June, 1959. This showed that the optimum π was significantly better than the double Tee and the foregoing analysis shows that both are very much better than the pi-circuit used by Murray and Landis.

The poor performance of the pi-network would have been much more apparent if the authors had considered the case where the boundary condition is given as a heat-transfer coefficient at the surface with a specified temperature for the ambient fluid rather than a specified surface temperature. This radiation type of boundary condition is a better test for a circuit because it represents the situation that occurs so frequently in practical heat transfer problems. It would be a serious mistake to conclude from the authors' results given in Figs. 2, 3, and 4 of the paper that the accuracy indicated for a network of 2, 4, or 8 modules would also obtain with the radiation type of boundary condition. It would also be a mistake to conclude that the results shown for $N = 1$ represent the best accuracy that can be obtained from a two-lump network since the conventional double Tee would be much more accurate and the optimum π better still.

Authors' Closure

It is apparent that there can be some misunderstanding on the objectives of work reported, and we should like to clarify these objectives in terms of the two points raised by Dr. Stephenson.

With regard to the first point, the authors did not study a passive network or passive networks. We did study the effects of spacewise lumping in solution of the one-dimensional diffusion equation:

$$\frac{\partial v}{\partial t} = \alpha \frac{\partial^2 v}{\partial x^2} \qquad (a)$$

Fig. 1(a) of the paper shows the commonly used lumping, or segmentation, of a homogeneous slab, where it is desired to determine the temperature histories at equally spaced points including those at the surface. In the usual electronic computer solution, the temperature at each interior point (or lump center) can be approximated in terms of the ordinary differential equation:

$$\left.\frac{dv}{dt}\right|_{N\Delta x} = \frac{k}{c\rho\Delta x}\left(\frac{v_{n+1} - v_n}{\Delta x} - \frac{v_n - v_{n-1}}{\Delta x}\right) \qquad (b)$$

At a boundary, the terms $(v_0 - v_{-1})/\Delta x$ or $(v_{N+1} - v_N)/\Delta x$ can be replaced by the appropriate term for conductive, convective, or radiative flow of an ambient, or can be replaced by a specified heat flow function $i(t)$.

Now, it is seen that the electrical network of Fig. 1(b) of the paper is an exact analog of the set of ordinary differential equations (b) if:

$$R_N = \frac{\Delta x}{k}, \quad C_N = c\rho\Delta x \qquad (c)$$

The question then, of separating the electrical network of Fig. 1(b) of the paper into pi or Tee elements is merely one of convenience. As stated in the paper, the separation into pi elements is the more convenient since temperature and heat flux at points of interest are directly analogous to voltage and current at the pi element terminals and since no special case of a half element need be established at the boundaries.

As regards the second of Dr. Stephenson's points, the authors did not attempt to determine the errors of lumping in one specific situation with real boundary conditions, but rather were more interested in the general effects of lumping in the slab without the effects of external boundary conditions. The results shown in Figs. 2, 3, and 4 of the paper indicate the effects of lumping in the slab wherein the first few terms of a polynomial expansion of a general surface temperature input condition are applied as the driving function.

We question that "That poor performance of the pi-network would have been much more apparent" if a convection-type boundary condition were introduced, since the introduction of such a boundary would tend to reduce the errors due to lumping and would make the solution less sensitive to the particular analog used. Indeed, as a boundary is introduced wherein very low Biot numbers obtain, we approach the condition of negligible internal resistance in the solid and obviously the criticality of the number of lumps or of the specific analog is reduced.

Additionally, the authors should like to take issue with Dr. Stephenson's comparison of the single pi element on Fig. 1(d) of the paper with his optimum pi-network or with a double tee element. A more appropriate comparison would be between two pi elements and the double tee. In this case, as shown by Murray[3] the A term of Dr. Stephenson's discussion is exactly the same as that for the double Tee, while the B term reduces to:

$$B_{Pi} = \frac{L}{k}\left(1 + 1.333\frac{L^2 s/\alpha}{3!}\right) \qquad (d)$$

We fail to understand the last sentence of the discussion, since the results shown in the paper for $N = 1$ indicate the effects of a single lump rather than a two-lump network as Dr. Stephenson suggests. For the example presented in the paper (i.e., $I_N = 0$) the results in Figs. 2 and 3 for $N = 2$ are exactly the same as those which would obtain for the double Tee. As shown by Table 1 of the paper, the transformed equations for this case are:

$$V_N = \frac{V_0}{\cosh N\psi} - I_N Z \sinh N\psi \qquad (e)$$

The terms $\cosh N\psi$ and $\tanh N\psi$ are the same for the double pi and the double Tee, and $I_N = 0$, eliminating the effects of Z, which is different for the two networks.

