Inequalities with $\pi N$ Polarization

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$\pi N$ polarization is investigated as a function of the relative magnitude and relative phase of the spin-non-flip and spin-flip amplitudes. Inequalities for the magnitude and phase in terms of the polarization are obtained and physical implications are discussed. From Schwartz’s inequality relations involving the polarization in integrated forms are obtained.

§ 1. Introduction

Until recently not much polarization data was available on $\pi N$ scattering over the entire angle region. But during the last years excellent experiments have been done at a large number of energies extending from the forward direction to the backward direction. The results pose a challenge to the theoretical models. In particular the medium angle region at high energies are difficult to treat with the Regge theory which has its natural domains only in the neighborhood of the forward and backward directions. Even in these regions the agreements are not completely satisfactory. But even the fits with the phase shifts, at least in some cases, are in gross disagreement with data. [See for instance Fig. 1 of Ref. 2) and Fig. 14 of Ref. 3).] Whether one is using a theoretical model or trying to fit the data with purely phenomenological parameters, there are some restrictions on the quantities on which the polarization depends. In § 2 we try to get a feeling for the behavior of the relative magnitude and relative phase of the spin-non-flip amplitude $f$ and the spin-flip amplitude $g$. It is hoped that the restrictions on these quantities may lead to a better parametrization of the polarization. From the inequality between the relative phase $(\sin \alpha)$ and the polarization it is seen that whenever these are equal the spin-flip and nonflip amplitudes are equal. There are angle regions over which these amplitudes seem to stay close to each other (see Fig. 5 as an example). On the other hand when the sine of the relative angle is larger than the polarization either the spin-flip amplitude is larger than the spin-non-flip amplitude or vice versa. An example for the first case is the $\pi^- p$ charge-exchange scattering; for the second case the forward region of the $\pi^+ p$ elastic scattering. This is confirmed by the largeness of the relative phase $(\sin \alpha)$ over the polarization in

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the forward direction (Fig. 5).

In § 3 we use Schwartz's inequality to derive inequalities between the cross sections and the polarization. Among those there are some which have to be satisfied by the existing data since they involve only the average differential cross section and the polarization. Both for the unintegrated and integrated inequalities the uniqueness question of the solution for the relative magnitude of the amplitudes has physical implications. Finally in § 4 we discuss the applications of the results.

§ 2. Relative amplitude and phase

The \( \pi N \)-polarization equation is

\[
P = \frac{2 \sin \theta \text{Im}(g^*f)}{|f|^2 + |g|^2 \sin^2 \theta}.
\]

(1)

Defining the relative phase between \( f \) and \( g \) as \( \alpha \) and the ratio

\[
\frac{|f|}{|g|} = r,
\]

Eq. (1) can be written as

\[
P = \frac{2 \sin \theta \sin \alpha}{r + (1/r) \sin^2 \theta}
\]

(2)

or

\[
\frac{\sin \alpha}{P} = \frac{1}{2} \left( \frac{r}{\sin \theta} + \frac{\sin \theta}{r} \right) = \frac{1}{2} \left( v + \frac{1}{v} \right),
\]

(3)

where

\[
\frac{\sin \theta}{r} = v.
\]

(4)

Equation (2) shows that the polarization depends only on the relative magnitude and the relative phase of the two amplitudes. In the physical region \( \sin \theta \geq 0 \), so that \( v \geq 0 \). The function \( (v + (1/v)) \) is positive with a minimum value of 2. Thus the quantity

\[
\frac{\sin \alpha}{P} \geq 1.
\]

(5)

That is, \( \sin \alpha \) and \( P \) have the same sign and

\[
|\sin \alpha| \geq |P|.
\]

(6)

