Linear and Angular Deformations in Three-Dimensional, Potential-Invariant Representations of the Actual Gravity Field of the Earth, and Applications

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Summary

The general expression of the second order tensor from which the linear and angular deformations induced by potential-invariant representations of the geopotential field on the normal ellipsoidal field is given. The application of the general results to a relevant class of correspondence laws between the ellipsoidal normal co-ordinates \( (\phi, \lambda) \) of the original points and their images leads to the explicit evaluation of the parameters defining the linear deformations quadric and of the angular distortions involved. Some properties of the correspondence between equipotential surfaces of the geopotential field and equipotential surfaces of the normal field induced by the previous class of potential-invariant representations are finally pointed out.

1. Introduction

It has been shown (Bocchio 1976) how a general, unequivocal definition of gravity anomalies in connection with potential-invariant representations of the geopotential field on the normal ellipsoidal field can be given starting from the metric properties of the normal field. In this paper the problem of evaluating the linear and angular deformations induced by the previous representations is studied and the results are applied to particular correspondence laws between the ellipsoidal normal co-ordinates \( (\phi, \lambda) \) of the original points and of their image.

A 'potential-invariant' correspondence between points \( P \) of the actual gravity field (\( W \)-field) and points \( Q \) of the normal ellipsoidal field (\( U \)-field) is defined as follows. If \( x^1 = \phi, x^2 = \lambda, x^3 = U \) denote the ellipsoidal normal or 'geodetic' co-ordinates of a point \( P \) and \( \Delta x^i (i = 1, 2, 3) \) are the differences between the geodetic co-ordinates of the image \( Q \) of \( P \) under the representation (\( Q \) is referred to in what follows simply as 'image point') and of the original point \( P \), a transformation is said to be 'potential-invariant' if

\[
U(Q) = W(P) = U(P) + T(P),
\]

where \( T(P) \) is the disturbing potential in \( P \). We have thus

\[
\Delta x^3 = U(Q) - U(P) = T(P).
\]
If \( y'(i = 1, 2, 3) \) denote the geodetic co-ordinates of the image point under the transformation

\[
y^i = x^i + F^i(P)
\]  

(1.2)

where \( F^i = \Delta x^i \) is the displacement vector, we have thus, in the case of potential-invariant representations, \( F^3(P) = T(P) \).

2. General relations

From (1.2) we have (the notation of tensor calculus has been used throughout),

\[
dP = dx^i = u^i ds, \quad u^i = \frac{dx^i}{ds}
\]

where \( u^i \) is the unit vector of the displacement \( dP \). The corresponding displacement in \( Q \) is given by

\[
dQ = dy^i = (\delta^i_r + F^i_P) dx^r
\]

(2.2)

where \( \delta^i_r \) is the Kronecker's symbol and the bar denotes partial differentiation with respect to \( x^r \). We have also

\[
dQ = \bar{u}^i d\bar{s}
\]

(2.3)

where \( \bar{u}^i \) and \( d\bar{s} \) are the transformed of \( u^i \) and \( ds \) respectively. We have thus

\[
u^i = \frac{dy^i}{d\bar{s}} = \frac{ds}{d\bar{s}} (\delta^i_r + F^i_P) u^r = \frac{(\delta^i_r + F^i_P) u^r}{m_u}
\]

(2.4)

where \( m_u = d\bar{s}/ds \) is the scale factor in the direction of \( u^i \). If we denote by \( \gamma_{ij} \) and \( \gamma_{ij}^Q \) the metric tensor of the \( U \)-field computed in \( P \) and \( Q \), respectively, it follows

\[
ds^2 = \gamma_{ij} dx^i dx^j
\]

(2.5)

\[
d\bar{s}^2 = \gamma_{ij}^Q dy^i dy^j.
\]

(2.6)

We have thus

\[
m_u = \frac{d\bar{s}}{ds} = (D_{rs} u^r u^s)^\frac{1}{2},
\]

(2.7)

where

\[
D_{rs} = \gamma_{ij} (\delta^i_r + F^i_P) (\delta^j_s + F^j_P)
\]

(2.8)

is a second order symmetric tensor. If \( u^i \) and \( v^i \) are two unit vectors in \( P \) and \( \alpha \) is the angle between them we have

\[
\cos \alpha = \gamma_{ij} u^i v^j,
\]