Despite the differences of opinion stated herein, it should be noted that Dr. Stephenson and we generally agree on the numbers of lumps required for usual engineering accuracy. Based on Stephenson and Mitalas,[4] and our work, the conclusion may be reached that four lumps usually lead to an acceptible solution if the Fourier number is not too small, or if the frequency of the driving function is not too high, while eight lumps appear to be more than satisfactory for most cases of transmission through a slab when internal temperature histories are required.

The authors would like to take this opportunity to correct two typographical errors which, unfortunately, persisted in the original paper. The ratios of heat transfer coefficients to conductivity in Equation (1) should be inverted to read k/h_1 and k/h_2, respectively. In systems 2 and 3 of Table 1 the number 2 should be replaced by the letter Z in the matrix equations.

Crack-Extension Force for a Part-Through Crack in a Plate[1]

E. G. CHILTON.[2] To show a relation between the author's "critical crack extension force" $\mathcal{G}_c$ and the fictitious surface tension T, first mentioned by Griffith and later used by Orowan, one may

[3] W. D. Murray, "Numerical and Machine Solutions of Problems in Transient Heat Conduction," doctoral thesis, New York University, N. Y., 1948.

[4] D. G. Stephenson and G. P. Mitalas, "Lumping Errors of Analog Circuits for Heat Flow Through a Homogeneous Slab," International Developments in Heat Transfer, 1961, pp. 28–38 (Available through The American Society of Mechanical Engineers).

[1] By G. R. Irwin, published in the December, 1962, issue of the JOURNAL OF APPLIED MECHANICS, vol. 29, TRANS. ASME, vol. 84, Series E, pp. 651–654.

[2] Stanford Research Institute, Menlo Park, Calif. Mem. ASME.

apply another energy criterion to the semielliptical crack. Thus, as soon as the change in potential energy due to an infinitesimal crack growth exceeds the surface-tension energy, the crack will become unstable and continue to grow.

The change in potential energy can be developed from Green and Sneddon's work[3] to be

$$W = \tfrac{4}{3}\pi c a^2(1 - \nu^2)\sigma^2/E\Phi$$

in the author's nomenclature.

The surface tension energy is simply

$$U = 2\pi c a T$$

(the 2 because the crack has two sides).

Then the crack will become unstable when

$$\frac{\partial}{\partial a}(W - U) = 0$$

which yields

$$a = a_{\text{crit}} = 3ET\Phi/4(1 - \nu^2)\sigma^2$$

or

$$T = 4a(1 - \nu^2)\sigma^2/3E\Phi$$

Comparing this with Equation (24) by letting $\varphi = \pi/2$ (the point where the crack will start to grow) and noting that $\mathcal{G} \to \mathcal{G}_c$ when $a \to a_{\text{crit}}$,

$$\mathcal{G}_c = \pi a(1 - \nu^2)\sigma^2/E\Phi^2$$

Hence

$$\mathcal{G}_c = (3\pi/4\Phi)T$$

Now Φ is just slightly larger than 1 as long as a/c is not too great. For instance, at $a/c = 0.3$, $\Phi \approx 1.10$ so that, at that value,

$$\mathcal{G}_c = (3\pi/4.4)T \approx 2.14T$$

It can be shown that a similar analysis for the "through" crack yields

$$\mathcal{G}_c = 2T$$

A mechanical method to produce semielliptical surface cracks in high-strength steels was developed some time ago by Stanford Research Institute but has not, as yet, been published. The method, being purely mechanical, does not affect the metallurgy of the steel in the immediate vicinity of the crack.

