Table 1a

The Geodetic Problem

Given: $P_{1j}, \gamma_{1j}, j = 2,3,4,5$ (see Fig. 2)

Find: M and A_1 such that $\angle A_1P_{1j}M = \angle \gamma_{1j}$

Method: As shown by Bottema, M and A_1 constitute a Burmester point-pair with P_{1j} as poles, M as centerpoint, and A_1 as circlepoint.

Procedure:

1 Assume an arbitrary point T_1 in position 1 of the moving plane.

2 Using the given poles P_{1j} and semirotations γ_{1j}, construct the corresponding points, T_j, $j = 2,3,4,5$, by vector additions in an arbitrary Cartesian co-ordinate system of axes.

3 Use $\mathbf{r} = \mathbf{OT}$ and $\varphi_j = 2\gamma_{1j}$ as input for the computer program #9.5.003 of Ref. [15].

4 Proceed according to Table 1 of the paper to determine the Burmester point-pairs, $M_uK_u^{(1)}$, $u = 1,2$ or $1,2,3,4$, if real.

5 By inspection of an approximate diagram, select the solution $M_u \equiv M$, $K_u^{(1)} \equiv A_1$, where M and A_1 are the points whose locations were to be found.

putations themselves. And finally, as the discussers have observed, the desirability of developing a few, versatile, general-purpose programs in the mechanisms field is becoming increasingly evident.

Concerning the pertinent questions raised by Professor Meyer zur Capellen, the answer seems to us as follows: In proceeding to five infinitesimal positions, the poles all coincide in the limiting position and the results of this paper either reduce to trivial identities or to indeterminate ratios, the evaluation of which would constitute a formidable task. Furthermore, the theory of this paper deals with motion in general, whereas much of R. Müller's pioneering work dealt with the four-bar linkage. An intriguing "inversion" of Professor Meyer zur Capellen's question is the following: Can any of R. Müller's results for infinitesimal displacements be extended to finite displacements? The investigation of this question will, it is hoped, be reported on in joint future work. A unified treatment of both finite and infinitesimal displacements would certainly be advantageous.

The thoughtful comments of Dr. Goodman are in line with, and supporting of, the comments on versatility made by Professors Denavit and Hartenberg. It is true that Roberts' theorem and inversion—techniques which the discusser has applied and explained with success—indeed extend the range of solutions in many problems involving mechanisms and constitute a powerful adjunct to the imagination. The investigation of the seven-bar, two-degree-of-freedom linkage is certainly far from exploiting all available design parameters. It represents, in fact, a mere beginning and was intended to illustrate a possible application of the computer program as well as to encourage further interest in its analysis.

In conclusion, the authors should like to state their conviction that the discussers have added to an unusual degree to the significance of the subject matter reported on in their present study.

Steady-State Behavior of Nonlinear Dynamic Vibration Absorber[1]

F. R. ARNOLD.[2] No doubt the authors have noted that the illustration in Fig. 1 is not appropriate for the particular differential equations (1) and (2) treated in the paper. Fig. 1 should have the damper connected between ground and m_1. Had the equations appropriate to damping between m_1 and m_2 been treated, the device of introducing the phase angle α in the disturbance term could not have been used and the approximating functions for the displacements of m_1 and m_2 would have had the form

$$x_1 \approx \bar{x}_1 = A_1 \cos \omega t + B_1 \sin \omega t \tag{1}$$

$$x_2 \approx \bar{x}_2 = A_2 \cos \omega t + B_2 \sin \omega t \tag{2}$$

The authors state that their paper extends previous work by determining an approximate solution for the case where both main and absorber springs are nonlinear. Actually, the extension lies in the inclusion of damping, since various combinations of nonlinear springs in undamped systems have been treated previously as listed in the bibliography of the authors' reference [3].

In describing the solution of the amplitude equations, the authors state that a maximum of nine possible solutions exists for each disturbance frequency. Actually, there are at mostonly three, since equations (15) and (16), or (17) and (18) are only of the third degree in the amplitudes. It is believed that the extraneous solutions have arisen from the computational methods which apparently involved squaring processes and hence yield some roots of the squared equations that will not satisfy the original equations.

