Exact Upper Bound for Pion-Nucleon Coupling Constant*

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An exact upper bound for pion-nucleon coupling constant $g$ which has been originally discovered by Geshkenbein and Ioffe has been rederived without assuming numbers of subtractions for spectral representation of $S_F'(p)$. Also, it has been proved that their bound is violated if and only if the ghost-like pole discussed by Goebel and Sakita appears, when $S_F'(p)$ has one zero point. It is computed that the bound leads to $g^2 \leq 20$ in the exact $SU(3)$ limit, which is very close to the experimental value $g^2 = 15$. The $d$ and $f$ coefficients of baryon-meson octet coupling constants are shown to be restricted to $0.30 \leq d \leq 0.94$. Also, the renormalization constant $Z_2$ of the nucleon must satisfy $0.85 \leq Z_2 \leq 0$, irrespective of any dynamical detail.

§ 1. Introduction and summary of principal results

Some years ago, Geshkenbein and Ioffe\textsuperscript{1} derived an exact bound for pion-nucleon coupling constant $g_{NN}$ in terms of physical mass values of pion and nucleon. Their result is (in unrationalized unit)

$$g_{NN}^2 \leq 85$$

which is somewhat larger than the experimental value of $g_{NN}^2 = 15$. The original derivation which was rather complicated was subsequently simplified by Ida.\textsuperscript{2} The main assumptions needed in deriving this bound (apart from some technical ones) are as follows:

(i) The Feynman propagator $S_F'(p)$ (or $A_F'(p)$) for the nucleon field (or spinless field) has no zero point at all, regarded as a complex analytic function of variable $i\not p$ (or $p^\mu$).

(ii) $S_F'(p)$ (or $A_F'(p)$) satisfies either unsubtracted or at most once-subtracted spectral representation of Kamefuchi-Umezawa-Lehman-Källen (hereafter referred to as KULK).

Geshkenbein and Ioffe have also attempted to show that the assumption (i) is physically reasonable. However, Goebel and Sakita\textsuperscript{3} subsequently demonstrated that their argument on this point was incorrect. We shall return to this point shortly.

It is the purpose of this note to re-investigate the problem by using a new method which has been found to be useful in various other problems.\textsuperscript{4,5,6} First,
we shall prove that the same bound for $g_{NN^*}$ can be derived without the assumption (ii). In other words, we need not specify numbers of subtractions needed for KULK representations of $S_{p'}(p)$ or $A_{p'}(p)$. Actually, the same conclusion will be valid even if $S_{p'}(p)$ or $A_{p'}(p)$ needs an infinite number of subtractions. Second, in the exact SU(3) limit, we show that our bound is considerably improved to $g_{NN^*}^2 \leq 4.5$, which is very close to the experimental value of $g_{NN^*} = 4.5$. Moreover, if we assume that a value of $g_{NN^*}^2$ will not change appreciably in the SU(3) limit, then we can show that $d/f$ ratio for baryon-meson octet coupling constants cannot be arbitrary. Indeed, they must satisfy the condition $0.30 \leq d \leq 0.94$. Especially, the negative value of $d$ is not possible. This inequality is consistent with the present experimental value of $d \approx 0.66$. We also computed a bound for $p-\pi-\pi$ coupling constant $g_{p\pi\pi}$ and found $g_{p\pi\pi}^2 \leq 45$ which should be compared to the experimental value of $g_{p\pi\pi}^2 = 15$. However, in the SU(3) limit, we will have $g_{p\pi\pi}^2 < 1.5$ which is smaller than the experimental value.

A bound for renormalization constant $Z_t$ of the nucleon is also found to give $0 \leq Z_t \leq 0.85$. In other words, the probability of finding a bare nucleon inside the physical nucleon is at most 85%. In the exact SU(3) limit, we can improve the bound to $0 \leq Z_t \leq 0.63$. These bounds for $Z_t$ are derived without assuming the ansatz (i).

Even when $S_{p'}(p)$ or $A_{p'}(p)$ has a zero point, we can still derive a weaker bound for the coupling constant $g_{NN^*}$ if the position of the zero point is known. Moreover, it has been proved without using any Feynman diagram argument that the bound of Geshkenbein and Ioffe is violated in this case if and only if the ghost-like pole discussed by Goebel and Sakita appears.

Last, we obtained an exact upper bound for asymptotic behavior of the vertex function even without assuming both ansätze (i) and (ii).

§ 2. Bounds for scalar particle interaction

To illustrate our method, we consider first the case of interaction coupling constant $g$ among three spinless particles which we label by indices $a$, $b$ and $c$ with masses $m_a$, $m_b$ and $m_c$, respectively. To be definite, we assume (otherwise so stated) that all these particles are stable. Then, the stability condition is expressed as

$$t_0 > (m_a)^2 > t_1,$$

where $t_0$ and $t_1$ are defined by

$$t_0 = (m_b + m_c)^2,$$

$$t_1 = (m_b - m_c)^2.$$ (2·2)

We can obtain a bound for the coupling constant $g$, if we assume the ansatz (i) but not the ansatz (ii) of the previous section. For that end, let us set $D(t)$
\[ D(t) = \frac{1}{(m_a^2 - t)} + \int_{t_0}^{\infty} d(m^2) \frac{\rho(m^2)}{m^2 - t} \]  

(2·3)

where the spectrum weight \( \rho(m^2) \) is defined by
\[ \rho(m^2) = (2\pi)^3 \sum_n |\langle 0| \phi_a(0) |n\rangle|^2 \delta^{(4)}(p - p_n) \]  

(2·4)

with \( -p^2 = m^2 \geq t_0 \) in terms of the renormalized interpolating field \( \phi_a(x) \) for the particle "a". As we shall prove shortly, we need not assume the validity of the unsubtracted representation Eq. (2·3). However, for simplicity, we assume it temporarily.

Now, as usual, we also assume that the lowest mass intermediate state contributing to the spectral weight \( \rho(m^2) \) is the two-particle state containing both particles \( b \) and \( c \). Then, the lowest integration value \( t_0 \) for \( m^2 \) in Eq. (2·3) is indeed the same quantity as that defined by Eq. (2·2), if there is no anomalous threshold. Also, as is well known,\(^1\) the positivity of the Hilbert space demands that we have, for \( t \geq t_0 \)
\[ \text{Im} \ D(t + i\epsilon) = \pi \rho(t) \geq (g^2/4) t^{-3} (t - t_0)^{3/2} (t - t_1)^{1/2} \times |\Gamma(t + i\epsilon)|^2 |D(t + i\epsilon)|^2, \]  

(2·5)

where \( g \) is the unrationlized renormalized coupling constant for three-particle interaction, and \( \Gamma(t) = \Gamma(t, m_b^2, m_c^2) \) is the corresponding proper vertex function normalized at \( t = m_a^2 \) with
\[ \Gamma(m_a^2) = 1. \]  

(2·6)

Note that Eq. (2·3) implies \( D(t) \) to be a real analytic function of \( t \) with a cut on real axis at \( \infty \geq t \geq t_0 \). Moreover, it satisfies the renormalization condition
\[ \lim_{t \to m_a^2} (m_a^2 - t) D(t) = 1. \]  

(2·7)

Actually, all we need in the following are these analyticity properties together with Eqs. (2·5), (2·6) and (2·7), and these will be valid, even if \( D(t) \) requires an arbitrary number of subtractions in writing down its spectral representation.