(2.9)

and, if the corresponding quantities in \( Q \) are considered,

\[
\cos \bar{\alpha} = \gamma_{ij}^Q u^i v^j,
\]

(2.10)

where

\[
\bar{v}^j = \frac{1}{m_u} (\delta^i_s + F^i_P) \bar{v}^s.
\]

(2.11)
The actual gravity field of the Earth

We obtain from (2.10), taking into account (2.4), (2.11),

\[ \cos \bar{a} = \frac{\gamma_{ij}}{m_u m_v} (\delta_{i}^{\prime} + F_{i}/a)(\delta_{j}^{\prime} + F_{j}/a) u^{i} v^{j} \] (2.12)

or, from (2.7), (2.8),

\[ \cos \bar{a} = \frac{D_{a q} u^{q} v^{q}}{(D_{a q} u^{q} v^{q})^{2/3}}. \] (2.13)

If \(\gamma_{ij}\) is given the knowledge of \(D_{a q}\) leads thus also to the evaluation of the angular deformation involved in the transformation (1.2). It can be shown that

\[
\begin{align*}
D_{11} &= \gamma_{11}^{q q}(1 + F^{1}/1)^{2} + \gamma_{22}^{q q}(F^{2}/1)^{2} \\
&\quad + \gamma_{13}^{q q}(T/1)^{2} + 2\gamma_{13}^{q q}(1 + F^{1}/1) T/1 \\
D_{22} &= \gamma_{11}^{q q}(F^{1}/2)^{2} + \gamma_{22}^{q q}(1 + F^{2}/2)^{2} \\
&\quad + \gamma_{13}^{q q}(T/2)^{2} + 2\gamma_{13}^{q q} F^{1}/2 T/2 \\
D_{33} &= \gamma_{11}^{q q}(F^{1}/3)^{2} + \gamma_{22}^{q q}(F^{2}/3)^{2} \\
&\quad + \gamma_{13}^{q q}(1 + T/3)^{2} + 2\gamma_{13}^{q q} F^{1}/3(1 + T/3) \\
D_{12} &= \gamma_{11}^{q q}(1 + F^{1}/1) F^{1}/2 + \gamma_{22}^{q q} F^{2}/3(1 + F^{2}/2) \\
&\quad + \gamma_{13}^{q q} T/1 T/2 + \gamma_{13}^{q q} [(1 + F^{1}/1) T/2 + F^{1}/2 T/1] \\
D_{13} &= \gamma_{11}^{q q}(1 + F^{1}/1) F^{1}/3 + \gamma_{22}^{q q} F^{2}/3 F^{2}/3 \\
&\quad + \gamma_{13}^{q q} T/1 (1 + T/3) + \gamma_{13}^{q q} [(1 + F^{1}/1)(1 + T/3) + F^{1}/3 T/1] \\
D_{23} &= \gamma_{11}^{q q} F^{1}/2 F^{2}/3 + \gamma_{22}^{q q}(1 + F^{2}/2) F^{2}/3 \\
&\quad + \gamma_{13}^{q q} T/2 (1 + T/3) + \gamma_{13}^{q q} F^{1}/2(1 + T/3) + F^{1}/3 T/2
\end{align*}
\] (2.14)

Given (list of symbols at the end of the paper)

\[
\gamma_{ij} \equiv \begin{pmatrix} \rho^2 & 0 & \rho f \\ 0 & N^2 \cos^2 \phi & 0 \\ \rho f & 0 & 1 + f^2 \end{pmatrix} \] (2.15)

we obtain, neglecting second order terms in the small quantities \(\tau, \zeta, \xi, \eta, f, \theta\) (as we have
done throughout the paper)