In arriving at a solution to the amplitude equations, the following procedure is suggested to avoid graphical methods and to permit any desired accuracy in obtaining data for plotting response diagrams:

1 First, select a value for phase angle α, say α_i, and use the relation

$$\sin \alpha = 2\xi\eta \frac{A_1}{s}$$

to obtain A_1/s as a function of η. A curve of A_1/s versus η^2 may be drawn immediately as the first step in plotting the response diagram corresponding to the selected phase angle.

2 Selecting a value for η^2, say η_i^2, and its corresponding $(A_1/s)_i$ use equation (15) of the paper to find at most three values of $(A/s)_{i(1,2,3)}$.

3 Use equation (16) to find a value for $(A_2/s)_{i(1,2,3)}$ corresponding to each $(A/s)_{i(1,2,3)}$.

4 The complete response diagram corresponding to the selected phase angle may thus be drawn before selecting another phase angle and repeating the process until a set of response diagrams for each of as many phase angles as desired is established.

It is of interest to note the physical significance of the authors' method compared with that of some other methods. The former determines an approximate solution of sinusoidal form with the approximate amplitude of such value that the energy dissipation over the approximate cycle equals the work input of the disturbance. The second Ritz method, a Fourier series approach, or a modified Duffing method applied to the same problem will yield approximate solutions such that the coefficients of the approximating functions will have values that make the resulting approximate solution come as close as possible to satisfying whatever *minimum principle* exists for the dynamical system being considered—even if the principle is unknown.

In conclusion, it is suggested that equation (26) can be misleading without an examination of the response diagram of the system since the equation merely gives conditions for which $|A_1/s| \equiv 1$. Actually, on each side of the value of η^2, for which $|A_1/s| = 1$, the amplitude may even be smaller rather than larger. What happens is a matter for determination from the whole response diagram rather than from Fig. 3 alone.

L. F. KREISLE.[3] The authors are to be congratulated for extend-

[1] By W. J. Carter and F. C. Liu, published in the March, 1961, issue of the JOURNAL OF APPLIED MECHANICS, vol. 28, TRANS. ASME, vol. 83, Series E, pp. 67–70.

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ing the study of the nonlinear dynamic vibration absorber with a differential equation of the Duffing type previously studied by Arnold [3], Atkinson [4], and Robertson [2].[4]

Apparently there is a discrepancy in Fig. 1 since the entire paper is based upon a physical system different from that indicated in the illustration. If a viscous damper (with viscous damping coefficient B) is inserted between the fixed wall and the mass m_1 in Fig. 1 as printed, and if the damper between masses m_1 and m_2 is deleted, the resulting physical system appears to be that investigated by the authors. It is on this assumption that the following remarks are made:

Evidently, also through a typographical error, a cube has been omitted in the last nonzero term of equation (8). This term should read $\beta^3A^3 \cos^3 \omega t$ and not $\beta^3A \cos^3 \omega t$.

In order for equation (15) to be correct as printed and consistent with previous equations, the terms ν, ν_1, and λ^2 must be defined as follows:

$$\nu = \frac{\beta^2 F_m{}^2 k_2}{k_1{}^3}$$

$$\nu_1 = \frac{\gamma^2 F_m{}^2}{k_1{}^2}$$

$$\lambda^2 = \frac{k_2}{\mu k_1}$$

This definition of λ^2 differs from that given in the nomenclature of the paper. If the foregoing three definitions are followed, then η^2/μ should be substituted for the $\mu\eta^2$ term appearing in the equations (18), (21), (22), (23), (24), (25), and the equation for $R(A_2/s)$ appearing before equation (20). In addition, the η^2-term of equation (26) should be revised to η^2/μ.

It is hoped that the data for the curves in Figs. 2 and 3 and the digital-computer results were obtained from correct equations and that the incorrect equations appearing in this paper resulted from typographical errors. Although the writer did not do the calculations necessary to verify it, the curves in Figs. 2 and 3 appear to be correct for $\mu = 10$ rather than $\mu = 0.1$ as stated in the figures. Apparently the authors have employed two different definitions of μ, one being the reciprocal of the other. With $\mu = m_1/m_2$ as defined in the nomenclature, then the suggested revisions in several equations and the indication in Figs. 2 and 3 that $\mu = 10$ appear desirable to make the paper completely consistent. In closure, this paper makes a significant contribution to the analysis of the Duffing-type nonlinear dynamic absorber.