Suppose that \( D(t) \) has no zero points at all in the cut \( t \)-plane (ansatz (i)). Then, \( \Gamma(t) \) will also be a real analytic function of \( t \) with the same cut at \( t_0 \leq t < \infty \), but now without any pole in the entire cut-plane. However, if \( D(t) \) has a zero point, then \( \Gamma(t) \) will acquire a pole exactly at the same point. Such a zero point of \( D(t) \) is obviously a CDD type pole\(^2\) for its inverse. Thus, we are assuming that the inverse of \( D(t) \) has no CDD pole. The case in which \( D(t) \) has one zero point will be treated in the end of this section. Now, if we note a trivial inequality
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\[ |D(t+ie)| \geq \text{Im } D(t+ie), \]

then we can rewrite Eq. (2.5) as

\[ k(t) \geq |F(t+ie)| \]

(2.8)
on the entire cut at \( \infty > t \geq t_0 \), where \( k(t) \) is given by

\[ k(t) = \left( 4/g^2 \right) t (t-m^2_a) (t-t_0)^{-1/2} (t-t_1)^{-1/2} \]

(2.9)
and \( F(t) \) is defined by

\[ F(t) = (m^2_a-t) D(t) (\Gamma(t))^4. \]

(2.10)

By its construction, \( F(t) \) is a real analytic function of \( t \) with a cut at \( t_0 \leq t < \infty \) and satisfies the bound Eq. (2.8) on the cut. Moreover, we have

\[ F(m^2_a) = 1. \]

(2.11)

Then, as we shall prove in the Appendix, we have two alternatives. Either \( |F(t)| \) increases exponentially at infinity just like

\[ |F(t)| \approx \text{constant} \times \exp[\alpha |t|^\beta], \quad (t \to \infty) \]

(2.12)
with \( \beta \geq 1/2 \) and \( \alpha > 0 \), at least along one straight radial direction toward the infinity, or \( |F(t)| \) behaves as

\[ |F(t)| \leq (4/g^2) |t|, \quad (t \to \infty) \]

(2.13)
at infinity in the cut plane. Moreover, for the case of the second alternative, we must have

\[ |F(t)| \leq |G(t)| \]

(2.14)
in the entire cut \( t \)-plane where \( G(t) \) is defined by

\[ G(t) = \exp \left\{ \frac{1}{\pi} (t_0-t)^{3/2} \int_{t_0}^{\infty} dt' (t'-t)^{-1} (t'-t_0)^{-1/2} \log k(t') \right\} \]

\[ = \frac{4}{g^2} \frac{(t_0)^{3/2} + (t_0-t)^{3/2}}{(t_0-t)^{3/2} + (t_0-t_1)^{3/2}}. \]

(2.15)

Hereafter, we reject the first alternative Eq. (2.12). Notice that so far inequality (2.13) is a special case of (2.14) and (2.15). However, as we shall see shortly, its validity is more general. Then setting \( t = m_a^2 \) in Eq. (2.14) and noting Eq. (2.11), we find

\[ g^2 \leq (\gamma_{\text{max}})^2, \]

(2.16a)

\[ (\gamma_{\text{max}})^2 = 16 \left( t_0 - m_a^2 \right)^{3/2} \left[ (t_0)^{3/2} + (t_0-m_a^{2})^{3/2} \right] \left( t_0 - m_a^2 \right)^{3/2} + (t_0-t_1)^{3/2}. \]

(2.16b)
This is exactly the same as has been obtained in Ref. 1). However, we emphasize the fact that in our derivation, we did not assume the ansatz (ii), as well as any
asymptotic behavior for $D(t)$, as long as the condition (2.13) is satisfied. Especially, even an exponential behavior for $D(t)$ at infinity is perhaps permissible. Such a possibility could happen for super-propagators of non-polynomial Lagrangian field theory. It is interesting to compare our method with that of Ref. 1. They proceed from the fact that $D(t)$ is a Herglotz function (hereafter referred to as H-function), i.e., it satisfies $\text{Im} \, D(t) \geq 0$ for $\text{Im} \, t \geq 0$. This can be easily shown if $D(t)$ satisfies the unsubtracted dispersion relation (2.3). Actually, $D(t)$ will still be a H-function, even if we need one subtraction for it. However, if $D(t)$ requires more than two subtractions, then we can no longer prove the same property. At any rate, assuming $D(t)$ to be a H-function, they show first an inequality

$$\frac{1}{\pi} \int_{-\infty}^{\infty} dt \frac{1}{(t-m_a^2)k(t)} |\Gamma(t+i\varepsilon)|^2 \leq 1,$$  \hspace{1cm} (2.17)

where $k(t)$ is the same function defined by Eq. (2.9). From this inequality, they derive the bound (2.16) by means of the Szegő theorem. However, it is worthwhile to remark that in order to apply this method, it is perhaps necessary for us to assume the polynomial boundedness of $|F(t)|$ at infinity, although this requirement can be relaxed somewhat.

Next, let us investigate the case in which $D(t)$ has zero points at $t=\Delta_j$; $(j=1, 2, \ldots, N)$. Then $\Gamma(t)$ will develop poles at the same points. As the result, $F(t)$ will have poles at $t=\Delta_j$ $(j=1, \ldots, N)$. Then we have to use

$$\bar{F}(t) = \prod_{j=1}^{N} (\Delta_j - t) F(t),$$ \hspace{1cm} (2.18a)

$$\bar{k}(t) = k(t) \prod_{j=1}^{N} |t-\Delta_j|.$$ \hspace{1cm} (2.18b)

This gives us (see the Appendix)

$$|\bar{F}(t)| \leq |\bar{G}(t)|,$$ \hspace{1cm} (2.19a)

$$\bar{G}(t) = G(t) \prod_{j=1}^{N} [(t_0-t)^{\nu_\Delta} + (t_0-\Delta_j)^{\nu_\Delta}]^p$$ \hspace{1cm} (2.19b)

in the entire cut plane. This replaces Eq. (2.14). If the numbers $N$ of zero points is finite, then our asymptotic formula Eq. (2.13) for $|F(t)|$ is still valid. Similarly, setting $t=m_a^2$ in Eq. (2.19), we find a weaker bound for $g^2$. However, since the result will be rather complicated, hereafter we shall restrict ourselves to the discussion of the case $N=1$, i.e., when $D(t)$ has exactly one zero at $t=\Delta < t_0$. Our formula for the bound of coupling constant is now

$$g^2 \leq (g_1)^2,$$ \hspace{1cm} (2.20a)

$$(g_1)^2 = (g_{\text{max}})^2 [(t_0-m_a^2)^{1/\nu} + (t_0-\Delta)^{1/\nu}] |\Delta - m_a^2|^{-1}.$$ \hspace{1cm} (2.20b)

Actually, we can improve our bound if we know the residue of $\bar{F}(t) = (\Delta - t) F(t)$...
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at $t=\Delta$. Notice that $\bar{F}(t)$ is a real analytic function of $t$ with a cut at $t_0 \leq t < \infty$. Moreover we know the value $\bar{F}(m_\pi^2) = (\Delta - m_\pi^2)$ at $t = m_\pi^2$. Then as we shall prove in the Appendix, we must have an inequality

$$\frac{g^2 - \varepsilon \theta (y)^3}{\varepsilon (y)^3 - \theta g^2} \leq \frac{g^2 + \varepsilon \theta (y)^3}{\varepsilon (y)^3 + \theta g^2},$$

