The $K_{\pi}$ Decay Form Factors in the Broken $SU(3) \times SU(3)$ Model

Subir Bose and P. Narayanaswamy

Department of Physics and Astronomy
Southern Illinois University at Carbondale, Carbondale, Illinois

*Physics Faculty
Southern Illinois University at Edwardsville, Edwardsville, Illinois

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The $K_{\pi}$ form factors are calculated in the $(3^*, 3) + (3, 3^*)$ model of broken chiral $SU(3) \times SU(3)$, without invoking any perturbative treatment of the symmetry breaking. The results are compared with other theoretical calculations and with available experimental data.

§ 1. Introduction

Some time ago, Weinberg$^1$ demonstrated that reasonable predictions can be obtained for the axial vector form factors describing the decay $K \rightarrow 2\pi + l + \nu$ on the basis of the current commutation relations and the hypothesis of partially conserved axial vector current (PCAC). Since then the $K_{\pi}$ form factors have been the subject of several calculations.$^2$ More recently, the broken chiral $SU(3) \times SU(3)$ symmetry has been used to study the effects of symmetry breaking in the $K_{\pi}$ decay.$^3,4,5$ These calculations employ a perturbative treatment around the $SU(3) \times SU(3)$ symmetric limit through the neglect of $O(\varepsilon^2)$ terms, $\varepsilon$ being the symmetry breaking parameter. However, the possibility of an enhancement from the higher order symmetry breaking terms cannot altogether be ruled out.$^6$ Despite the fact that the currently available data on the $K_{\pi}$ form factors are uncertain due to large experimental errors, it is important to know whether the higher order terms are enhanced and of such a sign as to spoil the agreement of the $O(\varepsilon)$ calculation with experiment; since, if it were so, it would indicate that a perturbative treatment of broken $SU(3) \times SU(3)$ is suspect and the agreement of the predictions to $O(\varepsilon)$ is rather fortuitous.

The $K_{\pi}$ matrix element plays a crucial role in the analysis of the $K_{\pi}$ form factors.$^5$ The calculations cited earlier$^5,6$ employ an approximate expression for

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$^1$ Since Weinberg's work, a number of calculations using hard pion current algebra, vector and axial vector meson dominance and chiral dynamics have been carried out.$^7$

$^2$ Gaillard$^5$ concludes that deviations in $K_{\pi}$ decay (i.e., a large $\varepsilon$ value) would imply large deviations from current algebra predictions for $K_{\pi}$ decay.

$^3$ Such a possibility has been shown to be a reality in the case of $K_{\pi}$ decay.$^9$
the $K_{\pi}$ matrix element valid to $O(\varepsilon)$, derived by Dashen and Weinstein. An essential point of Dashen and Weinstein's calculation is that the meson pole dominance of the divergence of the axial vector current is valid up to symmetry breaking terms $O(\varepsilon^2)$. This is the source for the perturbative calculation in various orders of $\varepsilon$. Furthermore, their analysis did not consider any specific model of symmetry breaking and thus the "$\sigma$-commutator" term contribution had to be ignored.

More recently, it has been shown by Deshpande that general low energy theorems on the $K_{\pi}$ form factors can be derived on the basis of the $(3, 3^*) + (3^*, 3)$ symmetry breaking model proposed by Gell-Mann, Oakes and Renner and the partial conservation of vector and axial vector currents. The results obtained are valid to second order in momenta but to all orders in the symmetry breaking. Deshpande's claim that the results are correct to all orders in symmetry breaking is, however, valid only if PCAC is taken to be an exact statement, i.e., in the usual sense of an operator identity. This viewpoint is different from what is invoked by Dashen and Weinstein, viz., that PCAC involves the neglect of $O(\varepsilon^2)$ terms. However, apart from this assumption, no perturbative development in $\varepsilon$ is necessary and the results are true to all orders in $\varepsilon$. Furthermore, the results of Dashen and Weinstein are reproduced when certain appropriate limits are taken.