Thus the polarization is a lower bound for the magnitude of the phase. The inequality (5) is a strong restriction on \( \sin \alpha \) and has important implications. It forces \( \sin \alpha \) not only to stay above \( P \) but to change its sign whenever \( P \) does. This forces \( \sin \alpha \) to go through zero whenever \( P \) does. Thus even if the spin-flip amplitude \( g \) may be zero at an angle still \( \sin \alpha \) must have the same zero if
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$P$ crosses it, because $\sin \alpha$ is the only quantity which can change its sign. In cases where $P$ goes almost from one extreme to the other (see Ref. 3) $\sin \alpha$ does not have much freedom and has to stay very close to $P$. Thus one wonders whether one is not measuring essentially the phase when measuring $P$. As we shall see this would mean that the spin-flip differential cross section and the non-flip cross section are equal over the range of the angle where $\sin \alpha$ and $P$ remain very close. Equation (3) is invariant under exchange of $r \leftrightarrow \sin \theta$. We also notice that the differential cross section

$$\frac{d\sigma}{d\Omega} = |f|^2 + |g|^2 \sin^2 \theta$$

is invariant under this exchange. Solving Eq. (3) for $r$ we find

$$r = \sin \theta \left( \frac{\sin \alpha}{P} \pm \sqrt{\left( \frac{\sin \alpha}{P} \right)^2 - 1} \right).$$

(7)

Defining

$$\frac{\sin \alpha}{P} = t \geq 1,$$

we see from

$$t \pm \sqrt{t^2 - 1} = t \left( 1 \pm \sqrt{1 - \frac{1}{t^2}} \right)$$

that

$$t + \sqrt{t^2 - 1} \geq 1$$

and

$$t - \sqrt{t^2 - 1} \leq 1,$$

so that the two solutions $r_+$ and $r_-$ satisfy

$$r_+ \geq \sin \theta,$$

(8)

$$r_- \leq \sin \theta.$$

(9)

In order not to disagree with the forward differential cross section and the optical theorem, we feel compelled to choose $r_+$. This is done at the expense of giving up the uniqueness of the solution and is based on the finiteness of $g$ at $\theta = 0$. With this choice

$$|f| \geq |g| \sin \theta$$

at least in the forward direction. Calling $t = 1/x$ we solve Eq. (7) for $y = (r/\sin \theta) (P/\sin \alpha)$.

$$y = 1 \pm \sqrt{1 - x^2}.$$
This is a circle (Fig. 1) centered at $y=1$ with the equation $(y-1)^2 + x^2 = 1$. Because $P/\sin \alpha \geq 0$ only the right half of the circle is physical. Since the range of $x$ is $0 \leq x \leq 1$ we find for the range of $y$

$$1 \leq y_+ \leq 2,$$

$$0 \leq y_- \leq 1.$$

For the choice of $r_+$, $1 \leq y \leq 2$ which is $y_+$. At $P/\sin \alpha = 1$ $y=1$ is uniquely defined. For all other $x$ values $y$ is double valued. Figure 1 can be used to find the limits of the relative magnitude of the amplitudes. With the help of the inequality

$$1 \geq |\sin \alpha| \geq |P|$$

we see that the part of the circle to the right of the line $x=P$ is the interval to which

$$y = \frac{r}{\sin \theta} \cdot \frac{P}{\sin \alpha}$$

is restricted. The following inequalities are obtained:

$$0 \leq y_- = \frac{r_-}{\sin \theta} \cdot \frac{P}{\sin \alpha} \leq 1,$$

$$1 \leq y_+ = \frac{r_+}{\sin \theta} \cdot \frac{P}{\sin \alpha} \leq 2.$$

These in turn give

$$0 \leq r_- \leq \sin \theta \sin \alpha \frac{P}{r},$$

$$\sin \theta \sin \alpha \frac{P}{r} \leq r_+ \leq 2 \sin \theta \sin \alpha.$$

$r_+$ and $r_-$ satisfy

$$r_+ r_- = \sin^2 \theta,$$  \hspace{1cm} (10·a)

$$r_+ + r_- = 2 \sin \theta \sin \alpha \frac{P}{r},$$  \hspace{1cm} (10·b)

$$r_+ - r_- = 2 \sin \theta \sin \alpha \frac{P}{r} \sqrt{1 - \frac{P^2}{\sin^2 \alpha}}.$$  \hspace{1cm} (10·c)