(2·21)

where $\varepsilon$, $\theta$ and $\tau$ are defined by

$$\varepsilon = \frac{\Delta - m_\pi^2}{|\Delta - m_\pi^2|}, \quad (2·22a)$$

$$\theta = \frac{(t_0 - m_\pi^2)^{\frac{3}{2}} - (t_0 - \Delta)^{\frac{3}{2}}}{(t_0 - m_\pi^2)^{\frac{3}{2}} + (t_0 - \Delta)^{\frac{3}{2}}}, \quad (2·22b)$$

$$\tau = 4(t_0 - \Delta)G(\Delta) = \bar{G}(\Delta) > 0. \quad (2·22c)$$

Before going into detail, we notice that the function $H(t) = (m_\pi^2 - t)^{-1}\bar{F}(t) = D(t) \times (F(t))^2$ represents the scattering amplitude for the reaction $b + \bar{c} \rightarrow b + \bar{c}$ in one particle exchange approximation. It has also the physical pole at $t = m_\pi^2$, in addition to the new pole at $t = \Delta$. Geshkenbein and Ioffe argued that such a new pole will be observable, and demanded therefore that we should not have such a new pole in the interval $m_\pi^2 < t < t_0$. However, Goebel and Sakita\(^5\) and subsequently Jin and MacDowell\(^1\) have shown that the presence of such a pole at $t = \Delta$ is actually unphysical and ghostlike with negative residue for $H(t)$, and will be cancelled by another Feynman diagram.

Here we shall prove that indeed, the new pole at $t = \Delta$ will be ghostlike if our bound $g^2 \leq (g_{\max})^2$ (see Eq. (2·16)) is violated. This fact is easy to see from Eqs. (2·21) and (2·22). First we notice that $\tau > 0$, $0 \leq \theta \leq 1$ and $|\varepsilon| = 1$. Now, suppose that we have

$$g^2 > (g_{\max})^2 = \theta (y)^3 \quad (2·23)$$

in contrast to (2·16). Then it is easy to prove

$$\varepsilon \bar{F}(\Delta) > 0 \quad (2·24)$$

on the basis of Eqs. (2·20) and (2·21). The condition (2·24) is equivalent to the fact that the residue of the scattering amplitude $H(t)$ at $t = \Delta$ is negative, i.e., ghost-like. Conversely, if the new pole at $t = \Delta$ corresponds to real particle-like pole for $H(t)$, then we have

$$\varepsilon \bar{F}(\Delta) < 0 \quad (2·24')$$

which leads to the original Geshkenbein-Ioffe bound $g^2 \leq (g_{\max})^2$ (see Eq. (2·16)) again. This is exactly the generalization of the result of Goebel and Sakita\(^5\) and of Jin and MacDowell\(^1\) without using any Feynman-diagram argument.

Returning to the original problem, we have derived our bound without assuming the ansatz (ii). However, in reality there seems to be a certain interre-
lation between two ansätze (i) and (ii). To see it, let us assume that $D(t)$ has a Regge behavior for $t \to \infty$, i.e.,

$$D(t) \approx \text{const } t^n. \quad (t \to \infty) \tag{2.25}$$

We emphasize here that our assumption is for the whole amplitude $D(t)$ with its phase rather than simply its magnitude $|D(t)|$. Now, let us compute the contour integral

$$J = \frac{1}{2\pi i} \oint dt \frac{D'(t)}{D(t)}$$

along the cuts and the large circle at infinity. As is well known, $J$ is simply the difference of numbers of zero points, $n$ and of poles, $n'$ of $D(t)$ in the entire cut plane. For our case, $D(t)$ has exactly one pole so that $n' = 1$. Therefore, $J$ is given by $J = n - 1$. On the other hand, we can directly compute the integral to obtain

$$J = \alpha + \frac{1}{\pi} [\delta(\infty) - \delta(t_0)],$$

where $\delta(t)$ is the phase of $D(t+is)$ on the cut, i.e.,

$$\delta(t) = \text{Arg } D(t+is). \quad (t \geq t_0)$$

Comparing both expressions, we find

$$n = 1 + \alpha + \frac{1}{\pi} [\delta(\infty) - \delta(t_0)] \tag{2.26}$$

if we assume the Regge behavior (2.25). Our formula Eq. (2.26) may be regarded as an analogue of Levinson's theorem. At any rate, since we know $	ext{Im } D(t+is) = \pi \rho(t) \geq 0$ on the cut, we must have $\sin \delta(t) \geq 0$ for $\infty > t_0$. Hence, if $\delta(t)$ does not change its value discontinuously on the cut, we must have

$$\pi \geq \delta(\infty) - \delta(t_0) \geq -\pi.$$

As a result, we find

$$2 + \alpha \geq n \geq \alpha. \tag{2.27}$$

Thus, for example, for $\alpha = 1/2$ which necessitates one subtraction for KULK representation of $D(t)$, we conclude that $D(t)$ must have at least one zero point. Therefore, in this case, the ansatz (i) automatically requires the ansate (ii). Of course, in this argument, we assumed that the phase $\delta(t)$ will not have any sudden discontinuity of an integral multiple of $2\pi$. Such a possibility could happen, if $D(t)$ has a double zero point (or double pole) on the cut. However, such a behavior for $D(t)$ is perhaps unphysical and unlikely. It may be that our assumptions of Regge behavior Eq. (2.25) is too restrictive. However, even accepting the validity of Eq. (2.25), our derivation has merit, since the bound
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(2.13) (which is correct even in the presence of zero points of \(D(t)\)) predicts then
\[
|\Gamma(t)| \leq \text{const}|t|^{-a/2}. \quad (t \to \infty)
\] (2.28)

Thus, when \(D(t)\) needs a subtraction for \(\alpha > 0\), then \(\Gamma(t)\) will satisfy unsubtracted dispersion relation.

In concluding this section, we shall make another application of our method, assuming again that \(D(t)\) has no zero point. Now suppose that \(D(t)\) represents the Feynman propagator function for pion. It has a cut starting with \(9\mu^2\) where \(\mu\) is the pion mass. Suppose that the cut of \(D(t)\) for \(4m_N^2 \leq t \leq 9\mu^2\) is negligible where \(m_N\) is the nucleon mass. Also, let us assume that the pion-nucleon vertex \(\Gamma(t)\) has no cut at all for corresponding interval. Then, using the method of Geshkenbein and Ioffe, Ida\(^b\) has proved that we must have \(g_{\pi N}^2 \leq 2\) for the pion-nucleon coupling constant. This is too small in comparison to the experimental value \(g_{\pi N}^2 = 15\), and suggests strongly that the cut from \(9\mu^2\) to \(4m_N^2\) is indeed not negligible. Instead of the approximation made above, let us assume that the cut of \(D(t)\) and \(\Gamma(t)\) are dominated by one-pion and one-rho-meson intermediate state. Pretending that the rho meson is stable, then a similar calculation gives us

\[
\text{Im} \ D(t+is) \geq \left(1/2m^2\right) \left(g_{\pi\rho}\right)^2 \frac{1}{t-t_0} (t-t_0)^{3/2} D(t+is)\Gamma(t+is)
\]

for the \(t \geq t_0 = (m+\mu)^2\), where \(m\) is the mass of the rho meson and \(t_0 = (m-\mu)^2\).