In this paper we extend Deshpande's procedure to a calculation of the $K_{\pi}^4$ form factors. Since we use the $(3, 3^*) + (3^*, 3)$ model of broken symmetry, we do retain the contributions arising from the $\sigma$-commutators. Although the calculations of Refs. 3 and 4 employ the model independent results of Dashen and Weinstein for the $K_{\pi}$ form factors, they have to resort to the GMOR model in the $K_{\pi}$ calculation. In contrast, we employ the GMOR model to calculate even the $K_{\pi}$ matrix element which enters in the analysis of the $K_{\pi}$ decay matrix element.

§ 2. The $K_{\pi}$ Form Factors

The axial vector $K_{\pi}$ form factors are defined by

$$
\langle \pi^+(p)\pi^-(q)|A_{\pi}^{K-}|K^+(k)\rangle
= (i/m_k) \left[ (p+q)_\mu F_1 + (p-q)_\mu F_2 + k'_\mu F_3 \right],
$$

where $m_k$ is the $K$-meson mass and $k'=k-p-q$. The form factors are related to the off-shell $K\pi$ scattering amplitude

$$
T^{K_{\pi}}(q, p, k, k') = \frac{m_p^2 - p^2}{F_1} (m_s^2 - q^2) \frac{m_s^2 - k^2}{m_p^2 - m_k^2} \frac{m_k^2 - k'^2}{F_2}
\times \int d^4x d^4y d^4z \exp(-ip\cdot x - iq\cdot y + ik\cdot z)
\times \left\langle D_\pi^{\star}(x) D_\pi^{\star}(y) D_\pi^{\star}(z) D_\pi^{\star}(0) \right\rangle_0
= A^+ - A^-,
$$

where $P$ is the Pauli operator.
The $K^i_4$ Decay Form Factors in the Broken $SU(3) \times SU(3)$ Model

where $A^\pm (s, u, t; p^2, q^2, k^2, k'^2)$ are the $t$-channel charge conjugate eigen amplitudes and PCAC is assumed:

$$D_i^t(x) = \partial_\mu A_\mu^t(x) = F_m x_i^t \phi_i(x),$$

(3)

$F_k$ denoting the $K$-meson decay constant. The $A^\pm$ amplitudes satisfy the crossing relations

$$A^\pm (s, u, t; p^2, q^2, k^2, k'^2) = \pm A^\pm (u, s, t; q^2, p^2, k'^2, k^2),$$

(4)

where

$$s = (k - q)^2, \quad u = (k - p)^2, \quad t = (p + q)^2.$$

The symmetry breaking is assumed to be described by the GMOR form:

$$H = H_0 - u_0 - c_u s$$

(5)

and the divergence relations following Eq. (5) are

$$\partial_\mu V_\mu^t(x) = D_i^t(x) = cf (k \cdot v_k(x),$$

$$\partial_\mu A_\mu^t(x) = D_s^t(x) = - (d_{ik} + c d_{ik}) v_k(x),$$

(6)

where $u_i$ and $v_i$ are scalar and pseudoscalar nonets belonging to the $(3, 3^*) + (3^*, 3)$ representation of $SU(3) \times SU(3)$. The charges

$$Q^t = \int V_i^t(x) d^3 x,$$

and

$$Q_s^t = \int A_i^t(x) d^3 x$$

(7)

satisfy the $SU(3) \times SU(3)$ current commutation relations. We will have occasion to use the following commutation relations:

$$[Q_i^s, D_i^t] = \frac{2\sqrt{2} - c}{3c} D_i^t,$$

$$[Q_i^s, D_s^t] = \frac{2c + 2\sqrt{2}}{3c} D_i^t.$$

(8)

By standard reduction techniques, the $K^i_4$ form factors can be related to the three-particles-on-shell limit of the $I = 3/2 K\pi$ amplitude:

$$A^+ (s, u, t; m^2_s, m^2_x, m^2_k, k^2) - A^- (s, u, t; m^2_s, m^2_x, m^2_k, k'^2)$$

$$= \frac{m^2_x - k^2}{F_k m^2_x} \langle \pi^+ (p) \pi^- (q) | D_s^t (0) | K^+ (k) \rangle$$

$$= \frac{m^2_x}{F_k m^2_x} \{ k' \cdot (p + q) F_t + k \cdot (p - q) F_t + k'^2 F_t \}.$$

