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Discussion

Extended Kane's Equations for Nonholonomic Variable Mass System[1]

Y. Pironneau[2]. The paper presents a generalization of Kane's equations to nonholonomic variable mass systems. The authors claim that the new equations are simpler than any of the others and give an illustrative example.

However it seems to the writer that even for the simple system of Fig. 1 the method leads to more complicated results. The (ideal) constraint

$$x^2 + y^2 = a^2$$

gives

$$x\dot{x} + y\dot{y} = 0.$$

Following Kane's method we add the linear relation

$$\alpha\dot{x} + \beta\dot{y} = u$$

supposedly to simplify that α and β are (arbitrary) constants. We have to calculate $\dot{x}$, $\dot{y}$, $\ddot{x}$, $\ddot{y}$ and

$$\mathbf{F}(P) \cdot \hat{\mathbf{V}}(P) = -mg \times (\beta x - \alpha y)^{-1}\hat{u}$$

$$ma(P) \cdot \hat{\mathbf{V}}(P) = m(\ddot{x}\dot{x} + \ddot{y}\dot{y}) = m[\ . \ . \ .]\hat{u}.$$

Finally we obtain the equation of motion

$$(x^2+y^2)(\beta x - \alpha y)^{-1}[\dot{u} + (\alpha x + \beta y)(\beta x - \alpha y)^{-2}u^2] + gx = 0.$$

(The result is more complicated of α and β are considered to be functions of x and y.)

This does not seem simpler than the equation obtained when using, for instance, Lagrange's multiplier

$$m\ddot{x} = \lambda x$$

$$m\ddot{y} = -mg + \lambda y$$

and, eliminating λ,

$$y\ddot{x} - x\ddot{y} = gx.$$

For a system with n generalized coordinates $q(q_i, i = 1, \ . \ . \ . \ , n)$ and m ideal constraints represented by m linear nonholonomic equations

$$a_{si}(q,t)\dot{q}_i = b_s(q,t) \quad (s = 1, \ . \ . \ . \ , m) \tag{1}$$

we can choose $p = n - s$ coordinates q_α as principal, the s others designated by q_j being secondary. The equations (1) can be written in the matrix form

$$A(q_\alpha, q_j, t)\underline{\dot{q}}^{(j)} + B(q_\alpha, q_j, t)\underline{\dot{q}}^{(\alpha)} = \underline{b}(q_\alpha, q_j, t) \tag{2}$$

$$\underline{\dot{q}}^{(j)} = -A^{-1}B\underline{\dot{q}}^{(\alpha)} + A^{-1}\underline{b} \tag{3}$$

The Euler-Lagrange equation of motion is

$$(E_i - Q_i - C_i)\hat{\dot{q}} = 0 \quad (i = 1, \ . \ . \ . \ , n) \tag{4}$$

where

$$E_i\dot{q} \triangleq \Sigma\mathbf{a}(P)m(P) \cdot \frac{\partial \mathbf{P}}{\partial q_i}\dot{q}_i,$$

$Q_i\hat{\dot{q}}_i$ is the virtual power of the given forces, and

$C_i\hat{\dot{q}}_i$ is the virtual power of the constraining forces, with

$$C_i\hat{\dot{q}}_i = C_\alpha\hat{\dot{q}} + C_j\hat{\dot{q}}_j.$$

This quantity is equal to zero if, after (3),

$$\underline{\hat{\dot{q}}}^{(j)} = -A^{-1}B\underline{\hat{\dot{q}}}^{(\alpha)} \tag{5}$$

In this case (4) can be rewritten

$$[\underline{E}^{(\alpha)} - \underline{Q}^{(\alpha)}]^T\underline{\hat{\dot{q}}}^{(\alpha)} + [\underline{E}^{(j)} - \underline{Q}^{(j)}]^T[-A^{-1}B\underline{\hat{\dot{q}}}^{(\alpha)}] = \underline{0} \tag{6}$$

and we obtain $p = n - m$ equations of motion

$$\underline{E}^{(\alpha)} - \underline{Q}^{(\alpha)} - [A^{-1}B]^T(\underline{E}^{(j)} - \underline{Q}^{(j)}) = \underline{0} \tag{7}$$

which, using (3), gives p differential equations of the second order in q_α whose coefficients are functions of q_α, q_j, and t. In some particular and very interesting cases these coefficients are only functions of q_α and t.