From the first relation together with the lower and upper limits of $r_+$ we find

$$\frac{1}{2} \sin \theta \frac{P}{\sin \alpha} \leq r_- \leq \sin \theta \frac{P}{\sin \alpha}.$$
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Also with the upper limit for $r_-$ we find

$$ r_+ \geq \sin \theta \frac{P}{\sin \alpha} . $$

Using the fact that $\sin \alpha/P \geq 1$ the following bounds are obtained:

$$ r_+ \geq \sin \theta \frac{P}{\sin \alpha} \geq \sin \theta |P| , $$

$$ r_+ \geq \sin \theta , \quad (11 \cdot a) $$

$$ r_+ \leq \frac{2 \sin \theta \sin \alpha}{P} \leq \frac{2 \sin \theta}{|P|} \leq \frac{2 \sin \alpha}{|P|} , \quad (11 \cdot b) $$

$$ r_+ \leq \frac{2 \sin \theta \sin \alpha}{P} \leq \frac{2 \sin \alpha}{|P|} , $$

$$ r_- \leq \sin \theta - \frac{P}{\sin \alpha} \leq \frac{1}{\sin \alpha} , $$

$$ r_- \leq \sin \theta , \quad (11 \cdot c) $$

$$ r_- \leq \sin \theta , \quad (11 \cdot d) $$

The bounds which can be experimentally established are $(11 \cdot a)$, $(11 \cdot b)$, $(11 \cdot c)$ and $(11 \cdot d)$. From Eq. $(10 \cdot c)$ we also find

$$ r_+ - r_- \leq \frac{2 \sin \theta}{|P|} \sqrt{1 - P^2} . $$

From this another upper bound for $r_+$ is obtained;

$$ r_+ \leq \sin \theta \left( \frac{2}{|P|} \sqrt{1 - P^2} + 1 \right) . $$

This is a better upper bound for $r_+$ for $P > 4/5$ whereas $(11 \cdot b)$ is a better bound for $P < 4/5$. The bounds $(11 \cdot b)$ and $(11 \cdot d)$ can be further improved by using directly the solution $(7)$. This gives

$$ r_+ \leq \sin \theta \left( \frac{1}{|P|} + \sqrt{\frac{1}{|P|^2} - 1} \right) , $$

$$ r_- \geq \sin \theta \left( \frac{1}{|P|} + \sqrt{\frac{1}{|P|^2} - 1} \right) . $$

In Fig. 2 we give the quantities $r_+ / \sin \theta$ and $r_- / \sin \theta$ as a function of polarization. We would like to point out that in these bounds only the polarization equation $(1)$ is used. As the differential cross section is given by

$$ \frac{d\sigma}{d\Omega} = |g|^2 (r^2 + \sin^2 \theta) , $$
A knowledge of the differential cross section in addition to the polarization makes it possible to put bounds on the magnitudes of $g$ and $f$. These bounds are best in the regions where the polarization is large.

One can try to use further information on $r$ and $\sin \alpha$. We tried to see whether the unitarity relations for $f$ and $g$ have simple implications on $r$ and $\sin \alpha$. The relative amplitude can be thought of as a single complex amplitude with the phase $\alpha$. Unfortunately the two unitarity integrals for the amplitudes $f$ and $g$ are coupled and do not reduce to a simple form for the relative amplitude. But we feel that in any model dependent or purely phenomenological parametrization of the quantities $(r, \sin \alpha)$ on which the polarization depends their bounds as well as their general behavior will be of use. In particular as we mentioned before the phase $\sin \alpha$ (Fig. 3) follows the general behavior of $P$ in the sense that it has the same sign as $P$, goes through the same zeros (with the possible exceptions of forward and backward directions when $P$ is zero at these points) and becomes 1 when $P$ is. As for $r$ we can summarize its behavior with the help of Fig. 4 as follows: If we accept that $g$ cannot be singular at $z = \pm 1$ we must assume that $r$ starts at $\theta = 0$ as $r = |f|/|g| > \sin \theta$. Once $r_+$ is chosen this way it is not possible for $r$ to cross $\sin \theta$ without making

Fig. 2. Upper limit of $r_+/\sin \theta$ and lower limit of $r_-/\sin \theta$ as a function of polarization.

