Inserting (19), (20), and the solution of (21)-(23) into (18) yields

$$y(x,t) = 8A\Big[\Big(\frac{x^3 - \pi^2 x + 6\sin x}{12}\Big)\sin\alpha t + \frac{\alpha t\cos\alpha t\sin x}{2} + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^3(n^4-1)}\,(n^2\sin\alpha n^2 t - \sin\alpha t)\sin nx\Big] \quad (24)$$

The equality of (1.6) and (24) follows since it can be shown that

$$\frac{2(x^3-\pi^2 x)}{3} - 2x\cos x + 9\sin x - \frac{2\pi\sinh x}{\sinh\pi} = 8\sum_{n=2}^{\infty}\frac{(-1)^{n+1}\sin nx}{n^3(n^4-1)} \quad (25)$$

Because the *M-G* method has wider applicability than the *E* method, it remains the standard method of solution for beams subjected to time-dependent boundary excitation. However, for problems involving sinusoidal excitation only, the *E* method provides an alternate solution.

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Author's Closure

H. D. Fisher's statement "that the *E* method is applicable exclusively to beam responses produced by sinusoid excitation" is not correct. The *E* method can be applied to boundary value problems with other boundary conditions. In the Brief Note it was not the author's intent to describe the most general case, but instead to show one illustration of the method. For more generality we have the following.

The existence of a form for the change of dependent variable depends on the properties of the functions that are prescribed on the boundaries. If these functions possess a finite number of linearly independent derivatives, then the form for the change of dependent variable should contain a linear combination of these functions and all of their linearly independent derivatives. However, in place of constants in this linear combination there should be functions of the variable held fixed on the boundary. Also, if the partial differential equation is separable as well as homogeneous, the particular product solutions can be determined. A comparison of these product solutions with the linear combination of the boundary functions and all of their derivatives will determine the need for either a bounded or unbounded form for the change of dependent variable. Thus, since the form for the change of dependent variable depends on the prescribed functions on the boundaries as well as the partial differential equation, the range of applicability of this method cannot be ascertained by generalizing the form of the change of dependent variable I used in the illustration of this method.

On the Formulation of Strain-Space Plasticity With Multiple Loading Surfaces[1]

J. Casey[2] and P. M. Naghdi[3]. We take exception to a number of points made in the paper of Yoder and Iwan [1], and especially to their claim that the stress-space and strain-space formulations of plasticity are equivalent. Although in [1] both single and multiple loading surfaces are employed, it suffices for the purpose of this discussion to consider only the case of single loading surfaces.

The possibility of using a strain-space (rather than a stress-space) formulation of plasticity has been mentioned by several authors in the past. However, the physical significance of the use of the strain-space formulation was first brought out in the paper of Naghdi and Trapp [2]. To elaborate, it was observed by Naghdi and Trapp [2] that the stress-space formulation of plasticity leads to unreliable results in any region such as that corresponding to the maximum point of engineering stress versus engineering strain curve for uniaxial tension of a typical ductile metal. After also observing that the stress-space formulation does not reduce directly to the theory of elastic-perfectly plastic materials, and that a separate formulation for the latter is required, Naghdi and Trapp [2] proposed an alternative strain-space formulation of plasticity which:

(*a*) is valid for the full range of elastic-plastic deformation; and

(*b*) includes as a special case, the theory of elastic-perfectly plastic materials.

The strain-space formulation was further elaborated in [3], which also contains a discussion of restrictions imposed on constitutive equations by a work assumption that was originally introduced in a strain-space setting by Naghdi and Trapp [4]. Additional related developments utilizing the strain-space formulation are contained in [5–7].

Yoder and Iwan [1, p. 774] state: "(Naghdi) did not establish equivalence between stress and strain space loading criteria . . . ." Actually, Naghdi and Trapp did undertake a comparison between the two independently postulated sets of loading criteria. They concluded that a correspondence between the two sets could be established for all conditions except that of loading from an elastic-plastic state. They observed [2, p. 792]: ". . . no general conclusion can be reached regarding the correspondence or equivalence of $\hat{g}>0$ (the loading criterion in strain space) and $\hat{f}>0$ (the loading criterion in stress space)."

Once a strain-space formulation is adopted, stress appears as a dependent variable, and it is conceivable that certain conditions in stress space might be induced by the conditions that are assumed in strain space. If this were indeed the case, then it would not be necessary, or even desirable, to postulate independent conditions in both strain space and stress space. This is the point of view that was taken by Casey and Naghdi [7], who showed that, in fact, the loading conditions in stress space are determined by those in strain space through the constitutive equations of the theory. However, the conditions induced in stress space during loading are not identical to those of the strain-space formulation, nor do they imply the loading conditions of the strain-space formulation[4].