From this, we find, as before,
\[
(g_{\pi\rho}^2) \leq 8m^3 \frac{[(t_0)^{3/2} + (t_0 - \mu^2)^{3/2}]}{[(t_0 - \mu^2)^{3/2} + (t_0 - t_1)^{3/2}]}
\]

which gives
\[
(g_{\pi\rho}^2) \leq 4.5.
\]

This value should be compared to the experimental value of \((g_{\pi\rho}^2) = 2.4\). This strongly suggests that indeed the pion propagator \(D(t)\) is dominated by \(\pi-\rho\) intermediate state. However, if we go into the exact \(SU(3)\) limit, we can improve our bound considerably. In that case, we can use \(m = 852\) MeV and \(\mu = 413\) MeV which have been determined from the standard \(SU(3)\) mass formula. Also, the addition of \(\bar{K}K^*\) intermediate state improves the right-hand side of Eq. (2.29) by a factor of 2/3. The result is

\[
(g_{\pi\rho}^2) \leq 1.5
\]

which is now smaller than the experimental value. However, we should not worry too much about the discrepancy, because, strictly speaking, we cannot use our method due to the instability of the rho meson and we must take into account the uncorrelated three pion cuts as well as resulting possible anomalous threshold below unitarity cut because of instability of rho meson.
§ 3. Baryon-meson coupling

In the previous section, we discussed in some detail the case where all three particles involved are spin-less. We will now consider the case where two particles \( a \) and \( b \) have spin 1/2 with the same parity while \( c \) represents a pseudoscalar meson. For simplicity, we also assume

\[
m_b \geq m_c ,
\]

i.e., that the spin 1/2 particle \( b \) has a heavier mass in comparison to the spinless particle \( c \). Moreover, it is convenient to set

\[
\alpha = (t_b)^{1/2} = m_b + m_c ,
\]
\[
\beta = (t_b)^{1/2} = m_b - m_c
\]

so that the stability condition Eq. (2·1) together with Eq. (3·1) is written as

\[
\alpha > m_a > \beta > 0 .
\]

Temporarily, we shall assume for simplicity that the fermion propagator \( S(p) = iS_F(p) \) for the particle "a" satisfies an unsubtracted KULK representation

\[
S(p) = \frac{1}{m_a + i\gamma p} + \left[ \int_a^\infty + \int_{-\infty}^a \right] dm \frac{\rho(m)}{m + i\gamma p}
\]

although this assumption is again not necessary in our final analysis. The spectral weight \( \rho(m) \) is now given by

\[
(2\pi)^3 \sum_n \langle 0| \psi_a(0)|n\rangle \langle n| \bar{\psi}_a(0)|0\rangle \delta(p-p_n)
\]
\[
= \frac{1}{2} \left( 1 - \frac{i\gamma p}{m} \right) \rho(m) - \frac{1}{2} \left( 1 + \frac{i\gamma p}{m} \right) \rho(-m)
\]

with \( m = (-p^2)^{1/2} \geq \alpha \) where \( \psi_a(x) \) is the renormalized interpolating operator for the particle \( a \). In this case, it is more convenient to use the variable \( w \) defined by

\[
w = (t)^{1/2} = (-p^2)^{1/2}
\]

and set

\[
S(p) = \frac{1}{2} \left[ 1 - \frac{i\gamma p}{w} \right] f(w) + \frac{1}{2} \left[ 1 + \frac{i\gamma p}{w} \right] f(-w).
\]

Then, comparing Eqs. (3·7) and (3·4), we find

\[
f(w) = \frac{1}{m_a - w} + \left[ \int_a^\infty + \int_{-\infty}^a \right] dm \frac{\rho(m)}{m - w} .
\]

Also, following Bincer, we introduce the proper vertex function \( \Gamma(w) \) by

\[
\langle 0| \psi_a(0) |b(q_1)c(q_2) \rangle = \frac{i\gamma}{V} \left( \frac{4\pi m_b}{2q_{12}q_{10}} \right)^{1/2} S(p)
\]
where $p = q_1 + q_2$, and $g$ is the renormalized baryon-meson coupling constant. The vertex function $\Gamma(w)$ is normalized at
\begin{equation}
\Gamma(m_a) = 1. \tag{3.10}
\end{equation}
As has been proved by Bincer,\cite{Bincer} the product $f(w)\Gamma(w)$ is a real analytic function of $w$ with cuts on real axes at $\alpha \leq w < \infty$ and at $-\alpha \geq w > -\infty$. Equation (3.8) shows also that $f(w)$ is a real analytic function of $w$ with the same cuts. Moreover, it has a pole at $w = m_a$ with
\begin{equation}
\lim_{w \to m_a} (m_a - w)f(w) = 1. \tag{3.11a}
\end{equation}
As we shall see shortly, this analyticity is all we require and we really need not assume the validity of the unsubtracted KULK representation (3.8). Moreover, the positivity of the Hilbert space demands
\begin{equation}
\text{Im} f(w + i\epsilon) = \pi \rho(\omega) \geq \frac{1}{k(w)} |w - m_a| |\Gamma(w + i\epsilon)f(w + i\epsilon)| \tag{3.11b}
\end{equation}
on the cuts both at $\infty > w \geq \alpha$ and at $-\alpha \geq w > -\infty$, where $k(w)$ is defined by
\begin{equation}
k(w) = (8/g^2) |w|^\alpha |w - m_a| |w - \alpha|^{-\alpha/2} |w - \beta|^{-\beta/2} |w + \alpha|^{-1/2} |w + \beta|^{-1/2}. \tag{3.12}
\end{equation}
Again our crucial assumption is that $S(p)$ as a function of $i\gamma \rho$ does not have a zero point. This ansatz is equivalent to demand that $f(w)$ has no zero point at all in the cut $w$-plane. Then, $\Gamma(w)$ will not have any CDD type pole. Noting the trivial inequality
\begin{equation}
|f(w + i\epsilon)| \geq \text{Im} f(w + i\epsilon),
\end{equation}
we find that the inequality (3.11b) gives us
\begin{equation}
k(w) \geq |F(w + i\epsilon)| \tag{3.13}
\end{equation}
on the cuts where $F(w)$ is defined by
\begin{equation}
F(w) = (m_a - w)f(w)(\Gamma(w))^\alpha. \tag{3.14}
\end{equation}
By our construction, $F(w)$ is a real analytic function of $w$ with cuts at $\infty > w \geq \alpha$ and at $-\alpha \geq w > -\infty$, and satisfies
\begin{equation}
F(m_a) = 1. \tag{3.15}
\end{equation}
Just as in the previous section, let us define another real analytic function $G(w)$ by
\begin{equation}
G(w) = \exp \left\{ \frac{1}{\pi} (\alpha^2 - w^2)^{\alpha/2} \left[ \int_\alpha^\infty - \int_{-\infty}^{-\alpha} dw' \frac{\log k(w')}{(w' - w)(w'^2 - \alpha^2)^{\alpha/2}} \right] \right\}, \tag{3.16}
\end{equation}
where the branch cuts of $(\alpha^2 - w^2)^{1/2}$ are so chosen that on the cut-free real interval $-\alpha < w < \alpha$, it represents a positive number. The explicit form of $G(w)$ will be calculated in the Appendix. As we shall prove also in the Appendix, we have two alternatives for $F(w)$. Either, it increases exponentially at infinity as

$$|F(w)| \leq \text{const} \exp \{|cw|^\gamma\} \quad (w \to \infty) \quad (3.17)$$

with $\gamma \geq 1$ and $c > 0$, at least along one straight radial direction toward the infinity. Or, it must satisfy at infinity

$$|F(w)| \leq \text{const}, \quad (w \to \infty) \quad (3.18)$$

in the cut $w$-plane. Moreover, in the second alternative to which we will restrict ourselves hereafter, we must have

$$|F(w)| \leq |G(w)| \quad (3.19)$$

in the entire cut $w$-plane. Just as in the previous section, the bound equation (3.18) holds valid actually even when $f(w)$ has a finite number of zero points. At any rate, setting $w = m_a$ in Eq. (3.19), and noting Eq. (3.15) it gives, after evaluating $G(m_a)$ (see the Appendix), the bound

$$g^2 \leq \langle g_{\max} \rangle^2 \quad (3.20)$$

and

$$\langle g_{\max} \rangle^2 = 4 \frac{(1 + x)^8(1 + y)^8}{x(1 + x^2)(x + y)^8(1 + xy)} \quad (3.21)$$

where $x$ and $y$ are given by

$$x = \frac{(\alpha - m_a)^{1/2}}{\alpha + m_a} = \frac{(m_b + m_e - m_a)^{1/2}}{m_b + m_e + m_a}, \quad (3.22a)$$

and

$$y = \frac{(\alpha - \beta)^{1/2}}{\alpha + \beta} = \frac{(m_c)^{1/2}}{m_c}. \quad (3.22b)$$

