Author's Closure

The author would like to thank Dr. Levin for his kind words and interesting comments and questions. If the results are to be used for an actual sandwich shell, then the values of h and t will, of course, be known. On the other hand, if the results are to be applied as an approximation to a uniform shell, it appears reasonable to relate the dimensions so that the maximum pure moment and maximum direct stress are the same. Thus if H is the thickness of the uniform shell and K the yield stress in shear

$$M_0 = (1/2)\,KH^2 = 4\,kht$$

$$N_0 = 2\,KH = 4kh$$

Obviously, then, the equivalent sandwich dimensions are

$$kh = KH/2, \quad t = H/4$$

Therefore, the maximum elastic load for the uniform shell is approximately

$$p^* = \sqrt{3}\,KH/a$$

A comparison of this with the critical buckling load shows that

$$p_{cr}/p^* = \frac{(E/K)\,(H/a)^2}{4\sqrt{3}\,(1-\nu^2)}$$

As an example, if

$$E = 30 \times 10^6$$
$$K = 2 \times 10^4$$
$$\nu = 0.3$$

then

$$p_{cr}/p^* = 238\,(H/a)^2$$

In other words, if H/a is less than 0.065, the shell will buckle before any part of it becomes plastic, while if H/a is greater than 0.065 it will become at least partially plastic at a lower load. For shorter shells the equation for p_{cr} must be modified to account for the built-in ends, while the value of p^* is independent of length. Therefore the shorter a shell is, the smaller is the value of H/a at which plasticity will occur before buckling.

A Matrix Solution for the Vibration Mode of Nonuniform Disks[1]

S. H. Crandall.[2] In extending the methods of Myklestad[3] and Thomson[4] to organize the more involved computations for rotating-disk structures, the author has made a valuable contribution. The separation of the elastic behavior into a primary bending stiffness in the matrix E_k and the secondary stiffnesses due to centrifugal forces and membrane action represented in the matrix L_k is interesting.

Although for simplicity of exposition the author has introduced the summary matrix H, the writer would suppose that in actual calculation one would obtain the X_k in sequence according to Equation [53] of the paper. This would provide a picture of the appropriate mode shape which would be very useful in selecting the next trial ω. In similar computations with the Holzer and Myklestad processes the writer has found it efficient[5] to apply Rayleigh's principle to the approximate mode shape obtained from one calculation in order to get the next trial frequency.

It occurs to the writer that when the matrixes E_k for a particular disk have been obtained, they may be used for other design calculations besides those of natural frequencies. For instance, if when the disk is rotating at speed Ω its axle is tipped at an angular velocity $\dot{\gamma}$ it will be subjected to a gyroscopic force field $2\dot{\gamma}\Omega r$ cos $(\phi - \Omega t)$ per unit mass. The resulting deformations and stresses could be computed by using the matrixes E_k and L_k for $n = 1$ and $\omega = \Omega$ and the recurrence formula

$$X_{k+1} = E_k(L_kX_k + G_k)$$

where $G_k = (0, 0, 0, 2\mu_k\dot{\gamma}\Omega r_k^4)$. For this kind of computation a trial-and-error process would not be necessary but two calculations across the disk would be required. Thus, if the disk were free at the outer edge, the solution would be that linear combination of the results for $X_1 = (1, 0, 0, 0)$ and $X_2 = (0, 1, 0, 0)$ which satisfied the inner-edge boundary conditions.