Fig. 3. The behavior of the relative phase in relation to the polarization.

Fig. 4. The behavior of the magnitude of the relative amplitude in relation to the polarization.
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\[\sin \alpha = P\] as can be seen from Eq. (2). We found that $|\sin \alpha|$ was an upper bound for $|P|$. Thus whenever $r$ remains in the neighborhood of $\sin \theta$, $\sin \alpha$ remains in the neighborhood of $P$. Since $r$ must approach $z = -1$ from $r > \sin \theta$ the number of times $r$ can cross $\sin \theta$ has to be even. Such crossings correspond physically to situations when the spin-flip amplitude becomes equal to the spin-non-flip amplitude (including $\sin \theta$) when the relative phase is not $90^\circ$. If such a cross-over happens at $90^\circ$ relative phase then the polarization is 1.

We investigate now $P$ in the neighborhood of 1. Starting from $P = (2r \sin \theta \times \sin \alpha) / (r^2 + \sin^2 \theta)$ we introduce

\[\sin \alpha - P = \varepsilon,\]
\[r - \sin \theta = \eta.\]

Then

\[P = 2\varepsilon \left\{ \left( \frac{\sin \theta}{\eta} \right)^2 + \left( \frac{\sin \theta}{\eta} \right) \right\}. \quad (12)\]

Since at $P=1$ $\sin \alpha = P$ and $r = \sin \theta$ both $\varepsilon$ and $\eta$ go to zero. On the other hand $\sin \theta \neq 0$. Thus if we do not want $P$ to blow up in (12) we must have

\[\lim_{\eta \to \theta_m} \frac{\varepsilon}{\eta} = \text{finite.}\]

This makes

\[\lim_{\eta \to \theta_m} \frac{\varepsilon}{\eta} = 0.\]

Since $P \to 1$ at $\theta = \theta_m$ we have

\[\lim_{\eta \to \theta_m} \frac{\varepsilon}{\eta} = \frac{1}{2 \sin^2 \theta_m}.\]

Let us now expand everything near $z = z_m$. We define $\sin \alpha = s$

\[s = 1 + s'(z - z_m) + \frac{1}{2} s''(z - z_m)^2 + \frac{1}{3} s''(z - z_m)^3 + \cdots,\]
\[P = 1 + P'(z - z_m) + \frac{1}{2} P''(z - z_m)^2 + \frac{1}{3} P''(z - z_m)^3 + \cdots,\]
\[r = \sin \theta_m + r'(z - z_m) + \frac{1}{2} r''(z - z_m)^2 + \frac{1}{3} r''(z - z_m)^3 + \cdots,\]
\[\sin \theta = \sin \theta_m + \left( -\frac{1}{\sqrt{1 - z_m^2}} \right)(z - z_m) + \frac{1}{2} \left( -\frac{1}{\sin^2 \theta_m} \right)(z - z_m)^2 + \cdots.\]

If $r \geq \sin \theta$ then at $z_m$ $r' = (\sin \theta)'$. Also $s' = P' = 0$ since $P = 1$ is a maximum
point.

\[ \eta = r - \sin \theta = \frac{1}{2} \left( r - \frac{1}{\sin \theta_m} \right) (z - z_m)^3, \]
\[ \varepsilon = \sin \alpha - P = \frac{1}{2} \left( s'' - P'' \right) (z - z_m)^3 + \frac{1}{3} \left( s'' - P'' \right) (z - z_m)^3 \]
\[ + \frac{1}{4!} \left( s''' - P''' \right) (z - z_m)^3 + \cdots. \]

Let us now write

\[ \lim_{z \to z_m} \varepsilon = \frac{1}{2 \sin \theta_m} \eta'. \]