Regardless of an apparent difference in form, our simpler formula, Eqs. (3.20) and (3.21), is equivalent to that of Refs. 1) and 2). Indeed for the case $m_a = m_b$, it reproduces the result of Ref. 2) after some calculations.

For the pion-nucleon coupling constant $g_{NN}$, we have to replace $g^2$ in Eq. (3.20) by $3\langle g_{NN} \rangle^2$, since we have to consider three Cartesian pions for summations in the intermediate state contribution for $\rho(m)$. Inserting the experimental masses of pion and nucleon, our formula, of course, reproduce the result of Ref. 1), i.e.,

$$\langle g_{NN} \rangle^2 \leq 85. \quad (3.23)$$

This value is somewhat larger than the experimental value of $\langle g_{NN} \rangle^2 = 15$. However, our numerical bound improves considerably, if we go into the exact $SU(3)$ limit. In that case, we have to take into account contributions from octets of baryon and meson intermediate states for $\rho(m)$. Then we can replace Eq. (3.20) by
Exact Upper Bound for Pion-Nucleon Coupling Constant

\[(4/3)(5d^2 + 9f^2) (g_{NN})^2 \leq (g_{\text{max}})^2 \quad (3\cdot24)\]

where \(d\) and \(f\) are standard \(d\) and \(f\) coefficients with \(d+f=1\) for couplings of baryon and meson octets. Noting that

\[\frac{(30/7)}{(4/3)(5d^2 + 9f^2)}\]

holds valid, irrespective of specific values of \(d\), this gives us

\[(g_{NN})^2 \leq 20, \quad (3\cdot25)\]

where we have used the common octet mass values \(m_a = m_b = 1154\) MeV for baryons and \(m_c = 413\) MeV for mesons, as has been determined from the standard \(SU(3)\) mass formula. We observe that this bound is a considerable improvement in comparison to (3·23). Actually, we can derive a better bound if we use specific values for \(f\) and \(d\). For example, if we had used \(f=1\), and \(d=0\), then we would have obtained \(g_{NN}^2 \leq 7.5\) which is one half smaller than the experimental values. If the value of \(g_{NN}^2\) does not change appreciably when we take the \(SU(3)\) limit, then we can use our bound (3·24) to determine a possible allowed range for \(d\). The result is

\[0.30 \leq d \leq 0.94. \quad (3\cdot26)\]

Especially, this excludes negative values for \(d\). The present experimental value is \(d \approx 0.66\), consistent with our bound. A similar method is also applicable to the \(N^* \rightarrow N\pi\) decay problem although we shall not discuss it here.

We may remark that the closeness of our upper bound (3·25) to the experimental value may suggest that contributions from all higher multi-particle intermediate states to spectral weight \(\rho(m)\) are perhaps negligible in accordance with the threshold dominance hypothesis of Li and Pagels. In passing, it is interesting to see that if \(\Gamma(w)\) has an accidental (or kinematical) zero point, say at \(w=0\), then we can improve our bound considerably. In that case, we can use \(\Gamma(w)/w\) instead of \(\Gamma(w)\). Then, we can easily show that our bound (3·20) is now modified into

\[g^2 \leq (g_{\text{max}})^2 \left(\frac{1-x}{1+x}\right)^2.\]

This will give us \(g_{NN}^2 \leq 28\) for the \(SU(2)\) case and \(g_{NN}^2 \leq 5\) in the \(SU(3)\) limit. So we may use this last fact as an argument against the existence of such an accidental zero of \(\Gamma(w)\) at \(w=0\).

We may also briefly consider the case in which particles "a" and "b" have the opposite parity. In such a case, we have to drop the presence of the factor \(\gamma_s\) in Eq. (3·9) for the definition of the vertex function \(\Gamma(w)\). Then, this changes the definition of \(k(w)\) in Eq. (3·12) by letting \(\alpha \rightarrow -\alpha\) and \(\beta \rightarrow -\beta\). The final result is that Eqs. (3·20) and (3·21) are now modified into

\[g^2 \leq (g_{\text{max}})^2,\]
We can apply this formula for \( Y_0^* \rightarrow B \bar{B} \) problem, if \( Y_0^* (1405 \text{ MeV}) \) will become stable in the exact SU(3) limit. This will be the case if the mass value 1405 MeV for the \( Y_0^* \) remains practically unchanged in the SU(3) limit. Then, our formula predicts

\[
g^2 \leq 0.55
\]

for the coupling constant of \( Y_0^* \rightarrow B \bar{B} \). In this derivation, we have replaced \( g^2 \) in Eq. (3.27) by \( 8g^2 \) because eight octet mesons enter in the intermediate summation for \( \rho (m) \). This bound for \( g^2 \) is still considerably larger than the experimental value of \( g^2 = 0.05 \) which is determined from the experimental width \( \Gamma (Y_0^* \rightarrow \Sigma \pi) \approx 45 \text{ MeV} \).

So far we have assumed that \( f(w) \) has no zero point at all. If \( f(w) \) has a zero point at \( w = \Delta \) with \(-\alpha < \Delta < \alpha\), then we can similarly derive a bound

\[
g^2 \leq (g_1)^2, \tag{3.28a}
\]

\[
g_1^2 = (g_{\text{max}})^2 / \theta, \tag{3.28b}
\]

\[
\theta = \left| \frac{(\alpha + \Delta)^{1/2} (\alpha - m_e)^{1/2} - (\alpha - \Delta)^{1/2} (\alpha + m_e)^{1/2}}{(\alpha + \Delta)^{1/2} (\alpha - m_e)^{1/2} + (\alpha - \Delta)^{1/2} (\alpha + m_e)^{1/2}} \right|. \tag{3.28c}
\]

Also, if we set

\[
\tilde{F} (w) = (\Delta - w) F(w), \tag{3.29}
\]

then we find (see the Appendix)

\[
g^2 - \varepsilon \theta g_1^2 \leq \tilde{F} (\Delta) / \tau \leq g^2 + \varepsilon \theta g_1^2 \]

\[
\varepsilon g_1^2 - \theta g^2 \leq \tilde{F} (\Delta) / \tau \leq \varepsilon g_1^2 + \theta g^2 \tag{3.30}
\]

where \( \varepsilon \) and \( \tau \) are given by

\[
\varepsilon = \frac{\Delta - m_e}{|\Delta - m_e|}, \tag{3.31a}
\]

\[
\tau = (2/\alpha) (\alpha^2 - \Delta^2) G(\Delta) > 0. \tag{3.31b}
\]

Just as in the previous section, we can prove that we have

\[
\varepsilon \tilde{F} (\Delta) \geq 0 \tag{3.32}
\]

depending upon whether we have

\[
g^2 \geq (g_{\text{max}})^2 = \theta (g_1)^2. \tag{3.33}
\]

In other words, the Geshkenbein-Ioffe bound is violated if and only if the new pole of \( \Gamma (w) \) at \( w = \Delta \) gives rise to the ghost-like pole for reactions \( b + \varepsilon \rightarrow b + \varepsilon \).