Author's Closure

The authors wish to thank Professors Arnold and Kreisle for their comments. As both of them have noted, Fig. 1 of the paper was in error—the damper should have been between ground and m_1. The authors agree with Professor Arnold that there should be at most only three amplitudes for each disturbance frequency.

Stress Distribution in a Rotating Spherical Shell of Arbitrary Thickness[1]

H. PORITSKY.[2] The problem of stresses in a rotating spherical shell appears to be a popular one nowadays. In addition to the three authors, the writer, too, has had occasion recently to solve the elastic equations for the rotating spherical shell. However, his results have not been published even as an internal company report, since the writer discovered[3] that this problem had been solved over 70 years ago by C. Chree. Chree's paper, entitled "The Equations of an Isotropic Elastic Solid in Polar and Cylindrical Co-ordinates, Their Solution and Application," was presented to the Cambridge Philosophical Society on October 31, 1887.[4] After obtaining general solutions of the elastic equations in spherical co-ordinates and treating several problems of a solid sphere, Mr. Chree solves the problem of a rotating spherical shell.[5]

ROBERT SCHMIDT.[6] This paper is a significant addition to the collection of solutions of the axially symmetric problems in the theory of elasticity. It is also of interest to the investigators in the theory of shells.

The writer believes that there is a simpler and more direct approach to this problem. It can be shown that the problem may be formulated by means of a harmonic function

$$\psi = \sum (A_n r^n + B_n r^{-n-1}) P_n(p) \tag{1}$$

and a biharmonic function

$$\Omega = \chi + \frac{(1+\nu)(11-17\nu)}{360(1-\nu)E} \gamma\omega^2 r^4, \tag{2}$$

where

$$\chi = \sum \left[C_n r^n + D_n r^{-n-1} - \frac{5 - 4\nu + n}{2(2n+3)(1-\nu)} A_n r^{n+2} + \frac{4 - 4\nu - n}{2(2n-1)(1-\nu)} B_n r^{-n+1} \right] P_n(p) + \frac{(1+\nu)\gamma\omega^2 r^4}{360E} \left[50p^4 - \frac{3(7-9\nu)}{1-\nu} \right] \tag{3}$$

is a solution of the equation

$$\nabla^2 \chi = -\frac{1}{1-\nu} \frac{\partial}{\partial r}(r\psi) - 4\psi + \frac{(1+\nu)\gamma\omega^2 r^2}{3E} \left[\frac{5}{2}(p^2 - \bar{p}^2) - \frac{1-2\nu}{1-\nu} \right] \tag{4}$$

The stress components are calculated by the formulas

$$\sigma_r = \frac{E}{1+\nu} \frac{\partial}{\partial r} \left(\frac{\partial \chi}{\partial r} + \frac{2-\nu}{1-\nu} r\psi \right) - \gamma\omega^2 r^2 \left[\frac{\nu}{3(1-\nu)} + \frac{1}{2}(p^2 - \bar{p}^2) \right], \tag{5}$$

$$\tau_{r\theta} = \frac{E}{1+\upsilon} \frac{\partial}{\partial \theta} \left(\frac{1}{r}\frac{\partial \chi}{\partial r} - \frac{\chi}{r^2} + \psi \right) + \frac{1}{3}\gamma\omega^2 r^2 p\bar{p}, \tag{6}$$

and so on. These results are similar to those obtained by Hampl.[7]

[4] References in paper.

[1] By M. A. Goldberg, V. L. Salerno, and M. A. Sadowsky, published in the March, 1961, issue of the JOURNAL OF APPLIED MECHANICS, vol. 28, TRANS. ASME, vol. 83, Series E, pp. 127–131.

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[3] A. E. H. Love, "Treatise on Mathematical Memory of Elasticity," McGraw-Hill Book Company, New York, N. Y., 4th ed., p. 255.

[4] Published in *Transactions of the Cambridge Philosophical Society*, vol. 14, 1889, pp. 250–481.

[5] Ibid., section 6, pp. 295–302.

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[7] Miloslav Hampl, "The Stress Problem of Axisymmetrically Loaded Thick Spherical Shells," *Der Stahlbau*, 1940, pp. 96–100.