

Lewis Research Center. The author acknowledges the enthusiastic support and discussion of this work by Dr. Bernard Hamrock, Structures and Mechanical Technologies Division, Tribology Branch, Lewis Research Center.

References

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APPENDIX

We wish to establish that $\partial_i q_n^* = 0$ for $n > 0$ is consistent with the Reynolds equation for q_n^* . This result, required in arriving at equation (9), expresses the fact that the expectation of the pressure gradient is just the constant smooth value $\partial_i q_0 = g_i$. By substitution of the series for q into equation (2), the general equation for q_n is obtained:

$$\partial_i \partial_i q_n = -3\partial_i [\lambda^- \partial_i q_{n-1}] - 3\partial_i [(\lambda^-)^2 \partial_i q_{n-2}] - \partial_i [(\lambda^-)^3 \partial_i q_{n-3}] \equiv R_n \quad (A1)$$

In terms of the Green function defined by equations (10) and (11), the solution for q_n is

$$q_n(x) = \iint G(x-x^1) R_n(x^1) dx^1 \quad (A2)$$

Recursive substitution of lower order q_k on the right of equation (A1) making use of equation (A2) leads to a series of terms in R_n of which the general form is an $(\ell - 1)$ -fold double integration with respect to $x^2, x^3, \dots, x^\ell, 2 \leq \ell \leq n$, of a product of $(\ell - 1)$ Green functions and n factors of λ^\pm distributed between the ℓ arguments x^1, x^2, \dots, x^ℓ . For pressure flow this general term may be explicitly written

$$\{ \dots \{ \partial_i^j [\lambda^- (x^1)]^{p_1} \partial_i^j G(x^1 - x^2) \} \{ \partial_i^j [\lambda^- (x^2)]^{p_2} \partial_i^j G(x^2 - x^3) \} \dots \{ \partial_i^j [\lambda^- (x^\ell)]^{p_\ell} g_r \} dx^2 dx^3 \dots dx^\ell$$

times a numerical factor depending on the set of integers p_m , which must satisfy $\sum_{m=1}^{\ell} p_m = n$. For shear flow, the final brace is replaced by $\{ \mu_0 [\lambda^- (x^\ell)]^{p_\ell - 1} \partial_i^j [\lambda^+ (x^\ell)] V_r / h^{*2} \}$ while the numerical factor is also changed.

Forming the expectation value of the integrand before performing the differentiation of each λ factor, a valid commutation of operations, we obtain a CF of order n be-

tween ℓ different sample points of the stochastic rough surface. By stationarity, this CF depends on the $\ell(\ell - 1)/2$ argument differences. Carrying out next the required differentiations of the CF with respect to x^2, x^3, \dots, x^ℓ followed by the $(\ell - 1)$ -fold double integration then leads to a result independent of x^1 , since the Green function derivatives also depend only on variable differences. It should be noted that this proof depends on the assumed rapid decay of the CF with distance, allowing the range of integration to be extended to infinity. The ∂_i^j operator now remaining to the left of the expression reduces it to zero. Thus $R_n^* = 0$ and hence by equation (A1) $\partial_i q_n^*$ is a constant which may be chosen zero. This choice, which keeps q_n^* bounded at infinity, establishes the desired result.

DISCUSSION

A. W. Bush⁵ and G. D. Hughes⁵

The author has produced an elegant asymptotic verification of the PC flow factors for small σ/h^* . The discussers have also used the Green function approach to obtain the flow factors in an analysis which is not restricted to small σ/h^* . (This work is not yet published.) The expressions derived by the discussers for the flow factors are:

$$\phi_s = \frac{3}{4\pi\sigma} I_1$$

$$\phi_x = \frac{1}{h^{*3}} \left(\langle h^3 \rangle + \frac{9}{4\pi} I_2 \right)$$

where

$$I_1 = \iint \ln \{ (x - \zeta)^2 + (y - \eta)^2 \} \left\langle \frac{1}{h(x,y)} \frac{\partial h}{\partial x}(x,y) \left(\frac{\partial \delta_1}{\partial \zeta}(\zeta,\eta) + \frac{\partial \delta_2}{\partial \zeta}(\zeta,\eta) \right) \right\rangle d\zeta d\eta$$

$$I_2 = \iint \ln \{ (x - \zeta)^2 + (y - \eta)^2 \} \left\langle \frac{1}{h(x,y)} \frac{\partial h}{\partial x}(x,y) h^2(\zeta,\eta) \times \left(\frac{\partial \delta_2}{\partial \zeta}(\zeta,\eta) - \frac{\partial \delta_1}{\partial \zeta}(\zeta,\eta) \right) \right\rangle d\zeta d\eta$$

and

$$h = \begin{cases} h^* + \delta_2 - \delta_1 & \text{if } h^* + \delta_2 - \delta_1 > 0 \\ 0 & \text{if } h^* + \delta_2 - \delta_1 \leq 0 \end{cases}$$

The integration ranges are over a micro region Ω over which h^* may be considered constant. However, since the correlation between (x,y) and (ζ,η) heights decays rapidly to zero as the separation between the points increases the integration ranges may be taken as the whole (ζ,η) plane.