Author's Closure

Professor Crandall has aptly pointed out that convergence of the successive frequency trials can be speeded by using Rayleigh's principle on the deflection shape generated at any intermediate guess. It should be noted that the deflection shape cannot be calculated directly in the course of calculating the residual determinant. When the calculation is initiated, the relationship between the deflection and slope at the free outer edge (for instance) is unknown. Therefore the calculation is performed in the form

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ h_{31} & h_{32} \\ h_{41} & h_{42} \end{bmatrix} = L_KE_{K-1}L_{K-1}\ldots E_kL_k\ldots E_1L_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

After testing the residual determinant for zero

$$\epsilon = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} \rightarrow 0$$

the actual state of deflection, slope, moment, and force at any intermediate station may be calculated by computing

$$X_k = E_kL_k\ldots E_1L_1 \begin{bmatrix} 1 \\ -h_{11}/h_{12} \\ 0 \\ 0 \end{bmatrix}$$

This would give at the clamped inner edge

$$X_K = \begin{bmatrix} 0 \\ (h_{21}h_{12} - h_{11}h_{22})/h_{12} \\ m_Kr_K^2 \\ v_Kr_K^3 \end{bmatrix}$$

Unless the residual determinant is zero, the slope at the inner edge will be nonzero and application of the Rayleigh principle must take into account this nonsatisfaction of the boundary conditions.

Professor Crandall points out an interesting extension in the application of the matrix treatment of disk problems in applying it to the problem of gyroscopic moments. The author has been working on a similar extension to the problem of steady-state,

[1] By F. F. Ehrich, published in the March, 1956, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 78, pp. 109–115.

[2] Associate Professor of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Mass. Mem. ASME.

[3] "New Method of Calculating Natural Modes of Uncoupled Bending Vibrations," by N. O. Myklestad, *Journal of the Aeronautical Sciences*, vol. 11, 1944, pp. 153–162.

[4] "Matrix Solution for the Vibration of Nonuniform Beams," by W. T. Thomson, Journal of Applied Mechanics, Trans. ASME, vol. 72, 1950, pp. 337–339.

[5] "Iterative Procedures Related to Relaxation Methods for Eigenvalue Problems," by S. H. Crandall, Proceedings of the Royal Society of London, England, series A, vol. 207, 1951, pp. 416–423.

axisymmetric pressure forces acting on disks acting in combination with the stiffening effect of disk rotation. The recurrence formula takes the form (with $\omega = 0$ and $n = 0$)

$$X_{K+1} = E_K(L_K X_K + P_K)$$

where $P_k = (0,0,0,p_k r_k^3)$ where p_k is the pressure force acting at any loading circumference.

In addition to the torsion calculation of Holzer and the beam or shaft calculation of Myklestad,[3] the author has found that the system of successive multiplication of elasticity and loading matrices is a very efficient format for formulation and calculation of a wide variety of one-dimensional problems in elasticity as well as two-dimensional problems reducible to one-dimensional form by separation of variables.

It may be of interest to add that experimental data obtained by the author using the methods of photoelasticity and the conducting-paper analogy are in close agreement with the results obtained analytically in the paper.[3] Some discrepancies occurring are always in the neighborhood of the hole's boundary where the assumptions of the theory may not be well justified. They can, however, be explained by the difficulty of making optical measurements very close to the rim of the hole due to the imperfection of the contact between the strip and the pin and the very narrow region in the vicinity of the rim of the hole which is hidden by the deformed pin. It is, therefore, not possible to determine experimentally the stresses at the rim exactly because in this region the stresses vary rapidly.

³ "La Distribution des Tensions autour d'un Trou de Rivet dues à un Effort appliqué sur ce Rivet," by P. S. Theocaris, Doctor's thesis presented at the University of Brussels, Belgium, in March, 1953.

The Stress Distribution in a Strip Loaded in Tension by Means of a Central Pin[1]

G. Mesmer.[2] This contribution to the question of the stresses in a pin-loaded strip is very interesting because of the unusual simplification of the problem; some of its results are encouraging for similar further calculations of loaded pins. Without mentioning it explicitly, the author's calculations are based upon the assumption that the pin (disk, loaded at its center) has the same thickness and equal elastic behavior as the strip, and that the front side of the disk, the pressure side, is rigidly connected with the inner surface of the hole in the strip. This is equivalent to assuming an infinitely high coefficient of friction between disk and hole. As the results are in agreement with some experimental data of similar cases for small values of $\lambda = d_i/d_0$, these assumptions seem to be reasonable, when the relatively wide strip does not deform very much. For larger values of λ, however, this is not valid, and one must assume certain relative tangential displacements between the deformable strip and the more rigid disk. In this case, the pressure distribution on the front half of of the disk changes; its maximum flattens or may even shift to the sides, and the "open slot" behind the disk becomes shorter.