(9)
We now appeal to the Weinberg-Khuri\(^{(10)}\) method and expand the amplitudes \(A^\pm\) in powers of the invariants, retaining terms up to quadratic in the invariants and satisfying crossing symmetry:

\[
A^+ = A + B(s+u) + Ct + D(s+u)^3 + Esu + Ft^2
+ Gt(s+u) + H(p^2 + q^2) (k^2 + k'^2),
\]

\[
(10)
\]

\[
A^- = \alpha(s-u) + \beta t(s-u) + \gamma(s^2 - u^2) + \delta(s-u)(p^2 + q^2)
+ \delta'(p^2 - q^2) (k^2 - k'^2).
\]

\[
(11)
\]

It is noted that the expansion (11) was used in Ref. 3) to determine the form factor \(F_2\). In the three-particle-on-shell limit, one has the kinematical relations

\[
k'^2 = s + u + t - 2m^2 - m_k^2,
2k' \cdot (p+q) = 2m^2 + 2m_k^2 - s - u - 2t,
2k' \cdot (p-q) = s - u.
\]

\[
(12)
\]

Utilizing relations (10), (11) and (12) in Eq. (9) and assuming the form factors to be constants, we compare the coefficients of \(s, u, t,\) etc., and obtain the following:

\[
A - 4m^4 H = 2F_1'(m^2 + m_x^2)(m^2 + 2m_x^2) - 2F_1'(m^2 + m_x^2)^2,
B + 2m^2 H = -F_0'(3m^2 + 4m_x^2) + 2F_1'(m^2 + m_x^2),
C + 2m^2 H = 3F_1'(m^2 + m_x^2) - F_0'(3m^2 + 4m_x^2),
D + \gamma = F_2' - F_2'/2,\quad E = 0,
2G = 4F_2' - 3F_2',\quad F = F_2' - F_2',
\frac{1}{2} F_2' m^2 = \alpha + 2\gamma m^2 + \gamma m_k^2 + 2\delta m_x^2,
2G - 2\beta = -3F_2' + F_2' + 4F_2',
\]

\[
(13)
\]

where

\[
F_2' = F_2 / (F_k m_k^2).
\]

In order to determine the form factors we proceed to compute the expansion coefficients \(A, B,\) etc. For this purpose, we consider various on shell limits of the amplitude in Eq. (2) and obtain the relations

\[
A^+(k^2, m^2, m_x^2, m_k^2, 0, k^2, m_k^2) = A^- (k^2, m^2, m_x^2, m_k^2, 0, k^2, m_k^2),
\]

\[
(14)
\]

\[
A^-(k^2, k^2, m_x^2, 0, m_x^2, k^2, m_k^2) - A^- (m^2, k^2, m_x^2, 0, m_x^2, k^2, m_k^2)
= \frac{-i(m^2 - k^2)}{F_1 F_k m_k^2} \frac{2\sqrt{2} - C}{3c} \langle \pi^- (q) | D K^- | K^- (q - k) \rangle,
\]

\[
(15)
\]

\[
A^+(p^2, m^2, m_k^2, p^2, m_x^2, m_k^2, 0) = A^- (p^2, m^2, m_k^2, p^2, m_x^2, m_k^2, 0),
\]

\[
(16)
\]

\[
A^+(m^2, p^2, m_k^2, p^2, m_x^2, 0, m_k^2) - A^- (m^2, p^2, m_k^2, p^2, m_x^2, m_k^2, 0, m_k^2)
\]

\[
(17)
\]
The $K_{14}$ Decay Form Factors in the Broken SU(3) $\times$ SU(3) Model

\begin{equation}
\frac{i(m^2 - q^2)}{F_X F_X m^2} \frac{2c + 2\sqrt{2}}{3c} \langle \pi^- (q) | D^{K^0} | K^- (p + q) \rangle.
\end{equation}

We now focus our attention on the matrix element

\[ \langle \pi^- (q) | D^{K^0} | K^- (q - k) \rangle. \]

References 3) and 4) employ an approximate expression for this matrix element derived by Dashen and Weinstein, which is valid to $O(\varepsilon)$, regardless of the model of symmetry breaking. However, since the GMOR model needs to be used in the calculation\(^{(5),(4)}\) we believe that it is of no special advantage to resort to a model independent result for this matrix element alone. With this viewpoint, we employ a different approach to compute this matrix element, based on the GMOR model of symmetry breaking. The method is that used by Deshpande\(^6\) in the analysis of $K_{14}$ form factors.