Also, we can state an analogue of Eq. (2.27) in our case. Suppose that
$f(w)$ behaves as a Regge power at infinity, i.e.,

$$f(w) \propto \text{const } \omega^{a_1}. \quad (w \to \infty) \tag{3.34}$$

Then, the number $n$ of zero points of $f(w)$ is restricted by

$$\alpha_1 + 3 \geq n \geq \alpha_1 - 1. \tag{3.35}$$

This is weaker in comparison to Eq. (2.27) of the previous section. This is due to the fact that we have here both left- and right-hand cuts. Especially, for $\alpha_1 = 1$ which requires two subtractions for $f(w)$, it may have no zero point at all.

§ 4. Bound for $Z_2$ and final comments

As we have proved in the previous sections, we can derive an absolute upper bound for $\eta^N_{NM}$, if $S(p)$ has no zero point or if the zero point of $S(p)$ does not induce a ghost-like pole in the vertex function.

In this section, we shall show that the magnitude of the nucleon renormalization constant $Z_2$ is also bounded. This fact has been already noted by several authors.\textsuperscript{11,15} However, to see its close connection with the results of previous sections, here we shall first reproduce some results of these authors. We notice that $Z_2$ is given by

$$(Z_2)^{-1} = 1 + \left( \int_{a}^{\infty} + \int_{-\infty}^{-a} \right) dmp(m). \tag{4.1}$$

Thus, using the inequality (3.11b) we find

$$(Z_2)^{-1} - 1 \geq \frac{1}{\pi} \left( \int_{a}^{\infty} + \int_{-\infty}^{-a} \right) dw \frac{1}{k(w) |w - m_a|} |H(w)|^2, \tag{4.2}$$

where we have set

$$H(w) = (m_a - w)f(w)\Gamma(w). \tag{4.3}$$

Now, $H(w)$ is a real analytic function of $w$ with the same cuts as $f(w)$, satisfying

$$H(m_a) = 1 \tag{4.4}$$

and $k(w)$ is the same function defined by Eq. (3.12). We also note that $H(w)$ will not have any poles at all even if $f(w)$ has zero points, since $H(w)$ is essentially the improper vertex function discussed by Bincer.\textsuperscript{18} At any rate, Eq. (4.2) leads to the inequality (see the Appendix)

$$(Z_2)^{-1} - 1 \geq \frac{4(\alpha^2 - w^2)}{[(\alpha - w)^2(\alpha + m_a)^2 + (\alpha + w)^2(\alpha - m_a)^2]^2} |H(w)|^2 \tag{4.5}$$

for real values of $w$ satisfying $|w| < \alpha$. Especially, setting $w = m_a$ and noting Eq. (4.4), this gives
where \((g_{\text{max}})^2\) is defined by Eq. (3.21). This is essentially equivalent to results of Refs. 11) and 15). For the pion nucleon interaction, we can replace \(g^2\) above by \(3g_{NN}^2\). Using the experimental value \(g_{NN}^2\), this gives

\[
0 \leq Z_s \leq 0.85.
\]

This implies that the probability of finding a single bare nucleon inside the physical nucleon is at most 85%, independent of any dynamical detail. In the exact \(SU(3)\) limit, we can improve the above bound to

\[
0 \leq Z_s \leq 0.63
\]

if we use \(g_{NN}^2 = 15\) and \(d/f = 2\). These numbers should be compared to the case of the deuteron where several authors\(^{13}\) have discovered \((Z_2)_D\) to be consistent with zero. This is regarded to imply\(^{11}\) that the deuteron is a bound state.

We can also derive a bound for the renormalization constant \(Z_s\) for spinless particle, which is given by

\[
(Z_s)^{-1} - 1 = \int_0^\infty dt \rho(t)
\]

in the notation of § 2. Analogously, we find

\[
(Z_s)^{-1} - 1 \geq \frac{g^2}{(g_{\text{max}})^2}
\]

where \((g_{\text{max}})^2\) is now given by Eq. (2.16b). Again, in this derivation, it is unnecessary to assume the absence of zero point for \(D(t)\). If we assume as in § 2 that \(D(t)\) is dominated by one pion and one rho intermediate state, then the renormalization constant \(Z_s\) for pion has now the upper bound\(^{19}\)

\[
0 \leq Z_s \leq 0.62,
\]

where we used \((g_{\text{max}})^2 = 4.5\) and \((g_{\rho \rho})^2 = 2.4\) as in Eq. (2.30).

Of course, if the nucleon (or pion) is really a bound state, then we expect to\(^{16}\) have \(Z_s = 0\) (or \(Z_2 = 0\)). Indeed, there is a recent attempt\(^{15}\) to prove \(Z_2 = 0\) exactly, assuming Regge and scale-invariant properties of elastic electron-proton scattering form factor together with an ansatz

\[
\lim_{q^2 \to -\infty} q^2 \sigma_L(q^2, \omega) = 0
\]

in the notation of Ref. 18). The proof proceeds as follows. West finds after some manipulations an inequality [see Eq. (13b) of Ref. 18)],

\[
(Z_s)^{-1} - 1 \geq \frac{\pi^2 \alpha_\rho \mathcal{F}(q^4)}{\mathcal{I}(q^4)},
\]

\[
\mathcal{I}(q^4) = -4 \int_1^\infty \frac{d\omega}{\omega} q^2 \sigma_L(q^2, \omega),
\]

where \(\mathcal{F}(q^4)\) is a function of \(q^4\) which is assumed to be finite and non-zero for
the limit \( q^2 \to -\infty \). Now he interchanges the order of the limit \( q^2 \to -\infty \) and of the integration with respect to \( \omega \) in Eq. (4·13b) to find

\[
\lim_{q^2 \to -\infty} I(q^2) = 0
\]  

(4·14)

in view of the ansatz Eq. (4·12). Then, Eq. (4·13a) indeed requires \( Z_\pi = 0 \) identically. However, this interchange of the limit and the integral is in general not justifiable. For example, consider a simple counter example

\[
\sigma_L(q^2, \omega) = \frac{1}{(q^2)^2 + (\log \omega)^3}
\]

which satisfies Eq. (4·12). However, it is easy to compute

\[
I(q^2) = 2\pi \neq 0, \quad (q^2 < 0)
\]

in contrast to Eq. (4·14), and we cannot conclude to have \( Z_\pi = 0 \). Therefore, we believe that the conclusion of Ref. 18) is not correct, unless modified.

Finally, we simply remark that Cooper and Pagels\(^{19}\) derive an inequality

\[
\left| \kappa e/2m \right| \leq 32 \int_{m+a}^{\infty} dw \left( \frac{w}{w^2 - m^2} \right)^{3/2} [\rho(w) + \rho(-w)]^{1/2}
\]  

(4·15)

for the anomalous magnetic moment \( \kappa \) of the proton. Using the Schwarz inequality, this gives us a bound for \( Z_\pi \)

\[
(Z_\pi)^{-1} - 1 \geq \left( \frac{\kappa e}{2m} \right)^2 \left( \frac{\mu^2 + 2m\mu}{16} \right)^{1/2} \frac{1}{m^2 + 4m\mu + 2\mu^2},
\]  

(4·16)

where \( m \) and \( \mu \) are masses of the nucleon and pion, respectively. However, this gives numerically a far worse result for \( Z_\pi \) in comparison to Eq. (4·7).