\[
\begin{align*}
\sigma_{11} &= \sigma_{11} + 2\rho^2 F^1 \left[ f + \left( \frac{1}{\gamma} \frac{\partial}{\partial U} \right) \right] + 2\rho T \frac{\partial \rho}{\partial U} \\
\sigma_{22} &= \sigma_{22} - \rho N F^1 \sin 2\phi + 2NT \cos^2 \phi \frac{\partial N}{\partial U} \\
\sigma_{33} &= \sigma_{33} + \frac{f F^1}{\gamma} \frac{\partial \rho}{\partial U} + 2T \frac{\gamma^2}{\gamma^2} \left( f \frac{\partial f}{\partial U} - \frac{1}{\gamma} \frac{\partial \gamma}{\partial U} \right) \\
\sigma_{13} &= \sigma_{13} + \rho F^1 \left[ \frac{\partial \rho}{\partial U} + \frac{1}{\gamma} + \frac{f}{\gamma} \left( \frac{1}{\gamma} \frac{\partial}{\partial \phi} \right) \right] \\
&\quad + \frac{\rho T}{\gamma} \left( \frac{\partial f}{\partial U} - \frac{f}{\gamma} \frac{\partial \gamma}{\partial U} + \frac{f}{\rho} \frac{\partial \rho}{\partial U} \right) \\
\sigma_{23} &= N F^2 \cos^2 \phi \left( \frac{1}{N} \frac{\partial N}{\partial U} + \frac{1+\tan \phi}{\gamma} \right) \\
\sigma_{12} &= 0
\end{align*}
\]

(2.16)

Since it can be shown that

\[
\cos A \sin Z + f \cos Z + \xi (\cos Z - f \cos A \sin Z)
\]

\[
u^1 = \frac{+ \varepsilon \sin A \sin Z (\sin \phi - f \cos \phi)}{\rho}
\]

\[
u^2 = \frac{- \sin A \sin Z + \varepsilon (\cos \phi \cos Z - \sin \phi \cos A \sin Z)}{N \cos \phi}
\]

\[
u^3 = - (\cos Z - \xi \cos A \sin Z - \varepsilon \cos \phi \sin A \sin Z) \gamma
\]

(2.17)

where A and Z are the astronomical azimuth and zenith distance of \(u^1\), by substituting (2.17) into (2.7) \(m_u\) can be expressed as a function of the directional parameters of \(u^1\) referred to the actual vertical.

3. Application

This section is concerned with the application of the general results to the case

\[
F^1 = F^2 = 0, \quad F^3 = T
\]

(3.1)

which characterizes the 'isozenithal' correspondence considered by Marussi (1973, 1974). Nevertheless, since we are concerned here with a first-order theory the results obtained for the transformation (3.1) are of more general interest. In fact, within the limits of a first-order approximation the class of potential-invariant transformations, which lead to second-order differences in position with respect to the isozenithal correspondence give the same final results. Two relevant examples of representations pertaining to the previous class are

\[
F^1 = \frac{f T}{\gamma \rho} \approx ft, \quad F^2 = 0, \quad F^3 = T
\]

(3.2)
The actual gravity field of the Earth

in which the correspondence between ellipsoidal co-ordinates takes place along the lines of force of the $U$-field, and

$$F^1 = \frac{(\xi + f)}{\gamma \rho} T \cong (\xi + f) \tau, \quad F^2 = \frac{\epsilon T}{\gamma N} \cong \epsilon \tau, \quad F^3 = T \quad (3.3)$$

in which the correspondence takes place along the lines of force of the $W$-field.

We obtain

$$\gamma_{11} = \gamma_{11}(1 - 2\tau)$$
$$\gamma_{22} = \gamma_{22}(1 - 2\tau)$$
$$\gamma_{33} = \gamma_{33}(1 - 4\tau)$$
$$\gamma_{13} = \gamma_{13}(1 - 3\tau)$$
$$\gamma_{12} = \gamma_{23} = 0 \quad (3.4)$$

and

$$D_{11} = \rho^2(1 - 2\tau)$$
$$D_{22} = N^2 \cos^2 \phi(1 - 2\tau)$$
$$D_{33} = \frac{1 + 2\xi}{\gamma^2}$$
$$D_{13} = \frac{\rho(f - \xi)}{\gamma}$$
$$D_{23} = -\frac{N\eta \cos \phi}{\gamma}$$
$$D_{12} = 0 \quad (3.5)$$

From (2.17) and (3.5) it follows

$$m_u = 1 - \tau \sin^2 Z + \xi \cos^2 Z + (\xi \cos A + \eta \sin A) \sin Z \cos Z. \quad (3.6)$$