This implies that \( s'' = P'' \), \( s'' = P'' \) at \( z = z_m \). We find

\[ \frac{1}{3} \left( s''' - P''' \right) = \frac{1}{\sin \theta_m} \left( r' + \frac{z_m}{\sin \theta_m} \right)^3. \]

Thus if \( r \) does not cross \( \sin \theta \) at \( P = 1 \), \( \sin \alpha \) and \( P \) stay together for a large interval and one is measuring essentially \( \sin \alpha \) when \( P \) is measured. If \( r \) crosses \( \sin \theta \) at the point where \( P = 1 \) in general we have \( r' = (\sin \theta)' \) and a similar calculation leads to the relation

\[ s'' - P'' = \frac{1}{\sin \theta_m} \left( r' + \frac{z_m}{\sin \theta_m} \right)^3. \]

§ 3. Inequalities in integrated form

The two independent amplitudes \( f \) and \( g \) of the \( \pi N \) scattering satisfy Schwartz’s inequality:

\[ \int_{-1}^{+1} |f|^4 dz \int_{-1}^{+1} \sin^3 \theta |g|^3 dz \geq \left| \int_{-1}^{+1} f^* \sin \theta g dz \right|^2 \]
\[ = \left| \int \sin \theta |f||g|e^{i\alpha} dz \right|^2 \]
\[ = \left| \int \sin \theta \cos \alpha |f||g|dz \right|^2 + \left| \int \sin \theta \sin \alpha |f||g|dz \right|^2. \]

Defining the spin-flip and non-flip differential cross sections as:

\[ \left( \frac{d\sigma}{d\Omega} \right)_S = S \] and \[ \left( \frac{d\sigma}{d\Omega} \right)_N = N \]

and the differential cross section as
we obtain a first inequality by ignoring the first term on the right-hand side:

$$\int N dz \int S dz \geq \frac{1}{2} \left| \int P D dz \right|^2.$$  (16)

Using $D = N + S$ this becomes

$$\left( \int D dz - \int S dz \right) \int S dz \geq \frac{1}{4} \left| \int P D dz \right|^2.$$  (17)

Calculating the roots of the quadratic equation for $\int S dz$ we find

$$\frac{1}{2} \left\{ \int D dz - \sqrt{\left( \int D dz \right)^2 - \left( \int P D dz \right)^2} \right\} \leq \int S dz$$

$$\leq \frac{1}{2} \left\{ \int D dz + \sqrt{\left( \int D dz \right)^2 - \left( \int P D dz \right)^2} \right\}.$$  (18)

or going back to physical quantities

$$\frac{1}{2} \left\{ \frac{\sigma^*}{2\pi} - \sqrt{\left( \frac{\sigma^*}{4\pi^2} \right)^2 - \left( \int P d\sigma/d\Omega \ dz \right)^2} \right\} \leq \int \left( \frac{d\sigma}{d\Omega} \right)_f dz$$

$$\leq \frac{1}{2} \left\{ \frac{\sigma^*}{2\pi} + \sqrt{\left( \frac{\sigma^*}{4\pi^2} \right)^2 - \left( \int P d\sigma/d\Omega \ dz \right)^2} \right\}.$$  (19)

An inequality between $S$ and $D$ can be found in unintegrated form directly from the defining equation of $P$.

$$\left( \frac{\cos \alpha}{\sin \alpha} \right)^2 = \frac{4 f^2 g^2 \sin^2 \theta}{P^2 D^2} - 1.$$  (20)

Here $f$ and $g$ stand for moduli of the amplitudes,

$$\frac{\cos \alpha}{\sin \alpha} = \frac{1}{PD} \sqrt{4S(D-S)-P^4D^4}$$  (21)

which implies

$$4S(D-S) \geq P^4D^2$$  (22)

or

$$\frac{1}{2}D(1 - \sqrt{1 - P^4}) \leq S \leq \frac{1}{2}D(1 + \sqrt{1 - P^4}).$$  (23)

