Journal of Applied Mechanics

Discussion

The Vibrating Beam With Nonhomogeneous Boundary Conditions[1]

H. D. Fisher[2]. The author is to be congratulated for presenting a new solution method (subsequently designated the E method) for a class of time-dependent boundary value problems. The purpose of this discussion is to explore the generality of the E method and to present a detailed comparison of it with the widely used Mindlin-Goodman method (denoted here the M-G method) which is the author's reference [1]. The discusser was motivated to investigate the present problem by related research [1-3] in second-order systems.

The range of applicability of the E method is readily ascertained by expressing (1.2 – here the first number refers to the author's paper and the second to the numbered equation in that paper) and the last of the boundary conditions as

$$y(x,t) = Y(x,t) + F(x)\,T(t) + G(x)\,t\dot{T}(t) \tag{1}$$

and

$$y_{xx}(\pi,t) = \beta R(t) \tag{2}$$

where β is an arbitrary constant, $T(t)$ and $R(t)$ are time-dependent functions, and $\cdot$ denotes the derivative with respect to time. Taking the required derivatives of (1) and substituting into (1.1), gives (1.3) and the previously derived coupled system

$$F''''(X) - F(X) = 2G(X) \tag{3}$$

$$G''''(X) - G(X) = 0 \tag{4}$$

if, and only if,

$$\ddot{T}(t) + \alpha^2 T(t) = 0 \tag{5}$$

Thus from (5)

$$T(t) = A_1 \sin\alpha t + A_2 \cos\alpha t \tag{6}$$

where A_1 and A_2 are arbitrary constants. Substituting into the initial boundary conditions gives the earlier equations constraining Y, F, and G with the exception that the final restriction on F is replaced by

$$F''(\pi) = \beta \tag{7}$$

This substitution also discloses that

$$R(t) = T(t) \tag{8}$$

Equation (8) demonstrates that the E method, which requires that (3) and (4) be satisfied, is applicable exclusively to beam responses produced by sinusoid excitation.

Following the author's procedure, the solution to the problem described in the foregoing is given by (1) with

$$Y(x,t) = \sum_{n=1}^{\infty} (A_{4n} \sin\alpha n^2 t + A_{5n} \cos\alpha n^2 t) \sin nx \tag{9}$$

$$F(x) = A_3 \sin x + \frac{\beta}{2}\left(\frac{\sinh x}{\sinh \pi} + \frac{x}{\pi}\cos x\right) \tag{10}$$

$$G(x) = \frac{\beta}{\pi} \sin x \tag{11}$$

Here A_3 is an arbitrary constant and

$$A_{4n} = \left(\frac{2}{\pi \alpha n^2}\right) \int_0^\pi Y_t(x,0) \sin nx\,dx \tag{12}$$

$$A_{5n} = \left(\frac{2}{\pi}\right) \int_0^\pi Y(x,o) \sin nx\,dx \tag{13}$$

with

$$Y(x,0) = -A_2 F(x) \tag{14}$$

$$Y_t(x,0) = -\alpha A_1 [F(x) + G(x)] \tag{15}$$

The author's solution for the case when $\beta = 4A\pi$ and $R(t) = \sin\alpha t$ is readily derived from (1) comparing (10) and the first of (1.5) yields

$$A_3 = -5A \tag{16}$$

and from (8) and (6)

$$A_1 = 1,\ A_2 = 0 \tag{17}$$

Employing (16) and (17) in (12)-(15) simplifies (1) to (1.6) as required.

To solve (1.1) subject to the boundary conditions given in the author's paper, the M-G method requires that

$$y(x,t) = Y(x,t) + H(x)\,T(t) \tag{18}$$

with

$$H(x) = \frac{2A}{3}(x^3 - \pi^2 x) \tag{19}$$

and

$$T(t) = \sin\alpha t \tag{20}$$

Here $Y(x,t)$ satisfies the nonhomogeneous PDE

$$\alpha^2 Y_{xxxx}(x,t) + Y_{tt}(x,t) = \frac{2\alpha^2 A}{3}(x^3 - \pi^2 x)\sin\alpha t \tag{21}$$

together with the homogeneous boundary conditions (1.4) and the nonhomogeneous initial conditions

$$Y(x,0) = 0 \tag{22}$$

$$Y_t(x,0) = \frac{2\alpha A}{3}(\pi^2 x - x^3) \tag{23}$$

[1] Edstrom, C.R., and published in the September, 1981 issue of the ASME Journal of Applied Mechanics, Vol. 48, pp. 669-670.

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Inserting (19), (20), and the solution of (21)-(23) into (18) yields

$$y(x,t)=8A\left[\left(\frac{x^3-\pi^2x+6\sin x}{12}\right)\sin\alpha t + \frac{\alpha t\cos\alpha t\sin x}{2}+\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^3(n^4-1)}(n^2\sin\alpha n^2 t - \sin\alpha t)\sin nx\right] \quad (24)$$

The equality of (1.6) and (24) follows since it can be shown that

$$\frac{2(x^3-\pi^2x)}{3}-2x\cos x+9\sin x-\frac{2\pi\sinh x}{\sinh\pi} = 8\sum_{n=2}^{\infty}\frac{(-1)^{n+1}\sin nx}{n^3(n^4-1)} \quad (25)$$

Because the *M-G* method has wider applicability than the *E* method, it remains the standard method of solution for beams subjected to time-dependent boundary excitation. However, for problems involving sinusoidal excitation only, the *E* method provides an alternate solution.

