On a Probabilistic Interpretation in the Theory with Indefinite Metric

Kan-ichi YOKOYAMA

Research Institute for Theoretical Physics
Hiroshima University
Takehara, Hiroshima-ken

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An immediate difficulty in the theory with indefinite metric is the appearance of negative probability, or the breakdown of the unitarity of the physical S-matrix. The conventional way to avoid such a difficulty aims at eliminating fictitious particles somehow from the whole S-matrix so as to lead to a unitary physical S-matrix. In the regularization theory, however, there is another kind of approach to avoid negative probability. In this paper we briefly study it. The present approach is essentially based on Bogolyubov's unitarization mechanism, but regularizing fields introduced can be treated by it as real fields describing physically possible particles without destroying unitarity. At first sight, like Bogolyubov's case, circumstances about causality seem to be unfavourable. In the present stage, however, no definite answer is given to causality.

We here take simple case in which an interaction Lagrangian $L_{\text{Int}}(A(x))$ with a neutral scalar field $A(x)$ is replaced by a regularized form $L_{\text{Int}}(A(x)+B(x))$ with a regularizing neutral scalar field $B(x)$. The field $A(x)$ satisfies the usual commutation relation, while the field $B(x)$ the anomalous one with an opposite sign to that of $A(x)$. The masses of $A(x)$ and $B(x)$ are different from each other. If we obtain the $S$-matrix of this system by the usual procedure, we cannot help dealing with negative probability due to the appearance of external $B$-particles.

Suppose now that the initial state of scattering always consists of two parts, one of which contains an even number of $B$-particles, say $|\text{in}(+)\rangle$, and the other of which contains an odd number of $B$-particles, say $|\text{in}(-)\rangle$:

$$|\text{in}\rangle = |\text{in}(+)\rangle + |\text{in}(-)\rangle. \quad \text{(1)}$$

Then, the final state of scattering is also composed of two parts $|\text{out}(+)\rangle$ and $|\text{out}(-)\rangle$:

$$|\text{out}\rangle = |\text{out}(+)\rangle + |\text{out}(-)\rangle. \quad \text{(2)}$$

The state $|\text{in}(-)\rangle$ and $|\text{out}(-)\rangle$ have negative norms.

Such an assumption for the composition of the initial state need not be realistic, and we understand it only as a heuristic step. A realistic interpretation will be given in the later step.

The essential difficulty of the negative probability problem comes from the fact that the unitarity of the $S$-matrix holds as a whole in the mixing of $|\text{(+)\rangle}$ and $|\text{(-)\rangle}$ states. In order to separate the roles of these two kinds of states, we apply the Bogolyubov method. The states $|\text{out}(+)\rangle$ and $|\text{out}(-)\rangle$ are generally expressed in the form

$$|\text{out}(+)\rangle = S_1|\text{in}(+)\rangle + S_2|\text{in}(-)\rangle, \quad \text{(3)}$$

$$|\text{out}(-)\rangle = S_1|\text{in}(+)\rangle + S_2|\text{in}(-)\rangle,$$

where $S_1 = (S_1 + S_2)/2$ and $S_2 = (S_1 - S_2)/2$, and $S_1$ is the formal $S$-matrix obtained by the usual procedure with the interaction Lagrangian $L_{\text{Int}}(A+B)$ while $S_2$ with $L_{\text{Int}}(A-B)$. Equation (3) is a direct consequence of $|\text{out}\rangle = S_1|\text{in}\rangle$.

We now apply Bogolyubov's nonlocal boundary condition in the form

$$|\text{out}(-)\rangle = U|\text{in}(-)\rangle, \quad \text{(4)}$$

where $U$ is an unknown unitary operator which conserves the even-odd character in its operation to states. Since $S_1$ and $U$ are unitary, $|\text{out}(+)\rangle$ and $|\text{in}(+)\rangle$ are connected with each other by unitary ope-
Substitution of (4) into (3) leads to
\[
|\text{out}(+)\rangle = \hat{S}|\text{in}(+)\rangle,
\]
where we have taken \(U = -1\) for simplicity. Although there may remain a problem to determine the explicit form of \(U\), we do not here touch upon it further.

If we apply another boundary condition
\[
|\text{out}(+)\rangle = U|\text{in}(+)\rangle
\]
instead of (4), we have
\[
|\text{out}(-)\rangle = \hat{S}|\text{in}(-)\rangle
\]
with the same \(\hat{S}\). Therefore, \(\hat{S}\) is unitary not only in the space of \(|(+)\rangle\) but also in that of \(|(-)\rangle\), that is, unitary over the entire vector space.

Once such an operator \(\hat{S}\) has been found, it enables us to take the following scheme: We can disregard the condition (4) or (6) and define \(\hat{S}\) as a physical \(S\)-matrix for arbitrary initial states. Since \(\hat{S}\) is invariant under the transformation \(B(x) \to -B(x)\), it conserves the even-odd character of states and gives
\[
\begin{align*}
|\text{out}(+)\rangle &= \hat{S}|\text{in}(+)\rangle, \\
|\text{out}(-)\rangle &= \hat{S}|\text{in}(-)\rangle,
\end{align*}
\]
separately for an initial state, the even-odd composition of which is arbitrary.

In the above equations the norm of the even states is positive definite, while that of the odd states negative definite. Therefore, we can take a probabilistic interpretation such that the norm of a state \(A\langle (+)\rangle + B\langle (-)\rangle\) is newly defined by
\[
|A|^2\langle (+)\rangle\langle (+)\rangle - |B|^2\langle (-)\rangle\langle (-)\rangle.
\]
By this definition, we can treat all states in a space with positive metric, and can interpret the regularizing field \(B(x)\) as one describing a physically possible particles as well as the field \(A(x)\).