[1] By P. S. Yoder and W. D. Iwan, and published in the December, 1981, issue of the ASME Journal of Applied Mechanics, Vol. 48, pp. 773–778.

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[4] For a summary of the relationship between the conditions in stress space and the loading criteria in strain space, see [7, Table 1].

Geometrically, during loading the yield surface in strain space is always moving outward locally, whereas the corresponding yield surface in stress space may concurrently be moving outward, inward, or may be stationary. It should therefore be clear that the stress-space and strain-space formulations are not equivalent.

Our preference for choosing the loading criteria of strain space as primary in [7] was motivated by the limitations of the stress-space formulation that were mentioned in the foregoing. This leads [7] to a characterization of strain-hardening in terms of a rate-independent dimensionless quotient $\hat{f}/\hat{g}$ (with $\hat{g}>0$). A parallel analysis based on the possibility of taking the loading criteria of stress space as primary would lead to a characterization in terms of $\hat{g}/\hat{f}$ (with $\hat{f}>0$) – obviously, this characterization would be inappropriate if elastic-perfectly plastic behavior were also to be included, as indeed it must.

If the loading criteria of strain space are adopted as primary, then certain features of the work of Yoder and Iwan [1] may be obtained as a special case of that of Casey and Naghdi [7]. However, a demonstration of this requires considerable mathematical details; and, in the interest of keeping this discussion brief, the additional developments will be provided elsewhere. Finally, it may be emphasized that the basic theory of [2-7] is not limited to infinitesimal deformation and is valid for finite deformation of elastic-plastic materials.

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Authors' Closure

In view of Naghdi's pioneering efforts in the area of strain-space plasticity, we appreciate the careful attention that he and Casey have devoted to our recent paper [1]. It occurs to us that many of the points they raise in regard to the subject paper stem not so much from disagreements of substance as from the differing points of view from which we approach the matter of a strain-space formulation.

As noted in the introduction to [1], the analytical work documented therein was motivated in large part by the quest for more accurate and more efficient computational algorithms for plasticity. A follow-up paper [2], soon to appear, will present some numerical results intended to demonstrate the inherent superiority of the strain-space theory for computational applications. To keep this point from being obscured, it was decided to sidestep, for the time being, the question of finite deformations and cast the analysis of [1] and [2] within the traditional framework of the small-strain theory. Plasticity has, as a matter of fact, been extended to handle finite deformations in a number of different ways, as outlined by McMeeking and Rice [3]. Their recommendations on the matter, recently elaborated on by Hughes and Winget [4], do not precisely parallel those of Naghdi [5-7]. Naghdi's approach, therefore, appealing as it might be, is not the only possibility at hand, nor is it somehow inextricably linked to the concept of strain-space plasticity.

In describing the relationship between the stress and strain-space formulations, much of our language was influenced by a desire to address issues involved in the computational implementation of plasticity theory. So far as the authors know, previous computational algorithms for plasticity theory have been based on the notion of loading surfaces in stress space. This is true whether the algorithms were designed to deal with strain-hardening, perfectly plastic, or strain-softening behavior. Accordingly, it seemed natural to use the expression *stress-space plasticity* in a rather broad sense that includes whatever formulations might be necessary to describe these three cases in terms of stress-space loading surfaces. It appears in retrospect that some readers of [1] may have been confused concerning the intended scope of the presentation. It should be emphasized that constitutive law [Γ] as presented in the paper is appropriate only for the case of strain hardening ($\Delta>0$). The stress-space formulations applicable to perfect plasticity ($\Delta=0$) and to strain softening ($\Delta<0$) are omitted from [1] for brevity but may be found in reference [8].

Never was there any intention to suggest that the strain-hardening constitutive law [Γ] could be used interchangeably with [*C*] in cases of perfect plasticity or strain softening. The authors wholeheartedly agree with Casey and Naghdi that statement (γ), or $\hat{f} > 0$ in their notation, would be a most unacceptable criterion for loading in cases of perfect plasticity or strain softening. It was never claimed that the loading conditions induced in stress space were "identical" to those in strain space. That they could not be identical is obvious because, as Casey and Naghdi point out, during loading the relaxation surface in strain space is moving outward locally while the yield surface in stress space may be moving outward, inward, or may be stationary. This conclusion in fact follows directly from an extension of the concepts presented in the authors' paper [8]. This observation does not in any way conflict with the idea of "equivalence" as used by the authors.