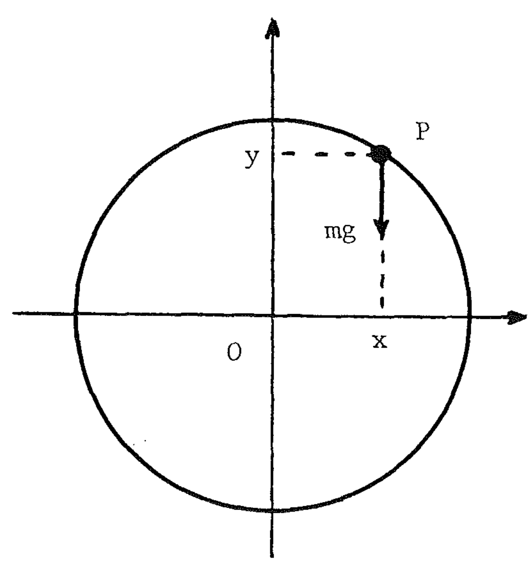

Fig. 1

[1] By Z. M. Ge and Y. H. Cheng, and published in the June, 1982 issue of the ASME JOURNAL OF APPLIED MECHANICS, Vol. 49, pp. 429–431.

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According to Kane's method, we have to add to the relations (2) p linear equations in order to obtain

$$\left|\begin{array}{c:c} A & B \\ \hdashline X & Y \end{array}\right| \left|\begin{array}{c} \dot{q}^{(j)} \\ \hdashline \dot{q}^{(\alpha)} \end{array}\right| = \left|\begin{array}{c} b \\ \hdashline u \end{array}\right| \quad (8)$$

or

$$\dot{\underline{q}}^{(j)} = -A^{-1}B\dot{\underline{q}}^{(\alpha)} + A^{-1}\underline{b} \quad (9)$$

$$\dot{\underline{q}}^{(\alpha)} = [Y - XA^{-1}B]^{-1}[u - XA^{-1}b], \quad (10)$$

and, for virtual velocities,

$$\hat{\dot{\underline{q}}}^{(\alpha)} = [Y - XA^{-1}B]^{-1}\hat{\underline{u}} \quad (11)$$

$$\dot{\underline{q}}^{(j)} = -A^{-1}B[Y - XA^{-1}B]^{-1}\hat{\underline{u}}. \quad (12)$$

Then the relations (6) become

$$[(\underline{E}^{(\alpha)} - \underline{Q}^{(\alpha)})^T + (\underline{E}^{(j)} - \underline{Q}^{(j)})^T(-A^{-1}B)][Y - XA^{-1}B]^{-1}\hat{\underline{u}} = \underline{0} \quad (13)$$

and we obtain p equations of motion

$$[(Y - XA^{-1}B)^{-1}]^T[\underline{E}^{(\alpha)} - \underline{Q}^{(\alpha)} - (A^{-1}B)^T(\underline{E}^{(j)} - \underline{Q}^{(j)})] = \underline{0} \quad (14)$$

which, using (9) and (10), give p differential equations of the first order in u but with coefficient functions of q_α, q_j and t. It seems difficult to consider (14) as simpler than (7).

For the example given in (1) there are eight parameters but only two, q_3 and q_4, appear explicitely in the six constraint equations (2). The authors choose two complementary equations

$$\dot{q}_4 = u_1 \quad (15)$$

$$\cos(q_3 + q_4)\dot{q}_1 + \text{Sin}(q_3 + q_4)\dot{q}_2 = u_2 \quad (16)$$

and obtain two differential equations of the first order in u_1 and u_2 with coefficient functions of q_3 and q_4.

In fact (16) can be rewritten (L being a constant) as

$$L(\sin\ q_4)^{-1}\cdot\dot{q}_3 = u_2. \quad (17)$$

Taking $q_1{}^{(\alpha)} = q_3$ and $q_2{}^{(\alpha)} = q_4$, the relations (15) and (17) correspond in (8) to

$$X = 0 \text{ and } Y = \begin{vmatrix} L(\sin q_4)^{-1} & 0 \\ 0 & 1 \end{vmatrix}.$$

It would have been simpler to take $X = 0$ and $Y = 1$, which is exactly the Lagrange method.