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**Appendix**

Here we shall prove several mathematical statements made in the previous sections. To maintain a generality, let us suppose that \( F(t) \) is a real analytic function of \( t \) [i.e., \( F^*(t^*) = F(t) \)] with cuts on real axis at \( \infty > t \geq a \) and at \( -b \geq t > -\infty \). Hereafter we assume without loss of generality that \( a \) and \( b \) are positive, i.e., \( a > 0 \) and \( b > 0 \), since we can shift the origin \( t = 0 \) suitably, if necessary, by a translation \( t \to t' = t + \Delta \).

Suppose that on the cuts, we have

\[
|F(t \pm i\epsilon)| \leq k(t),
\]  

(A·1)
where \( k(t) \) is a given non-negative function defined on the cuts. Now, we define \( G(t) \) by

\[
G(t) = \exp \left\{ \frac{1}{\pi} (a-t)^{1/2} (b+t)^{1/2} \left[ \int_{a}^{b} \frac{\log k(t')}{(t'-t)|t'-a|^{1/2}|t'+b|^{1/2}} \, dt' \right] \right\},
\]

(A·2)

assuming that the integral exists. Here, we choose the cuts of \((a-t)^{1/2}\) and \((b+t)^{1/2}\) in such a way that they are real and positive in the cut-free real interval \(-b < t < a\). Then, \( G(t) \) is a real analytic function of \( t \) with the same cuts as \( F(t) \), and satisfies an inequality

\[
|G(t \pm i\varepsilon)| = k(t) \geq |F(t \pm i\varepsilon)|
\]

(A·3)
on the cuts. Since by construction \( G(t) \) has no zero points in the entire cut-plane, \( R(t) \) given by

\[
R(t) = \frac{F(t)}{G(t)}
\]

is also a real analytic function of \( t \) with the same cut structure as \( F(t) \). Moreover, on the cuts, it satisfies

\[
|R(t \pm i\varepsilon)| \leq 1
\]

(A·5)

in view of Eq. (A·3). Now, we would like to apply a version of Lindelöf-Phragmen theorem, due to Nevanlinna.\(^{20}\) First, since \( R(t) \) is real analytic, i.e., \( R^*(t^*) = R(t) \), it is sufficient to consider the situation only in the upper half \( t \)-plane, and set \( \alpha \) to be the maximum of \( |R(t)| \) on the real interval \(-b \leq t \leq a\), i.e.,

\[
\alpha = \max_{-b \leq t \leq a} |R(t)|.
\]

(A·6)

Then setting

\[
\beta = \max(\alpha, 1),
\]

(A·7)

we find

\[
|R(t + i\varepsilon)| \leq \beta
\]

(A·8)
on the entire real axis. Then, Nevanlinna's theorem\(^{20}\) tells us the following: When we set

\[
M(r) = \max_{0 \leq \theta \leq \pi} |R(re^{\iota \theta})|,
\]

(A·9)
we have two possibilities:

(i) \( |R(t)| \) increases so rapidly for \( |t| \rightarrow \infty \) that we have

\[
\lim_{r \rightarrow \infty} \inf \left\{ \frac{1}{r} \log M(r) \right\} = \delta > 0,
\]

(A·10)
or

(ii) we have \( \delta = 0 \) and

\[
|R(t)| \leq \beta
\]  

(A.11)

in the _entire_ upper-half plane. In our applications, \( k(t) \) has a simple polynomial boundness property for \( t \to \infty \). This implies that \( |G(t)| \) has the same property for \( |t| \to \infty \) as we shall see shortly. Then, translating the content of Nevanlinna's theorem in our case, we can prove two alternatives, Eqs. (3.17) and (3.18).

For the more interesting second alternative, we can do better. Since \( R(t) \) is real analytic, then the inequality (A.11) should hold in the entire cut-plane. Then, we must have \( \beta = 1 \). To show it, suppose contrarily that we have \( \beta = \alpha > 1 \). Then, in view of Eqs. (A.6) and (A.11), \( |R(t)| \) takes its maximum value \( t_1 \) at an interior point in the interval \(-b < t < a\) where \( R(t) \) is analytic. Hence by the maximum modulus theorem, \( R(t) \) must be constant with modulus \( \beta \). However, this contradicts the inequality (A.5) on the cut. Therefore, we must have

\[
|R(t)| \leq 1
\]  

(A.12)

in the _entire_ cut plane. This reproduces Eq. (3.19) of § 3.

When \( F(t) \) has no left-hand cut, then we have to modify our argument slightly. In that case, letting \( b \to \infty \), \( G(t) \) of Eq. (A.2) will become

\[
G(t) = \exp\left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} dt' \left[ \frac{\log k(t')}{(t' - t) (t' - a)^{1/\alpha}} \right] \right\}
\]  

(A.13)

which is again a real analytic function with a cut at \( a < t < \infty \) and satisfies on the cut

\[
|G(t \pm ie)| = k(t) \geq |F(t \pm ie)|.
\]

We set now

\[
\xi = (t - a)^{1/\alpha}
\]  

(A.14)

which maps the entire cut \( t \)-plane onto the entire upper half \( \xi \)-plane. Defining \( \tilde{R}(\xi) \) by

\[
\tilde{R}(\xi) = R(t) = F(t)/G(t),
\]

then \( \tilde{R}(\xi) \) is an analytic function of \( \xi \) in the upper half \( \xi \)-plane and satisfies

\[
|\tilde{R}(\xi)| \leq 1
\]

on the real axis. Hence, we can apply again the Nevanlinna's theorem to obtain two possibilities, Eqs. (2.12) and (2.14) of § 2. We notice that a simple example of a case (i) function is given by

\[
R(t) = \exp[\alpha (a - t)^{1/\alpha}], \quad (\alpha > 0)
\]

which satisfies \( |R(t)| = 1 \) on the cut but increases exponentially at infinity along
negative axis.

To evaluate the explicit form of $G(t)$, it is more convenient to map the entire cut-plane into the interior of the unit circle, $|z|<1$ by the transformation,

$$
\left( \frac{t-a}{t+b} \right)^{1/2} = i \left( \frac{a}{b} \right)^{1/2} \frac{1-z}{1+z}.
$$

(A·15)

This maps both upper right and left cuts in $t$-plane onto the upper semi-circle, $|z|=1$ and $\pi \geq \arg z \geq 0$, while both lower right and left cuts are now transformed onto the lower semi-circle, $|z|=1$ and $2\pi \geq \arg z \geq \pi$. Also, three points $t=a$, $t=0$ and $t=-b$ are mapped into $z=1$, 0 and $-1$, respectively. The above results also apply to the case $b=\infty$.