It is seen that for full polarization $(d\sigma/d\Omega)_f = \frac{1}{2} \cdot d\sigma/d\Omega$. To find improved bounds from Schwartz’s inequality let us keep the first term in inequality (14). With the help of Eq. (21) this becomes

$$\int D dz \int S dz - \left( \int S dz \right)^2 \geq \frac{1}{4} \left| \int 4S(D-S)-P^4D^4 \ dz \right|^2 \frac{1}{4} \left| \int P D dz \right|^2.$$  (24)
To separate the $S$ integration from square root we use the following inequalities which are based on the positiveness of the quantities involved and the fact that the square root of a quantity between 0 and 1 is larger than itself.

$$\int \sqrt{4S(D-S)-P^2} D^3 dz = \int D^3 (1-P^2) - (D-2S)^3 dz$$

$$= \int D \sqrt{1-P^3} \sqrt{1 - \frac{(D-2S)^3}{D^3 (1-P^3)}} dz$$

$$\geq \int D \sqrt{1-P^3} \left(1 - \frac{(D-2S)^3}{D^3 (1-P^3)}\right) dz \geq \int D \sqrt{1-P^3} \left(1 - \frac{|D-2S|}{D \sqrt{1-P^3}}\right) dz$$

$$= \int \left(D \sqrt{1-P^3} - |D-2S|\right) dz = B,$$

$$|B|^3 = \left( \int D \sqrt{1-P^3} dz \right)^3 + \left( \int |D-2S| dz \right)^3 - 2 \left( \int D \sqrt{1-P^3} dz \right) \left( \int |D-2S| dz \right)$$

$$= \left( \int D \sqrt{1-P^3} dz \right)^3 + \left( \int D dz \right)^3 - 4 \int D dz \int S dz + 4 \left( \int S dz \right)^3$$

$$\equiv 2 \int D dz \int D \sqrt{1-P^3} dz \pm 4 \int S dz \int D \sqrt{1-P^3} dz$$

$$= \left( \int D (1 \mp \sqrt{1-P^3}) dz \right)^3 - 4 \int D (1 \mp \sqrt{1-P^3}) dz \int S dz + 4 \left( \int S dz \right)^3.$$  

Here the upper signs are for $D-2S>0$, and the lower signs for $D-2S<0$. Inequality (14) is now

$$\int D dz \int S dz - \left( \int S dz \right)^2 \geq \frac{1}{4} \left( \int D (1 \mp \sqrt{1-P^3}) dz \right)^3 - \int D (1 \mp \sqrt{1-P^3}) dz \int S dz$$

$$+ \left( \int S dz \right)^3 + \frac{1}{4} \left( \int PD dz \right)^3.$$  

This has the form

$$ax - x^3 \geq \frac{1}{4} c^3 - cx + x^3 + \frac{1}{4} d^3.$$

Solving for the roots we find

$$\frac{1}{2} \left\{ \int D dz \mp \frac{1}{2} \int D \sqrt{1-P^3} dz \right\} - \sqrt{\frac{1}{2} \left( \int D dz \right)^3 - \frac{1}{4} \left( \int D \sqrt{1-P^3} dz \right)^3 - \frac{1}{2} \left( \int PD dz \right)^3} \leq \int S dz$$

$$\leq \frac{1}{2} \left\{ \int D dz \mp \frac{1}{2} \int D \sqrt{1-P^3} dz \right\} + \sqrt{\frac{1}{2} \left( \int D dz \right)^3 - \frac{1}{4} \left( \int D \sqrt{1-P^3} dz \right)^3 - \frac{1}{2} \left( \int PD dz \right)^3}.$$  

(25)
We shall show that this improved bound does not imply that \( r \geq \sin \theta \) everywhere even if we choose the \((-)\) sign which would imply partial use of \( r_+ \) together with Schwartz's inequality. For this all we have to show is that the long square root is larger than \( \frac{1}{2} \int D \sqrt{1-P^4} dz \) thus overcompensating its decreasing effect. This can be seen by using Schwartz's inequality on the functions \( \sqrt{D} \sqrt{1-P} \) and \( \sqrt{1+P} \).