Authors' Closure

The Kane's equation and its derivation in Pironneau's simple example are wrong, since the coordinates used in Kane's equation are generalized coordinates that were mistaken for the mutually dependent Cartesian coordinates in the example. The correct procedure of the formulation of Kane's equation for this example is as follows. Let the radius of the ring be r and the angle between the OP and x axis be θ which is taken as a generalized coordinate.

$$\dot{q} = \dot{\theta} = u \quad (1)$$

$$\mathbf{V}^P = u(-r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j}) \quad (2)$$

The partial velocity

$$\mathbf{V}_p^P = -r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j} \quad (3)$$

$$\mathbf{a}^P = r(-\dot{u}\sin\theta - u^2\cos\theta)\mathbf{i} + r(\dot{u}\cos\theta - u^2\sin\theta)\mathbf{j} \quad (4)$$

The generalized active force

$$F = \mathbf{V}_p^P \cdot \mathbf{R} = -rmg\cos\theta \quad (5)$$

The generalized inertia force

$$F^* = \mathbf{V}_p^P \cdot (-m\mathbf{a}^P) = -r^2m\dot{u} = -r^2m\ddot{\theta} \quad (6)$$

The Kane's equation is

$$F + F^* = 0 \quad (7)$$

$$r\ddot{\theta} + g\cos\theta = 0 \quad (8)$$

It is true that for such a simple example, the method of Lagrange's multiplier is simpler than Kane's equation and Lagrange's equation. Furthermore, in this case, the most convenient method is not the method of Lagrange's multiplier but the Newton's equation of motion. By taking the projection of Newton's equation on the tangent line of the circle at P, we can immediately obtain (8). However, this fact has not presented an obstacle to the superiority of Lagrange's equations over Newton's equations and the method of Lagrange's multiplier for complicated problems, just as it cannot present an obstacle to the superiority of Kane's equations over other methods in many cases of complicated problems.

Both (14) and the derivation are far from Kane's equations and their derivation, although for specific problems the final equations of motion obtained by (14) and by Kane's equations may be the same, just as the same equation (8) can be obtained by several methods. The original contribution of Kane's method lies in the introduction of partial velocities, generalized active forces, and generalized inertia forces, which all disappeared in the derivation of (14), and the application of D'Alembert's principle as the foundation without the use of virtual displacements (or velocities), while the foundation of (14) is D'Alembert-Lagrange's principle, which had been miscalled Euler-Lagrange equation of motion in Pironneau's statement, with the concept of virtual velocities used. By the statement "it would have been simpler to take $X = 0$ and $Y = 1$, which is exactly the *Lagrange method*," the lead author has reason to say that (7) in the statement is the varied and unfinished form of Woronetz's equations of motion [1] and (14) the varied and unfinished form of Hamel's equations of motion [2, 3], which was compared with the former.

Equation (16) is superior to (17) in that the physical significance of the former, of which left side is the velocity (projection) of Q, is more apparent than (17). Equations (15) and (16) are superior to $u_1 = \dot{q}_4$, $u_2 = \dot{q}_3$ in that first, the final equations obtained are simpler; and second, they have very interesting practical importance *together*. Let $u_1 = \dot{q}_4 = \dot{\phi}$, $u_2 = v$, the equations of motion in our paper may be rewritten as

$$\left.\begin{array}{l} L\ddot{\phi} + v\,\dot{\phi}\cos\phi + \dot{v}\sin\phi = 0 \\ \sigma\,\ddot{\phi}\sin\phi + \pi\,v\,\dot{\phi}\sin\phi\cos\phi \\ \quad + (\tau + \tau_1\sin^2\phi + \tau_2^2\cos^2\phi)\,\dot{v} + C(t)\,\dot{m}\cos\phi = 0 \end{array}\right\} \quad (9)$$

We have two equations of two variables ϕ, v, where ϕ is the turning angle of the car and v the velocity of the car which, taken together, properly characterize the motion of the car and in which we are most interested.

The choice of generalized speeds affords an opportunity to select the parameters of practical interest as the generalized speeds that can be often described by closed system of differential equations (in our case, two equations of two variables) so that the specific problem can be greatly simplified. This is also an advantage of Kane's method. Although the calculation of accelerations is an additional task, yet as a

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whole, Kane's method is simpler than other methods in many cases of complicated problems.

References

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