The expectations appearing in the above expressions are obtained by numerical simulation of surfaces using the method proposed by Patir [1]. Contacts ($h=0$) are dealt with by ensuring $\partial h/\partial x$ has a stronger zero at these points. An asymptotic expansion of these expressions for small σ/h^* allows the expectations to be expressed as derivatives of the autocorrelation function and the Author's results are obtained. *Although the Author has obtained correct results there are some important criticisms to be discussed.*

The contribution to the solution for q_1 from the complementary function has been neglected in equation (12). This term is present as a boundary integral on Γ , the boundary of Ω , and involves boundary conditions for the pressure on the

⁵Department of Mathematics and Statistics, Teesside Polytechnic, Middlesbrough, U.K.

micro bearing. The importance of the boundary condition has been a source of considerable controversy (author's reference [6] and Tonder's discussion of author's reference [2]).

The discussor's approach uses Greens functions in conjunction with a two scale analysis which distinguishes the fast (roughness) distance scale and the slow (mean) scale. It can be shown that the flow factors are independent of the boundary conditions. The reason is as follows. Let $q = q^* + q'$ where $q^* = \langle q \rangle$ and q' is the random component. Then q' may be expressed in terms of two Greens function integrals, one over Ω and the other a boundary integral over Γ . The second integral involves values of q' on Γ . These are uncorrelated with height values within Ω so that the term $\langle h^3 / 12\mu_0 \partial_i q' \rangle$ will have no contribution from the boundary integral. This proves that the PC choice of $q' = 0$ on the boundary involves no loss of generality.

Equation (6) asserts that q_0 satisfies Laplace's equation from which it is *wrong to conclude that $\partial_i q_0$ is constant*. It is also wrong to state that roughness leaves the expected pressure gradient unchanged from its smooth value.

The expected pressure is invariant under the transformation $\delta_{1,2} \rightarrow -\delta_{1,2}$ which ensures $\langle q_i \rangle = 0$ for i odd. Thus

$$q^* = \langle q \rangle = q_0 + \epsilon^2 \langle q_2 \rangle + O(\epsilon^4)$$

but $\langle q_2 \rangle$ is not in general constant so $\partial_i q^* \neq \partial_i q_0$.

If the term $\epsilon^2 \langle \partial_i q_2 \rangle$ which was erroneously set to zero is included with the leading term $\partial_i q_0$ and the term $\epsilon^2 \partial_i q_0$ replaced by $\epsilon^2 \partial_i (q_0 + \epsilon^2 \langle q_2 \rangle)$ with error $O(\epsilon^4)$ then the expectation of equation (8) becomes

$$\frac{h^3}{12\mu_0} \{ \partial_i (q_0 + \epsilon^2 \langle q_2 \rangle) + \epsilon^2 [3 \langle (\lambda^-)^2 \rangle \partial_i (q_0 + \epsilon^2 \langle q_2 \rangle) + 3 \langle \lambda^- \partial_i q_1 \rangle] + O(\epsilon^3) \}$$

The author's subsequent analysis is correct providing q_0 is replaced by $q_0 + \epsilon^2 \langle q_2 \rangle$ (this is the expected pressure q^* to $O(\epsilon^2)$). The flow factors can then be obtained by equating the coefficients of the gradient of q^* with those of the right hand side of equation (4).

Reference

1 Patir, N., "A Numerical Procedure for Random Generation of Rough Surfaces," *WEAR*, Vol. 47, No. 2, 1978, p. 263.

K. Tonder⁶

The author is to be congratulated for a paper that is well written. In particular the clear presentation of the roughness problem in hydrodynamic lubrication is very useful.

However, the discussor has a few objections to the paper. The first is a very minor one, and deals with the use of the term "Flow Factor Method." All models involving Reynolds-type equations relating mean pressure and averaged film heights might be termed flow factor methods with equal justification.

It is therefore suggested that the term only be used for numerically determined flow and shear quantities, as probably intended by Patir and Cheng.

The second objection is serious—the discussor cannot accept the general validity of the results obtained. The mathematical assumptions and approximations are so subtle that it may be better to argue by physical reasoning. First, as the author points out himself, the models of Elrod and of

Bush and Gibson are essentially the same as his own and hence cannot be used for mutual verification.

