have

\[ [M^{(b)}, Z] = -\frac{\partial Z}{\partial t}, \]  

(A16)
and

\[ [M_\pm, Z] = \pm e^{\pm u} \frac{\partial Z}{\partial t}. \]  \hspace{1cm} (A17)

We believe that Eqs. (A10) - (A17) compete with some of those in Ref. 5). See also Ref. 7). However, if we take into account also the "night" face, Eq. (A7) should be replaced by

\[ \left( \mathcal{H}^{(1)}, \mathcal{H}^{(3)}, \mathcal{H}^{(3)} \right) = \frac{1}{4\pi} (\cos u, \sin u, 1) \begin{bmatrix} \frac{\partial Z}{\partial t} & \frac{\partial Z^+}{\partial t} & \frac{\partial Z}{\partial s} \\ \frac{\partial Z^+}{\partial t} & -\frac{\partial Z}{\partial t} & \frac{\partial Z^+}{\partial s} \end{bmatrix}. \]  \hspace{1cm} (A18)

References