The authors submit that [*C*] and [Γ], as specified in their paper, can be used interchangeably in cases of strain hardening provided the model parameters are interrelated through equations (18)-(20). In this regard it would perhaps be helpful to discuss in more detail the reasons for using the word *equivalence* to describe this relationship. It should be noted that σ^R and ϵ^P, as defined through equations (5) and (15), are merely alternative indications of the deviation from Hookean linearity. Until some assumptions are made regarding the manner in which they evolve as deformation proceeds, their use cannot in any way impose restrictions on the mechanical response. Similarly, the loading functions $\hat{F}$ and $\hat{\Phi}$ are powerless to influence the material behavior until either [*C*] or [Γ] is activated. Thus it is possible at any stage during the deformation to compute values for both D and Δ, based on the stress and strain history, without in any way influencing the relationship between the *next* stress and strain increments. Now, it is shown in [8] that if $\Delta>0$ and [*C*] is valid, then so is [Γ]. The scheme of the proof is as follows:

$$
\begin{array}{rcl}
(a) & \Rightarrow & (\alpha)\ ; \\
(\beta) \text{ and } (\gamma) & \Rightarrow & (b) \text{ and } (c) \\
 & \Rightarrow & (d) \\
 & \Rightarrow & (\delta)\ ; \\
\text{not } (\beta) \text{ or not } (\gamma) & \Rightarrow & \text{not } (b) \text{ or not } (c) \\
 & \Rightarrow & (e) \\
 & \Rightarrow & (\epsilon)\ .
\end{array}
$$

In a similar way, if $\Delta>0$, $[\Gamma] \Rightarrow [C]$. Analogous results can also be obtained for cases of perfect plasticity ($\Delta=0$) and strain softening ($\Delta<0$) [8] – provided the stress-space formulations are properly framed. This is what the authors mean by the word *equivalent*.

We apologize if the use of this word has proved misleading to some readers of the subject paper. The main point to be stressed is that in designing a computational algorithm for plasticity, one is at liberty to work from either the stress or strain-space version, whichever is more convenient, since the two approaches can, with certain restrictions, be made to yield the same physical behavior. The computational experience reported in [2] and [8] lends additional support to the interchangeability of these two formulations.

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The Annular Membrane Under Axial Load[1]

Robert Schmidt.[2] Let us introduce auxiliary notations x, y, z, and p defined by

$$r_2^2 x = r^2, \quad 4Dy = r^2 N_r, \tag{1a,b}$$

$$2^{3/2} tz = [3(1-\nu^2)]^{1/2} r\beta, \tag{1c}$$

$$2^{5/2} \pi E t^4 p = [3(1-\nu^2)]^{3/2} r_2^2 P, \tag{1d}$$

where $D = Et^3/12(1-\nu^2)$, and the remaining symbols have the meaning assigned to them in the Note under discussion. With these notations, the nonlinear differential equations [1] governing moderately large axisymmetric deflections of circular plates become [2, 3]

$$x^2 y'' = -z^2, \quad x^2 z'' = yz + px, \tag{2a,b}$$

where primes denote derivatives with respect to x.

For a membrane, the foregoing equations reduce to

$$x^2 y'' = -z^2, \quad yz = -px, \tag{3a,b}$$

and finally to Schwerin's form [4]

$$y^2 y'' = -p^2, \tag{4}$$

which possesses closed-form general implicit and special explicit solutions [4, 5], both of which were published by Schwerin in 1929 [4].[3] Needless to say, Schwerin was rather proud of his discovery, which doubled the number of known closed-form solutions in the theory of slack membranes from one to two. The authors of the present Note have simply rediscovered Schwerin's special solution, which is valid for $\nu = 1/3$.

Much later, closed-form solutions, similar to Schwerin's, were obtained by Jahsman, Field, and Holmes [5] for a prestretched axisymmetrical membrane, and by E. Reissner for a spherical membrane.

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[3] It should be noted that equation (4) is analogous to the differential equation $r^2 \ddot{r} = \text{const.}$, describing the free fall of two spherical bodies toward each other.

[1] By D. J. Allman and E. H. Mansfield, and published in the December, 1981, issue of the ASME JOURNAL OF APPLIED MECHANICS, Vol. 48, pp. 975–976.

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The Annular Membrane Under Axial Load[1]

C. W. Bert[2]. The authors are to be congratulated for obtaining a closed-form solution for a nonlinear problem which has not been solved previously in the context of the Föppl theory [1] used. The limitations of the Föppl theory have been discussed by Junkin and Davis [2] in comparison with an exact membrane theory obtained from Budiansky's shell equations [3]. In reference [2], it was shown that Föppl's theory is valid only when the ratio of the deflection to the outside radius is small compared to unity. Nevertheless, using order-of-magnitude considerations, reference [2] obtained, for $\nu = 1/3$, a closed-form solution equivalent to that obtained by the authors.

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