The perturbation method findings are thus only supported by the Patir-Cheng results to which they are rather similar, a fact that may or may not be coincidental. To assist us in the arguments, we may use the ratio λ of the mean wavelength along and across the direction of motion. We see that λ coincides with the Peklenik number γ for random roughness and equals the ratio of fixed roughness spacings in a regular pattern.

All current models for Reynolds roughness agree for $\lambda=0$ and $\lambda=\infty$. We may illustrate the deviations by considering $\lambda=1$, and, specifically, isotropic roughness which has been treated by Patir-Cheng and by the discussor [2]. Whereas the former obtain a decrease of pressure flow relative to the smooth case, the latter obtains an increase. The Patir-Cheng boundary conditions are *physically* incorrect, permitting no sideways flow at local boundaries. This clearly reduces the net through-flow in the pressure gradient direction since lubricant meeting very narrow gaps in one region is prevented from finding open channels in a neighbouring region. This problem is avoided by the use of the discussor's random boundary conditions allowing such local flow. As expected this enhances the pressure flux. This seems to support the discussor's result rather than those of Patir and Cheng.

Because of the exponent 3 of the quantity H^3 in the expression for pressure flow, the latter is considerably more increased in a wide gap (thick film) than reduced in a gap correspondingly narrow. This indicates an increased pressure flow in the presence of isotropic roughness. However, as pointed out by Patir and Cheng, this is counteracted by the lubricant flow having to pass around obstacles. The magnitude of this latter effect can be estimated from calculations on regular patterns. Configurations where the deepest channels are only available through such flowing around asperities have been described by the author [3]. The effective lengthening of the flow path should correspond reasonably well to that of isotropic roughness and indeed, the flow and friction values are rather close to the discussor's results for isotropically rough surfaces.

The above arguments affect the present paper somewhat indirectly. However, the following deals with the paper directly.

There is no reason to exclude crossed striated roughness from the structures treated by the paper. Indeed, the author himself points out that for pressure flow in the x -direction, transverse furrows on one surface will just cancel the effect of longitudinal furrows on the other if their variances are in the ratio 1 to 2.

This is found by combining the results for the two roughness structures considered separately. However, according to equat. (20) this can come about only for $\gamma=2$. The discussor fails to see how this is reflected by the expression $\gamma = \sqrt{\beta/\alpha}$, or from the expressions following equat. (14).

The above arguments seem to support the discussor's belief that the current perturbation results leave out important aspects of roughness effects. The reason could be that the assumptions and approximations of the analyses, though seemingly reasonable, may not be permissible. Another possibility is the truncation of the expansion series, despite the author's assertions to the contrary.

Additional References

2 Tønder, K., "Simulation of the Lubrication of Isotropically Rough Surfaces," *ASLE Trans.*, Vol. 23, July 1980, pp. 326-332.

3 Tonder, K., "A General Reynolds-Type Equation for Rough Surfaces, Based on Numerical Simulation," Winter Annual Meeting, ASME, Lubr. Div. Nov. 1980, pp. 193-211.

⁶The University of Trondheim, the Norwegian Institute of Technology, Trondheim, Norway.

Author's Closure

The author is grateful to the discussers for raising several important mathematical and physical points treated too briefly in the body of the paper. First, Bush and Hughes offer their hitherto unpublished flow factor results for comparison with those given here. The similarities are most encouraging but it is instructive also to examine the differences. At first glance, the asymptotic forms of the flow factors seem to agree with equations (16)–(18), particularly with regard to the appearance of the two ACFs, ρ_c and ρ_D , in determining ϕ_x and ϕ_s , respectively. The integrals I_1 and I_2 differ however from those of equation (16) in one significant respect, namely the placement of the ∂_i (or $\partial/\partial x$) operator. Integrating equation (16) by parts, ∂_i indeed “passes through” G to operate upon the ACF just as in I_1 and I_2 but the integrated term arising at Γ , the boundary of the area of integration, cannot be ignored because of the singularity in G . In fact, if the asymptotic forms of I_1 and I_2 are evaluated using the model ACF of equation (19) they diverge, reducing to equation (16) only when the boundary term is retained to compensate for this divergence. Thus, while agreeing with Bush and Hughes that the integral on Γ must be considered, it only enters explicitly in their formulation. In contrast, by using the present approach “the author has obtained correct results” while avoiding direct reference to a complementary function.

The other singularities appearing in the full (nonasymptotic) expressions for I_1 and I_2 are associated with contact spots where $h \rightarrow 0$. As mentioned under “Outlook,” contact alters the ACFs and it is certainly necessary to allow for changes in asperity shape as one contribution to this effect. The author looks forward to published details of how, beyond changing the slope, the profile and flow patterns are affected by contact.