Author's Closure

In the discusser's comments the basic element of crack extension is assumed to be an incremental addition to the minor axis of the ellipse, a, with the major axis, c, held fixed. The discusser assumes that instability will occur when the total strain energy release is equal to the total gain of surface energy. This may be written

$$\int \mathcal{G}\, dA = \int 2T\, dA$$

where dA is the local area increment of crack extension. The integral extends along the entire crack border and across the assumed crescent-shaped increment of crack extension. $\mathcal{G}$ is a function of position along the crack border, as discussed in the paper.

The equation implies that only the average energy balance along the crack border is significant. In contrast, when the behavior is ideally brittle and all sections of the leading border of the crack resemble a notch of extreme sharpness, crack extension can be expected to occur first where $\mathcal{G}$ is maximum in accordance with a relationship based upon local rather than average conditions. Using the Griffith theory with this latter viewpoint, instability is always governed by $\mathcal{G} = 2T$, where both $\mathcal{G}$ and T refer to the same local segment of crack border.

Real materials usually possess less than ideal brittleness and will develop crack extension according to a relationship in which some degree of averaging is implicit. However, the appropriate type of averaging would not be uniform in all problems and we do not know what area of applicability exists for the special averaging procedure suggested by the discusser.

The author has been hopeful that readers of his paper would notice that the degree of localization one can assume is important both in the mathematical definition of $\mathcal{G}$ and in the relationships governing extension of real cracks. This topic deserves careful study and the author is grateful to the discusser for directing attention to it.

[3] A. E. Green and I. N. Sneddon, "The Distribution of Stress in the Neighborhood of a Flat Elliptical Crack in an Elastic Solid," *Proceedings of the Cambridge Philosophical Society*, vol. 46, 1950, pp. 159–164.

An Experimental Determination of the Minimum Reynolds Number for Instability in a Free Jet[1]

A. A. PUTNAM.[2] This report, though short, is interesting not only because of its bearing on the little studied problem of the minimizing Reynolds number for instability of an axially symmetric jet, but because a technique was used which, on occasions, can cause the appearance of unexpected phenomena. The technique is that of using dilute NaOH and phenolphthalein tracer in the reservoir, and dilute HCl in the test tank. While it is possible that the use of a downward pointing jet, or the low Reynolds number itself, might have prevented one such phenomenon from occurring, which we observed in a similar apparatus,[3] the designation of unsteady flow in the paper as one in which a series of ring vortices appears indicates that further study might be desirable.

Our apparatus was similar to that described but on a larger scale. The test tank was 9 in. dia by 18 in. long. The reservoir was a farm-type watering tank. Rotameters were used to measure flow rates. Poiseuille flow-approach tubes ranging from $^1/_2$ to $^{15}/_{16}$ in. dia discharged upward into the test tank. Sodium carbonate was added to the reservoir water, so the phenolphthalein would give a red indication, and nitric acid was added to the test tank. Over the range of Reynolds numbers studied, from 220 to 1680 (visual limitation in observing and counting), periodic vortices were observed to be produced. An example is shown by a figure by Putnam.[4] The visual count data on frequency were correlated by use of a Strouhal number.

Two facts from the study are pertinent to the present paper: (*a*) The use of a neutral density-dye tracer in place of the acid-alkaline reaction to show the phenomenon resulted in a coherent, smooth jet extending to about 20 diameters before it broke up, at Reynolds number of 700. At Reynolds number of 2000, the breakup occurred at about 6 jet diameters. Downstream of the breaking, the surface was rough with a very fine roughness compared to the jet diameter. (*b*) The data appeared to follow a consistent pattern relative to Strouhal number only when account was taken of the heat released by the acid-alkaline reaction, as well as the Reynolds number. It would appear that the heat re-

[1] By Andrus Viilu, published in the September, 1962, issue of the JOURNAL OF APPLIED MECHANICS, vol. 29, TRANS. ASME, vol. 84, Series E, pp. 506–508.

[2] Battelle Memorial Institute, Columbus, Ohio.

[3] A. A. Putnam, "Vortex Ring Produced by Buoyancy Effects in the Laminar Flow From a Tube," International Battelle Report on Basic Research in Fluid Dynamics, 357–1700, March, 23, 1962.

[4] A. A. Putnam, "Vortex Shedding from the End of a Tube as a Result of Buoyancy Effects," *Nature*, vol. 192, 1961, p. 1277.