In a similar way, if $\Delta > 0$, $[\Gamma] \Rightarrow [C]$. Analogous results can also be obtained for cases of perfect plasticity ($\Delta = 0$) and strain softening ($\Delta < 0$) [8] – provided the stress-space formulations are properly framed. This is what the authors mean by the word *equivalent*.

We apologize if the use of this word has proved misleading to some readers of the subject paper. The main point to be stressed is that in designing a computational algorithm for plasticity, one is at liberty to work from either the stress or strain-space version, whichever is more convenient, since the two approaches can, with certain restrictions, be made to yield the same physical behavior. The computational experience reported in [2] and [8] lends additional support to the interchangeability of these two formulations.

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The Annular Membrane Under Axial Load[1]

Robert Schmidt.[2] Let us introduce auxiliary notations x, y, z, and p defined by

$$r_2^2 x = r^2, \quad 4Dy = r^2 N_r, \tag{1a,b}$$

$$2^{3/2} tz = [3(1-\nu^2)]^{1/2} r\beta, \tag{1c}$$

$$2^{5/2} \pi E t^4 p = [3(1-\nu^2)]^{3/2} r_2^2 P, \tag{1d}$$

where $D = Et^3/12(1-\nu^2)$, and the remaining symbols have the meaning assigned to them in the Note under discussion. With these notations, the nonlinear differential equations [1] governing moderately large axisymmetric deflections of circular plates become [2, 3]

$$x^2 y'' = -z^2, \quad x^2 z'' = yz + px, \tag{2a,b}$$

where primes denote derivatives with respect to x.

For a membrane, the foregoing equations reduce to

$$x^2 y'' = -z^2, \quad yz = -px, \tag{3a,b}$$

and finally to Schwerin's form [4]

$$y^2 y'' = -p^2, \tag{4}$$

which possesses closed-form general implicit and special explicit solutions [4, 5], both of which were published by Schwerin in 1929 [4].[3] Needless to say, Schwerin was rather proud of his discovery, which doubled the number of known closed-form solutions in the theory of slack membranes from one to two. The authors of the present Note have simply rediscovered Schwerin's special solution, which is valid for $\nu = 1/3$.

Much later, closed-form solutions, similar to Schwerin's, were obtained by Jahsman, Field, and Holmes [5] for a prestretched axisymmetrical membrane, and by E. Reissner for a spherical membrane.

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[1]By D. J. Allman and E. H. Mansfield, and published in the December, 1981, issue of the ASME JOURNAL OF APPLIED MECHANICS, Vol. 48, pp. 975–976.

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[3] It should be noted that equation (4) is analogous to the differential equation $r^2\ddot{r} = $ const., describing the free fall of two spherical bodies toward each other.

The Annular Membrane Under Axial Load[1]

C. W. Bert[2]. The authors are to be congratulated for obtaining a closed-form solution for a nonlinear problem which has not been solved previously in the context of the Föppl theory [1] used. The limitations of the Föppl theory have been discussed by Junkin and Davis [2] in comparison with an exact membrane theory obtained from Budiansky's shell equations [3]. In reference [2], it was shown that Föppl's theory is valid only when the ratio of the deflection to the outside radius is small compared to unity. Nevertheless, using order-of-magnitude considerations, reference [2] obtained, for $\nu = 1/3$, a closed-form solution equivalent to that obtained by the authors.

[1]By D. J. Allman and E. H. Mansfield, and published in the December, 1981, issue of the ASME JOURNAL OF APPLIED MECHANICS, Vol. 48, pp. 975–976.

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Even in the context of the Föppl theory, the second of equations (2), is not quite correct, since it involves an extra derivative with respect to radius. The correct equation is

$$r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right)\right]+\frac{1}{2}Et\left(\frac{dw}{dr}\right)^2=0 \qquad (1)$$

Then equation (6) becomes

$$c\beta^2(\beta-2)r^{\beta-2}+\frac{1}{2}Et\ k^2\alpha^2r^{2\alpha-2}=0 \qquad (2)$$

Fortunately, the solution obtained by the authors, as manifested in their equations (3), (4), and (7), satisfies the preceding corrected equations and thus is still correct.

The lower order in equation (1) follows directly from the appropriate compatibility equation for this problem:

$$\frac{d}{dr}(r\epsilon_\theta)-\epsilon_r=-\frac{1}{2}\left(\frac{dw}{dr}\right)^2 \qquad (3)$$

since the strain-displacement relations are:

$$\epsilon_r=\frac{du}{dr}+\frac{1}{2}\left(\frac{dw}{dr}\right)^2;\quad \epsilon_\theta=u/r \qquad (4)$$

To compare results with an existing numerical solution [4] (not based on Föppl's theory), the authors' equations can be cast in the following dimensionless form

$$w_{\max}/r_2=(6f)^{1/3}(1-\rho^{2/3}) \qquad (5)$$

$$(\sigma_r)_{\max}/E=(9/16)^{1/3}f^{2/3}\rho^{-2/3} \qquad (6)$$

where $\rho=r_1/r_2$, $f=P/2\pi Etr_2$. It is interesting to note that although Lidin's analysis [4] is not based on Föppl's theory, Lidin obtained the same form of the variation of $w_{\max}/r_2$ and $(\sigma_r)_{\max}/E$ with f as in equations (5) and (6). However, there is some difference between the respective dependencies on ρ, as can be seen in the following tabulation:

	Values of $w_{\max}/r_2f^{1/3}$		Values of $(\delta_r)_{\max}/Ef^{2/3}$	
Radius ratio ρ	Allman & Mansfield	[4]	Allman & Mansfield	[4]
0.1	1.426	1.239	3.832	3.906
0.3	1.003	0.9042	1.842	1.735
0.5	0.6724	0.6849	1.310	1.201
0.7	0.3846	0.4287	1.047	0.8697
0.9	0.1233	0.2394	0.8856	0.5420

It is noted that Lidin's result is purportedly valid for any Poisson's ratio, yet it is independent of Poisson's ratio, which seems unusual from a physical viewpoint.

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Author's Closure

The authors are pleased to note the interest shown in their paper by Professor Bert and they are indebted to him for the valuable comparison of results with earlier work.

In response to his discussion, however, it must be restated here that the second of equations (2) is the correct Föppl equation (which is of fourth order) for the annular membrane under axial load. The third-order equation derived from first principles by Professor Bert is recognized as the first integral of the Föppl equation with the constant of integration zero, as is necessary for a solution of equation (6) to be obtained.

A subsequent paper [1] by the first author, which uses a variational formulation of Föppl's theory to obtain approximate solutions for the annular membrane under axial load, will also be published soon. Numerical results are presented there for a range of values of Poisson's ratio including the value of one-third considered in the present work.

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