The remaining points raised by Bush and Hughes concern the pressure boundary condition. It is gratifying to find that not only $\langle q \rangle$ but also q itself may without loss of generality be assigned a fixed value at the boundary. Taken in conjunction with the result (see Appendix) that $\langle q \rangle$ satisfies Laplace's equation, this leads to the present conclusion that it is sufficient, though admittedly not necessary, to hold $\partial_s \langle q \rangle$ at its smooth value. While depending upon the stationarity of the surface roughness, this requirement has the effect of a boundary condition. Clearly a different boundary condition could be imposed, such as holding longitudinal flow at its smooth value, but no physical objection remains to the present form, adopted as conforming most naturally to the concept of pressure flow factor.

Passing from these more mathematical comments, the discussion of Tonder deals with the physical depiction of flow between rough boundaries. Since neither the physical model nor its mathematical solution are directly questioned, what seems to be required here essentially is a qualitative picture making the behaviour of the flow factors, particularly with respect to γ , more acceptable at an intuitive level. The mathematical subtleties of the model make this no easy matter since, of course, they are provided as links at those very points in the chain of argument where intuition fails, in this case in visualizing modifications of flow in presence of random obstacles. Re-emphasis and slight extension of “Discussion” may help, however.

Tonder correctly identifies the exponent 3 of H^3 as the key to understanding random flow but then applies a non sequitur. That $\langle h^3 \rangle$ exceeds $\langle h \rangle^3$ even for rough surfaces of zero mean is a simple consequence of averaging. But it is true no matter how roughness is distributed over the surfaces and therefore sheds no light on the dependence of flow on the ACF, i.e., on γ . The essence of the pressure flow mechanism is that the random orientation of asperities requires the fluid to sample a continuum of averaging processes for h^3 , ranging from $\langle h^3 \rangle$ for pure longitudinal roughness to $\langle h^{-3} \rangle^{-1}$ in pure transverse cases. The net effect of these processes is governed by the ACF. The observation that the former average exceeds $\langle h \rangle^3$ by exactly half the amount by which the latter average is less than $\langle h \rangle^3$, loosely suggests that the transverse dimension of an asperity hinders flow twice as effectively as the longitudinal dimension enhances it. This is a partial explanation of why an asperity, or more strictly a surface, with $\gamma = 2$ has no effect on flow according to equation (20). This equation supplies, to second order, the behavior of the averaging process for h^3 in cases intermediate between pure longitudinal and transverse roughness, a behavior which only the most refined intuition could supply unaided. It is perhaps surprising, for example, that flow depends only on the ratio of two perpendicular correlation lengths and not on their individual magnitudes relative to the dimensions of the model bearing. A moment's reflection however shows that these dimensions are effectively infinite since the finite spatial average has been replaced by the ensemble average. Other details of asperity shape, while necessarily playing some role, apparently do not enter to this order of approximation. The generally weak dependence of flow factors on the detailed form of the ACF is noted already in “Discussion.”

In describing the PC boundary condition as “physically incorrect,” Tonder refers to “local boundaries” which apparently means “contacting asperities”. As such, this PC condition does over-restrict the flow and some increase in ϕ_x for any γ is to be expected. However, very little contact occurs for $H > 3$ so that this effect will certainly be negligible at larger H . It is difficult to estimate whether or not it would be large enough at any H to cause enhanced pressure flow in the isotropic case but certainly at small H the opposing effect of the connectivity of contact spots becomes overwhelmingly more important.

Some misunderstanding seems to have arisen with regard to flow factors in presence of two-sided roughness. Following equation (14) it is shown that the pressure factor splits into the sum of two parts when the two roughnesses are uncorrelated. Clearly an equation analogous to equation (22) for ϕ_s will then apply also to ϕ_x , replacing subscript s by x and the difference by the sum. It will involve two independent γ values, one from each surface, and it was such an expression that was used in the statement of the effect of crossed furrows under “Discussion.” To describe a two-rough-surface case by a single γ parameter, we might introduce the definition $(\gamma + 1)^{-1} \equiv (\gamma_1 + 1)^{-1} (\sigma_1/\sigma)^2 + (\gamma_2 + 1)^{-1} (\sigma_2/\sigma)^2$, which correctly yields $\gamma = 2$ in the crossed furrows case.

Finally we return to Tonder's first objection concerning use of the term “flow factor method.” Not willing to speculate upon what PC intended for this term, the author is nevertheless agreeable to a change in the title of this paper. Let the definite article be changed to the indefinite and the compromise should be acceptable to all parties.