References

1 Fisher, H. D., Cepkaukas, M. M., and Chandra, S., "Solution of Time-Dependent Boundary Value Problems by the Boundary Operator Method," *International Journal of Solids and Structures*, Vol. 15, 1979, pp. 607–614.

2 Fisher, H. D., "Solution of a Generalized One-Dimensional Wave Equation by the Boundary Operator Method," *Journal of Sound and Vibration*, Vol. 79, No. 2, 1981, pp. 316–318.

3 Fisher, H. D., "A Generalized Wave Equation in Finite Domains," *Journal of the Engineering Mechanics Division, American Society of Civil Engineers*, Vol. 108, No. 1, 1982, pp. 155–163.

Author's Closure

H. D. Fisher's statement "that the *E* method is applicable exclusively to beam responses produced by sinusoid excitation" is not correct. The *E* method can be applied to boundary value problems with other boundary conditions. In the Brief Note it was not the author's intent to describe the most general case, but instead to show one illustration of the method. For more generality we have the following.

The existence of a form for the change of dependent variable depends on the properties of the functions that are prescribed on the boundaries. If these functions possess a finite number of linearly independent derivatives, then the form for the change of dependent variable should contain a linear combination of these functions and all of their linearly independent derivatives. However, in place of constants in this linear combination there should be functions of the variable held fixed on the boundary. Also, if the partial differential equation is separable as well as homogeneous, the particular product solutions can be determined. A comparison of these product solutions with the linear combination of the boundary functions and all of their derivatives will determine the need for either a bounded or unbounded form for the change of dependent variable. Thus, since the form for the change of dependent variable depends on the prescribed functions on the boundaries as well as the partial differential equation, the range of applicability of this method cannot be ascertained by generalizing the form of the change of dependent variable I used in the illustration of this method.

On the Formulation of Strain-Space Plasticity With Multiple Loading Surfaces[1]

J. Casey[2] and P. M. Naghdi[3]. We take exception to a number of points made in the paper of Yoder and Iwan [1], and especially to their claim that the stress-space and strain-space formulations of plasticity are equivalent. Although in [1] both single and multiple loading surfaces are employed, it suffices for the purpose of this discussion to consider only the case of single loading surfaces.

The possibility of using a strain-space (rather than a stress-space) formulation of plasticity has been mentioned by several authors in the past. However, the physical significance of the use of the strain-space formulation was first brought out in the paper of Naghdi and Trapp [2]. To elaborate, it was observed by Naghdi and Trapp [2] that the stress-space formulation of plasticity leads to unreliable results in any region such as that corresponding to the maximum point of engineering stress versus engineering strain curve for uniaxial tension of a typical ductile metal. After also observing that the stress-space formulation does not reduce directly to the theory of elastic-perfectly plastic materials, and that a separate formulation for the latter is required, Naghdi and Trapp [2] proposed an alternative strain-space formulation of plasticity which:

(*a*) is valid for the full range of elastic-plastic deformation; and

(*b*) includes as a special case, the theory of elastic-perfectly plastic materials.

The strain-space formulation was further elaborated in [3], which also contains a discussion of restrictions imposed on constitutive equations by a work assumption that was originally introduced in a strain-space setting by Naghdi and Trapp [4]. Additional related developments utilizing the strain-space formulation are contained in [5–7].

Yoder and Iwan [1, p. 774] state: "(Naghdi) did not establish equivalence between stress and strain space loading criteria . . . ." Actually, Naghdi and Trapp did undertake a comparison between the two independently postulated sets of loading criteria. They concluded that a correspondence between the two sets could be established for all conditions except that of loading from an elastic-plastic state. They observed [2, p. 792]: ". . . no general conclusion can be reached regarding the correspondence or equivalence of $\hat{g}>0$ (the loading criterion in strain space) and $\hat{f}>0$ (the loading criterion in stress space)."

Once a strain-space formulation is adopted, stress appears as a dependent variable, and it is conceivable that certain conditions in stress space might be induced by the conditions that are assumed in strain space. If this were indeed the case, then it would not be necessary, or even desirable, to postulate independent conditions in both strain space and stress space. This is the point of view that was taken by Casey and Naghdi [7], who showed that, in fact, the loading conditions in stress space are determined by those in strain space through the constitutive equations of the theory. However, the conditions induced in stress space during loading are not identical to those of the strain-space formulation, nor do they imply the loading conditions of the strain-space formulation[4].

[1] By P. S. Yoder and W. D. Iwan, and published in the December, 1981, issue of the ASME JOURNAL OF APPLIED MECHANICS, Vol. 48, pp. 773–778.

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[4] For a summary of the relationship between the conditions in stress space and the loading criteria in strain space, see [7, Table 1].