We assume that the matrix element of the divergence of the vector current is dominated by the $\pi$-meson pole:\(^{(8)}\)

\[ \langle \pi^- (q) | D^{K^0} | K^- (q - k) \rangle = iG(k^2)F_X m^2 / (m^2 - k^2), \]

where $F_X$ and $m_X$ are the decay constant and mass of the $\pi$-meson. We assume that $G(k^2)$ can be expanded as

\[ G(k^2) = g_0 + k^2 g_1 + \cdots \]

since it has no poles.\(^{(9)}\) Following Dashen and Weinstein, we also write

\[ \langle \pi^- (q) | D^{K^0} | K^- (q - k) \rangle = i(a_0 + k^2 a_1). \]

From the relations (18) and (19), on comparing coefficients, we have

\[ a_0 = g_0 F_X, \]

\[ a_1 = (g_0 + g_1 m^2) F_X / m^2. \]

To determine $g_0$ and $g_1$, we consider the off-shell continuation of (18) to obtain

\[ \frac{F_X m^4}{m^2 - k^2} G(q^2, (q - k)^2, k^2) \]

\[ = \left[ (q - k)^2 - q^2 \right] f_+(q^2, (q - k)^2, k^2) + k^2 f_-(q^2, (q - k)^2, k^2) \]

\[ + \left( m^2 - q^2 \right) \left( m^2 - (q - k)^2 \right) / (F_X F_X m^2 m^2) \]

\[ \times \left\{ \frac{2\sqrt{2} - c}{2\sqrt{2} + 2c} \int \frac{m^4 p_x^{(0)}(m^2)}{m^2 - q^2} dm^2 - \frac{2\sqrt{2} + 2c}{2\sqrt{2} - c} \int \frac{m^4 p_x^{(0)}(m^2)}{m^2 - (q - k)^2} dm^2 \right\}, \]

where $f_\pm$ are the off-shell $K_{14}$ form factors and $p_x^{(0)}$ and $p_y^{(0)}$ are the spin-0 part of the spectral functions. Assuming $G(q^2, (q - k)^2, k^2)$ to be a sufficiently smooth function in the region of interest, we arrive at\(^{(6)}\)

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\(^{(4)}\) The kappa meson dominance of the divergence of the strangeness changing vector current is now widely used.\(^{(11)}\)
\( G(q^2, (q-k)^2, k^2) \approx \frac{G(0,0,0) + k^4 \frac{\partial G(0,0,0)}{\partial k^2}}{} + q^2 \frac{\partial G(0,0,0)}{\partial q^2} + (q-k)^2 \frac{\partial G(0,0,0)}{\partial (q-k)^2} \). (24)

We may further set \( q^2 = (q-k)^2 = 0 \) since we are interested merely in the \( k^4 \) dependence. From (19), (21), (22), (23) and (24), we obtain, using pole saturation of the spectral functions:*)

\[
a_0 = \frac{2\sqrt{2}-c}{2\sqrt{2}+2c} \frac{m^4_{\pi} F_{\pi}/F_{\pi}}{m^4_k F_k/F_k}, \]
\[
a_1 = \frac{1}{2} \left[ F_k/F_{\pi} - F_{\pi}/F_k + \frac{4\sqrt{2}+c}{3c} \frac{F_{\pi}^2}{F_k F_k} \right]. \quad (25)
\]

In deriving this we have also used the relation

\[
f_{\pi}(0,0,0) = \left( F_{\pi}^2 - F_{\pi}^2 + \frac{4\sqrt{2}+c}{3c} \frac{F_{\pi}^2}{F_k F_k} \right) \right] 2F_{\pi} F_k
\]

derived in Ref. 8). The coefficients \( a_0 \) and \( a_1 \) obtained above, determine the matrix element (20) up to terms quadratic in the momentum.