Setting now

$$
F(t) = f(z), \quad G(t) = g(z), \quad R(t) = B(z),
$$

(A·16)

then $f(z)$, $g(z)$ and $B(z)$ are real analytic functions of $z$ in $|z|<1$. Moreover, after some calculations, we can rewrite Eq. (A·2) as

$$
G(t) = g(z) = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{e^{i\theta} - z} \log W(\theta) \right\}
$$

(A·17)

where $W(\theta)$ is the value of $k(t)$ on the cuts, i.e.,

$$
W(\theta) = k(t), \quad t = \frac{2ab}{(b-a) + (b+a)\cos \theta}.
$$

(A·18)

From Eq. (A·17), we see that $g(z)$ is an outer function.\(^{(21)}\)

Also, since we have $|R(t)| \leq 1$ in the entire cut plane (excluding the case (i) of Nevanlinna's theorem), this implies

$$
|B(z)| \leq 1
$$

(A·19)

inside $|z| \leq 1$. Let $\lambda$ be an interior point, i.e., $|\lambda| < 1$, and define\(^{(3)}\) $A(z)$ by

$$
A(z) = \frac{1-\lambda z}{z-\lambda} \frac{B(z) - B(\lambda)}{1 - B^*(\lambda) B(z)},
$$

(A·20)

excluding for a while, the case in which $B(z)$ is a constant with unit modulus. By its construction, $A(z)$ is an analytic function in $|z|<1$. Moreover, on the unit circle $|z|=1$, it satisfies an inequality

$$
|A(e^{i\theta})| \leq 1, \quad (0 \leq \theta \leq 2\pi)
$$

(A·21)

in view of Eq. (A·19). Therefore, by the maximum modulus theorem, we have

$$
|A(z)| \leq 1
$$

(A·22)

in $|z| \leq 1$. If we choose especially both $\lambda$ and $z$ to be real, then $B(z)$ and $B(\lambda)$ are also real because of the reality of $B(z)$. In that case, the inequality (A·22) will be rewritten as
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\[ \frac{B(\lambda) + \theta(z)}{1 + \theta(z)B(\lambda)} \geq B(z) \geq \frac{B(\lambda) - \theta(z)}{1 - \theta(z)B(\lambda)} , \quad (A\cdot23) \]

where \( \theta(z) \) is given by

\[ \theta(z) = \left| \frac{z - \lambda}{1 - \lambda z} \right| \leq 1 . \quad (A\cdot24) \]

Equation (A\cdot23) is an improvement over the inequality (A\cdot19). Also, it holds valid for the case where \( B(z) \) is a constant with unit modulus. Now we change the variable \( z \) into \( t \) with \( B(z) = \frac{F(t)}{G(t)} \) where \( F(t) \) and \( G(t) \) are defined by Eqs. (2\cdot18a) and (2\cdot19b) with \( N=1, \) and \( \Delta_1=\Delta. \) Then, (A\cdot23) immediately leads to Eq. (2\cdot21) if we notice \( \frac{\tilde{F}(m_2^\pm)}{\tilde{G}(m_2^\pm)} = \frac{g_1^2}{g_2^2} \) and if we choose \( z \) and \( \lambda \) to correspond to \( t=\Delta \) and \( t=m_2^\pm, \) respectively. Similarly, we can derive Eq. (3\cdot30). The inequality (A\cdot23) has been also applied for \( K_{l\ell} \)-decay problems by several authors.\(^5\)\(^2\)\(^2\)\(^1\)

Analogously, we can extend our method to Meiman's inequality of \S\ (4). Suppose that \( F(t) \) satisfies now

\[ \frac{1}{\pi} \left[ \int_{-\infty}^{\infty} dt k(t) |F(t + i\epsilon)|^2 \right] \leq I^2 , \quad (A\cdot25) \]

instead of Eq. (A\cdot1), where \( k(t) \) is again a given non-negative function defined on the cuts of \( F(t). \) After making the mapping Eq. (A\cdot15), we can rewrite (A\cdot25) as

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta \tilde{W}(\theta) \left| f(e^{i\theta}) \right|^2 \leq I^2 , \quad (A\cdot26) \]

where \( \tilde{W}(\theta) \) and \( f(z) \) are defined by

\[ \tilde{W}(\theta) = W(\theta) \frac{2ab(a+b)\left| \sin \theta \right|}{[(b-a) + (b+a) \cos \theta]^2} , \quad (A\cdot27) \]

\[ f(z) = F(t) . \quad (A\cdot28) \]

In the above, \( W(\theta) \) is the same function as in Eq. (A\cdot18). Then following the method described elsewhere,\(^3\)\(^6\) it is not difficult to prove the following inequalities:

\[ \frac{4ab}{a+b} G(0) |F(0)|^2 \leq I^2 , \quad (A\cdot29) \]

\[ |F(0)|^2 + \left| \frac{2(a-b)}{a+b} F(0) + \frac{2ab}{a+b} \frac{G'(0)}{G(0)} F(0) + \frac{4ab}{a+b} F'(0) \right|^2 \leq \frac{a+b}{4ab} \frac{I^2}{G(0)} , \quad (A\cdot30) \]

where \( G(t) \) is the same function given by Eq. (A\cdot2). When we let \( b \to \infty, \) then
these formulas reproduce immediately results of Refs. 6) and 23). If we want to evaluate bounds of $F(t)$ at $t = A$ (rather than at $t = 0$), then we have simply to shift the origin of $t$ by setting $\tilde{F}(t) = F(t + A)$ and apply our inequality to $\tilde{F}(t)$ instead of $F(t)$. Then, it gives results of § 4.

Now, in our applications, $k(t)$ has a simple form

$$k(t) = c \prod_{j=1}^{N} (t - \gamma_j)^{n_j},$$  \hspace{1cm} (A·31)

where $c, \gamma_j$, and $n_j$ are real constants with $-b \leq \gamma_j \leq a$. In this case, we can easily evaluate the explicit form of $G(t)$. For this purpose, it is sufficient to consider the case $k(t) = |t - \gamma|$, $(a \geq \gamma \geq -b)$. Then, using Jensen-Poisson integral formula as has been explained in Ref. 23), we compute

$$\exp \left\{ \frac{1}{\pi} \frac{(a-t)^{1/3}}{(b+t)^{1/3}} \left[ \int_{-\infty}^{\infty} - \int_{-\infty}^{-y} dt' \frac{\log|t' - \gamma|}{(t' - t)(t' - a)^{1/3}(t' + b)^{1/3}} \right] \right\}$$

$$= \frac{1}{a+b} (u + v)^{1/2}, \quad (a \geq \gamma \geq -b)$$  \hspace{1cm} (A·32)

where $u$ and $v$ are given by

$$u = (a-t)^{1/3}(b+t)^{1/3},$$

$$v = (a-\gamma)^{1/3}(b+\gamma)^{1/3}. \hspace{1cm} (A·33)$$

Then, we can easily compute various integrals of §§ 3 and 4 by this integral formula. Also, letting $b \rightarrow \infty$ in Eq. (A·32), it leads to

$$\exp \left\{ \frac{1}{\pi} (a-t)^{1/3} \int_{a}^{\infty} \frac{1}{(t'-t)(t'-a)^{1/3}} \log|t' - \gamma| \right\}$$

$$= \left[ (a-t)^{1/3} + (a-\gamma)^{1/3} \right]. \quad (\gamma \leq a) \hspace{1cm} (A·34)$$

This formula has been used to compute explicit form for $G(t)$ of § 2, (Eq. (2·15)). We may easily check that the absolute values of the right-hand side of Eq. (A·32) reproduces exactly $|t - \gamma|$ for values of $t$ on the cuts at $\infty > t \geq a$ and $-b \geq t > -\infty$. Also a similar remark applies to Eq. (A·34).

After having written this paper, the author has received a preprint by Creutz.\textsuperscript{10} He proves by positivity of $\text{Im} f(w + i\varepsilon)$ on the cut that the function $f(w)$ defined in § 3 has at most $2(n + 1)$ zero points, if $f(w)$ requires $2n$ or $2n + 1$ subtractions. However, nothing can be said for minimum numbers of zero points.

References


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