If we consider the transformation

$$x = \frac{\sin Z \sin A}{m_u}$$
$$y = \frac{\sin Z \cos A}{m_u}$$
$$z = \frac{\cos Z}{m_u} \quad (3.7)$$

we have

$$m_u = \frac{1}{\sqrt{(x^2 + y^2 + z^2)}} = \frac{1}{r} ,$$
and

\[(1-2\tau)x^2 + (1-2\tau)y^2 + (1+2\zeta)z^2 + 2\eta \times \zeta + 2\xi yz = 1 \quad (3.8)\]

is thus the equation of the ellipsoid of linear deformations (denoted in what follows simply as E) referred to rectangular co-ordinates centered in P (Fig. 1). The principal directions of E are found by solving the third degree equation in the unknown \(k\)

\[\begin{vmatrix}
1-2\tau-k & 0 & \eta \\
0 & 1-2\tau-k & \zeta \\
\eta & \zeta & 1+2\zeta-k
\end{vmatrix} = 0. \quad (3.9)\]

The solutions turn out to be

\[
\begin{aligned}
k_1 &= 1-2\tau \\
k_2 &= 1-\tau+\zeta+\sqrt{[\tau+\zeta]^2+\theta^2]} \\
k_3 &= 1-\tau+\zeta-\sqrt{[\tau+\zeta]^2+\theta^2}]\end{aligned} \quad (3.10)
\]

Substituting the above values in the system

\[
\begin{cases}
(1-2\tau-k)l + \eta n = 0 \\
(1-2\tau-k)m + \zeta n = 0 \\
\eta l + \zeta m + (1+2\zeta-k)n = 0
\end{cases} \quad (3.11)
\]

in which the unknowns \(l, m, n\), are the direction cosines of the principal axes, it follows

\[
\begin{aligned}
\frac{l_1}{m_1} &= -\frac{\zeta}{\eta}, \quad n_1 = 0 \\
\frac{l_2}{m_2} &= \frac{\eta}{\zeta}, \quad n_2 = \tau+\zeta+\sqrt{[\tau+\zeta]^2+\theta^2]} \\
\frac{l_3}{m_3} &= \frac{\eta}{\zeta}, \quad n_3 = \tau+\zeta-\sqrt{[\tau+\zeta]^2+\theta^2}]\end{aligned} \quad (3.12)
\]

from which the astronomical azimuth \(A_0\) of the (\(\bar{y}, \bar{z}\)) principal plane (Fig. 1) and the zenith distance \(Z_0\) of the \(\bar{z}\)-axis can be deduced. The equation of E referred to its principal axes (\(\bar{x}, \bar{y}, \bar{z}\)) is thus

\[
(1-2\tau)\bar{x}^2 + [1-\tau+\zeta+\sqrt{[\tau+\zeta]^2+\theta^2}]\bar{y}^2
\]

\[
+ [1-\tau+\zeta-\sqrt{[\tau+\zeta]^2+\theta^2}]\bar{z}^2 = 1 \quad (3.13)
\]

and therefore the values of the scale factor in the principal directions of E are given by

\[
\begin{aligned}
m_x &= \sqrt{(1-2\tau)} \\
m_y &= \sqrt{(1-\tau+\zeta+\sqrt{[\tau+\zeta]^2+\theta^2]}) \\
m_z &= \sqrt{(1-\tau+\zeta-\sqrt{[\tau+\zeta]^2+\theta^2}]})\end{aligned} \quad (3.14)
\]

\(a = 1/m_x, b = 1/m_y, c = 1/m_z\) being the semi-axes of E. We observe that from (3.12) it follows that the \(\bar{x}\)-axis lies on the (\(x, y\))-plane (\(W\)-plane in Fig. 1).
Fig. 1. Linear deformations ellipsoid.
As far as the computation of angular deformations is concerned we obtained from (2.13)
\[
\cos \bar{\alpha} = \frac{1}{m_u m_v} \left[ \cos \alpha - 2\tau (\gamma_{11} u^1 v^1 + \gamma_{22} u^2 v^2) + \frac{2z}{\gamma} u^3 v^3 - \frac{\rho \xi}{\gamma} (u^1 v^3 + u^3 v^1) - \frac{N \eta \cos \phi}{\gamma} (u^2 v^3 + u^3 v^2) \right].
\] (3.15)