We now proceed to use Eq. (20) in the relations (14) \sim (17). On comparing the coefficients, we arrive at the following relations:

\[
A + B m_k^2 + C m_{\pi}^2 + D m_{\pi}^4 + F m_{\pi}^4 + m_k^2 m_k^2 (G + H)
\]

\[
= - (a m_k^2 + \beta m_{\pi}^2 m_k^2 + \gamma m_{\pi}^4 + \delta m_k^2 m_k^2 + \delta' m_{\pi}^2 m_k^2)
\]
\[
= (2\sqrt{2} - c) m_k^2 a_0 / (6c m_k F_{\pi} F_k), \quad (26)
\]

\[
B + 2D m_k^2 + E m_{\pi}^2 + (G + H) m_{\pi}^2
\]

\[
= \alpha + \beta m_{\pi}^2 + m_{\pi}^2 (\delta + \delta')
\]
\[
= (2\sqrt{2} - c) (a_1 m_{\pi}^2 - a_0) / (6c m_{\pi} F_{\pi} F_k), \quad (27)
\]

\[
D = \gamma = (c - 2\sqrt{2}) a_1 / (6c m_{\pi} F_{\pi} F_k), \quad (28)
\]

\[
A + B m_{\pi}^2 + C m_k^2 + D m_k^2 + F m_k^4 + (G + H) m_k^2
\]

\[
= - (a m_k^2 + \beta m_{\pi}^2 m_k^2 + \gamma m_{\pi}^4 + (\delta + \delta') m_k^2 m_k^2)
\]
\[
= (2\sqrt{2} + 2c) m_k^2 a_0 / (6c m_{\pi} F_{\pi} F_k), \quad (29)
\]

\[
B + 2D m_{\pi}^2 + E m_k^2 + (G + H) m_k^2
\]

\[
= \alpha + \beta m_k^2 + (\delta + \delta') m_k^2
\]
\[
= (2\sqrt{2} + c) (a_1 m_k^2 - a_0) / (6c m_k F_{\pi} F_k)
\]
\[
= (2\sqrt{2} + c) (a_1 m_k^2 - a_0) / (6c m_k F_{\pi} F_k)
\]
\[
(30)
\]

and

\[
D = -(2\sqrt{2} + 2c) a_1 / (6c m_{\pi} F_{\pi} F_k). \quad (31)
\]

*) The decay constants \( F_{\pi}, F_k \) and \( F_{\pi} \) are defined by

\[
\langle 0 | A_{\pi}^\pi | \pi^+(p) \rangle = i F_{\pi} p_{\pi}, \quad \langle 0 | A_{\pi}^k | k^+(p) \rangle = i F_k p_{\pi} \quad \text{and} \quad \langle 0 | V_{\pi}^k | k^0(p) \rangle = \sqrt{2} i F_k p_{\pi}.
\]
The $K_\mu$ Decay Form Factors in the Broken SU(3) $\times$ SU(3) Model

From the relations (28) and (31), we have

$$m_k^2/m_\pi^2 = (2\sqrt{2} - c)/(2c + 2\sqrt{2}),$$

which enables us to determine the value of $c$:

$$c = \sqrt{2}(m_\pi^2 - m_k^2)/(m_k^2 + \frac{1}{2}m_\pi^2).$$

(32)

Using observed values of $m_\pi$ and $m_k$, this gives $c = -1.25$. Comparing this with the derivation in Ref. 3), it should be observed that (32) does not involve the neglect of any higher order symmetry breaking effects and is a result of our derivation valid to second order in the invariants. This value is also observed to agree with the conclusion of GMOR.

Finally, we can now combine relations (13) and (26) $\sim$ (31). Eliminating the constants $A$, $B$... and $\alpha$, $\beta$... we obtain the $K_\mu$ form factors:

$$F_1 = -\frac{m_k(2\sqrt{2} - c)}{3cF_\pi(2m_k^2 - m_\pi^2)}[2a_0 + 3a_1m_\pi^2],$$

$$F_2 = (m_k/F_\pi)[a_0/(m_k^2 - m_\pi^2) - a_1(2c + 2\sqrt{2})/3c],$$

$$F_3 = -\frac{m_k(2\sqrt{2} - c)}{3cF_\pi(2m_k^2 - m_\pi^2)}[a_0 + a_1(m_\pi^2 + m_k^2)],$$

(33)

where $a_0$ and $a_1$ are determined by (25) and (32).