\[
\int D(1-P) dz \int D(1+P) dz \geq \left( \int D \sqrt{1-P^4} dz \right)^2,
\]

\[
\frac{1}{2} \left( \int D dz \right)^2 - \frac{1}{2} \left( \int P D dz \right)^2 \geq \frac{1}{2} \left( \int D \sqrt{1-P^4} dz \right)^2,
\]

\[
\frac{1}{2} \left( \int D dz \right)^2 - \frac{1}{4} \left( \int D \sqrt{1-P^4} dz \right)^2 - \frac{1}{2} \left( \int P D dz \right)^2 \geq \frac{1}{4} \left( \int D \sqrt{1-P^4} dz \right)^2. \tag{26}
\]

Thus one should allow \( r \) to alternate between the two roots \( r_+ \) and \( r_- \). Thus allowing for this sign change in \( r \) within the interval \(+1\) to \(-1\) in the variable \( z \) we go back to the inequality following the relation (24) and write a new inequality using

\[
D \geq |D-2S| \quad \text{and} \quad D \geq S.
\]

We find

\[
|B|^2 \geq \left( \int D dz - \int D \sqrt{1-P^4} dz \right)^2 - 4 \int D dz \int S dz + 4 \left( \int S dz \right)^2,
\]

\[
\int D dz \int S dz - \left( \int S dz \right)^2 \geq \frac{1}{4} \left( \int D dz - \int D \sqrt{1-P^4} dz \right)^2 - \int D dz \int S dz
\]

\[-\sqrt{\frac{1}{2}} \left( \int D dz \right)^2 + \int D dz \int D \sqrt{1-P^4} dz - \frac{1}{2} \left( \int D \sqrt{1-P^4} dz \right)^2 - \frac{1}{2} \left( \int P D dz \right)^2 \}
\]

\[
\leq \int S dz \leq \frac{1}{2} \left\{ \int D dz
\]

\[-\sqrt{\frac{1}{2}} \left( \int D dz \right)^2 + \int D dz \int D \sqrt{1-P^4} dz - \frac{1}{2} \left( \int D \sqrt{1-P^4} dz \right)^2 - \frac{1}{2} \left( \int P D dz \right)^2 \}
\]

\[
+ \frac{1}{2} \left( \int D dz \right)^2 + \int D dz \int D \sqrt{1-P^4} dz - \frac{1}{2} \left( \int D \sqrt{1-P^4} dz \right)^2 - \frac{1}{2} \left( \int P D dz \right)^2 \}
\]

\[
\left(27\right)
\]

The upper and lower limits for the total spin-flip cross section are then found to be
§ 4. Conclusions

We have studied the behavior of the magnitude and phase of the relative amplitude. The results can be used for testing different models of polarization like Regge model, direct channel resonances or absorption model to find out how the magnitude and phase are reproduced by these models, whether they violate some of the bounds and which of these quantities needs improvement.

They can also be used to test the existing data. These tests will be discussed separately. Here we give as a representative example Fig. 5 which shows a comparison of the bound $\sin \alpha / P \geq 1$ with data. $\sin \alpha$ is constructed from the Saclay phase shifts and $P$ are taken from Ref. 3) ($T_\pi = 1.935$ GeV). The angle regions where $\sin \alpha / P (f \gg g \sin \theta)$ (0.6 < $z$ < 1), where the bound is violated (0.45 < $z$ < 0.6) and where $\sin \alpha \approx P (f \approx g \sin \theta)$ (−0.3 < $z$ < 0.2) are clearly observed.

One can also use the inequalities and the general behavior of $r$ and $\sin \alpha$ to parametrize them. As an example the fact that $\sin \alpha$ shares the zeros and ones of $P$ may be used at least in those regions to write $\sin \alpha - P = k \sin \pi P$ where $k$ is a suitable parameter. The inequalities in integrated form can be used to put upper and lower bounds on the total spin-flip and nonflip cross sections. The excellent data tables of Ref. 3) are especially useful for this purpose.

References