As the directions of the principal stresses σ_1 and σ_2 are only approximately tangential and radial to the pin, their numerical values are not exactly the values of the pressure and the circumferential stress. The integral of $\sigma_1\, dy$ in the horizontal cross section is not necessarily equal to the load but must be larger. This effect may be partly responsible for the errors found by the author.

The writer hesitates to use the expression "exact" solution of a mechanical problem for solutions of this type. Most of them are exact only in the sense that they fulfill the two-dimensional equations for equilibrium and compatibility, assuming certain boundary conditions. These assumptions, however, are quite arbitrary and are, in fact, only approximations of the real physical problem, convenient for a rigorous mathematical treatment.

Author's Closure

The author appreciates the interest of Professor G. Mesmer and his valuable discussion of the paper. As is pointed out, the paper assumes that the diameter of the hole does not exceed the half width of the strip ($\lambda = a/b \leqslant 0.5$). The reason for this restriction is that the convergence of the series expressing the stress function ceases to be good for values of λ larger than 0.5 and that, in practice, λ would rarely be greater than 0.5.

¹ By P. S. Theocaris, published in the March, 1956, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 78, pp. 85–90.

² Head, Department of Applied Mechanics, Washington University, St. Louis, Mo.

Studies in Dynamic Photoelasticity[1]

A. J. Durelli.[2] The authors should be congratulated for developing Tuzi and Nisida's method of streak photography. Fig. 7 in their paper shows a striking corroboration of the theoretical thinking on wave reflection. The streak method, however, is subjected to very important limitations. The interpretation of the fringe pattern is complicated and the coverage of the field is restricted to a thin zone. Fig. 3 shows that the authors, with their technique, have not been able to detect the wave-propagation phenomenon in a disk under impact. The reaction at the end of the diameter opposite to the one on which the impact is applied takes place when only the $1^1/_2$-order fringe appears at the point of contact, and the rest of the disk is not stressed sufficiently to produce an optical response.

The writer and some of his associates have developed, in the past year and a half, a "whole field" method of dynamic photoelasticity, in conjunction with a grid method of dynamic-strain measurements. A plastic of the epoxy family, exhibiting a low modulus of elasticity, was especially developed for this purpose, and the photographs were taken using the commercially available Fastax camera. The description of these methods will be submitted for publication shortly. An example of the results that can be obtained is shown in Fig. 1 of this discussion. The point of the disk to which the load is applied shows fringes up to the ninth order before the reaction at the other end begins to develop; and the stress distribution can be seen within the entire field of the disk. The 60-microsec interval shown in the authors' disk-streak photograph is covered by ten frames like the ones shown in the writer's figure. It is obvious that this whole-field technique allows not only a much easier interpretation of the photoelastic patterns, but also a more precise determination of the wave propagation.

Another example of the kind of work that can be conducted using this method is shown in Fig. 2 of this discussion. Here the phenomenon of wave propagation in a rectangular strut, rigidly supported at one end and under a uniformly distributed impact at the other, is illustrated. Both photoelasticity and grids were used in this study of wave propagation. Two fixed wires (shown near the center of each photograph) were placed in the field to be used as a reference for measurements and to correct for film shrinkage. Some features of the wave-propagation phenomenon can be observed in these whole-field photo-

¹ By M. M. Frocht and P. D. Flynn, published in the March, 1956, issue of the Journal of Applied Mechanics, Trans. ASME, vol. 78, pp. 116–122.

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