For numerical estimates, we use $F_\pi m_k \approx 2.73$, $F_k/F_\pi \approx 1.3$, $F_\pi/F_\pi = 0.3$. These give

$$F_1 \approx 3.56, \quad F_2 \approx 3.4, \quad F_3 \approx 2.5.$$ (34)

It should be stressed at this point that the calculations of the form factors have been carried out under the assumption of constant form factors. As has been pointed out by Weinberg, although $F_1$ and $F_2$ can safely be assumed to be constants, the form factor $F_3$ is a rapidly varying function of $k'^2$ due to a $K$-meson pole. This pole does not appear in $F_1$ or $F_2$ and there is therefore no reason to believe that $F_1$ or $F_2$ would vary rapidly. We may thus explicitly incorporate the $K$-meson pole structure in $F_3$ and rewrite Eq. (9) as

$$A^+(s, u, t; m_\pi^2, m_\pi^2, m_k^2, k'^2) - A^-(s, u, t; m_\pi^2, m_\pi^2, m_k^2, k'^2)$$

$$= \frac{(k'^2 - m_k^2)}{F_km_k^2} \left[(k' \cdot (p + q)F_3 + k' \cdot (p - q)F_2 + \frac{k'^2F_3m_k^2}{(m_k^2 - k'^2)}\right],$$ (35)

where $F_3$ corresponds to the residue of $F_3$ at $m_k^2$. It should be observed that the structure used above for $F_3$ is merely an ansatz similar to an assumption made in a calculation of the $K_\mu$ form factors using Veneziano model. The calculation for the form factors can now be repeated using (35) instead of (9). The final results are

$$F_1 = F_2 = (m_k/F_\pi)[a_0/(m_\pi^2 - m_k^2) - a_1(2c + 2\sqrt{2})/3c]$$
and
\[
\bar{F}_3 = -\frac{(2\sqrt{2}-\epsilon)}{3cF_km_k}[a_0 + (m^2 + m_k^2)a_i]. \tag{36}
\]

The corresponding numerical estimates become
\[
F_1 = F_2 = 3.4, \quad \bar{F}_3 = 4.8. \tag{37}
\]

§ 3. Discussion

It is of interest to compare the results obtained here for the $K\pi_4$ form factors with those of the $O(2)$ calculations of Lane.\textsuperscript{(b)} Lane's values are
\[
F_1 = F_2 = 3.95, \quad F_3 = -10.1 \frac{(k' \cdot p)}{m^2 - k'^2}. \tag{38}
\]

Our results for $F_1$ and $F_2$ are thus only slightly different from that of Lane, although somewhat smaller. The latter fact may indicate, as was stated in the introduction, that the higher order symmetry breaking effects are of such a sign as to decrease the value of the form factors. The calculation of deAlwis predicts only $F_2$ and the result is the same as in (38).

The values for $F_1$ and $F_2$ obtained by Weinberg,\textsuperscript{(b)} from current algebra, are
\[
F_1 = F_2 = 3.7
\]
although some of the other calculations listed in Ref. 2) have values of $F_1$ and $F_2$ considerably larger than those obtained from $SU(3) \times SU(3)$ calculations.

Experimentally, practically nothing is known about $F_3$. For the form factors $F_1$ and $F_2$, by fitting the experimental data, Behrends et al.\textsuperscript{(b)} find
\[
F_1 = 5.6 \pm 0.6, \quad F_2 = 5.5 \pm 1.2. \tag{39}
\]
The other fit by Ely et al.\textsuperscript{(b)} has one of their solutions roughly consistent with the fit of Behrends et al. At any rate, the experimental results do not seem to be very conclusive to date and hence a serious comparison of theory with experiment for the $K\pi$ form factors must only be tentative.

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References

The $K_{4\pi}$ Decay Form Factors in the Broken $SU(3) \times SU(3)$ Model

9) M. Gell-Mann, R. J. Oakes and B. Renner, Phys. Rev. 175 (1969), 2195; to be referred to as GMOR in the text.
    see also Refs. 6) and 8).
    Experimentally, there is some evidence of a $0^+ \not{K}\pi$ resonance around 1100 MeV: see T. G. Trippe et al., Phys. Letters 28B (1963), 203.
12) The values of $F_{k}/F_{\pi}$ and $F_{k}/F_{\pi}$ are taken from the work of Y. Y. Lee, Nuovo Cim. 64A (1969), 474.