It is interesting to observe that when \( Z = \pi/2 \) it follows from (3.6) that the scale factor in the \( W \)-plane is given by
\[
m_u = 1 - \tau;
\] (3.16)
i.e. it is azimuth-independent. This means that the correspondence between equipotential surfaces of the \( W \)-field (\( W \)-surfaces) and equipotential surfaces of the \( U \)-field (\( U \)-surfaces) is conformal in character for the potential-invariant transformations considered in this section. The latter property can be deduced also from (3.15). Confining ourselves to surface unit vectors on a \( W \)-surface the pertinent components of \( u^i, v^i \) are obtained by substituting \( Z = \pi/2 \) into the general expression of a unit vector (2.17). We have thus
\[
\cos \bar{\alpha} = \frac{1}{m_u m_v} \left[ \cos \alpha - 2\tau (\gamma_{11} u^1 v^1 + \gamma_{22} u^2 v^2) \right].
\] (3.17)

If we observe that, apart from second-order terms,
\[
\tau (\gamma_{11} u^1 v^1 + \gamma_{22} u^2 v^2) = \tau \cos \alpha,
\] (3.18)
we obtain from (3.17), taking into account that on a \( W \)-surface \( m_u = m_v = 1 - \tau \),
\[
\cos \bar{\alpha} = \cos \alpha.
\] (3.19)

If we denote with \( \sigma \) the geodesic curvature of a surface line on a \( W \)-surface and with \( \bar{\sigma} \) the transformed curvature on the corresponding \( U \)-surface, we have by Schols theorem (Hotine 1969),
\[
\bar{\sigma} = \exp (-\mu)(\sigma - \mu/\alpha v^a), \quad (\alpha = 1, 2),
\] (3.20)
where \( \mu = \lg m_u \) and \( v^a \) is the unit normal to the surface line on the \( W \)-surface. It can be shown that
\[
\mu/\alpha = \frac{T/\alpha}{3T - W},
\] (3.21)
where \( T/\alpha \) is the surface gradient of the disturbing potential. From (3.21) we deduce that the directions of steepest variation of the scale factor and of the disturbing potential coincide on a \( W \)-surface. Since it can be shown that
\[
\begin{align*}
u^1 &= -N \cos \phi (\sin A - \varepsilon \cos A \sin \phi) \\
u^2 &= \rho (\cos A + \varepsilon \sin A \sin \phi),
\end{align*}
\] (3.22)
we have thus
\[
\bar{\sigma} = \frac{(W - T)}{W - 2T} \sigma + \frac{T/\alpha v^a}{W - 4T} = \frac{(W - T)}{W - 2T} \sigma + \frac{\gamma \rho N \cos \phi}{W - 4T} (\xi \sin A - \eta \cos A).
\] (3.23)
We finally point out that if the surface line is locally geodesic in $P$, i.e. if $\sigma(P) = 0$, and crosses $P$ with the astronomical azimuth

$$A = \arctg \frac{\eta}{\zeta}, \quad (3.24)$$

we have from (3.23) that the corresponding surface line on the $U$-surface is also locally geodesic in the image point, i.e. $\delta(Q) = 0$.

**List of symbols**

- $W$ actual potential;
- $U$ normal potential;
- $T$ disturbing potential;
- $\gamma$ normal gravity;
- $\tau = \frac{T}{U}$ relative anomaly in the potential;
- $\zeta = \frac{\Delta g}{\gamma}$ relative anomaly in gravity ($\Delta g$ is the gravity anomaly);
- $\xi, \eta = \epsilon \cos \phi$ deflections of the vertical;
- $\phi, \lambda$ geodetic latitude and longitude;
- $\rho, N$ radius of curvature of the meridian and of the normal section perpendicular to it;
- $A, Z$ astronomical azimuth and zenith distance;
- $\theta = \sqrt{(\xi^2 + \eta^2)}$ total deflection of the vertical;
- $E$ ellipsoid of linear deformations;
- $r$ radiusvector of $E$;
- $f = \frac{\partial \lg \gamma}{\partial \phi}$.

**References**