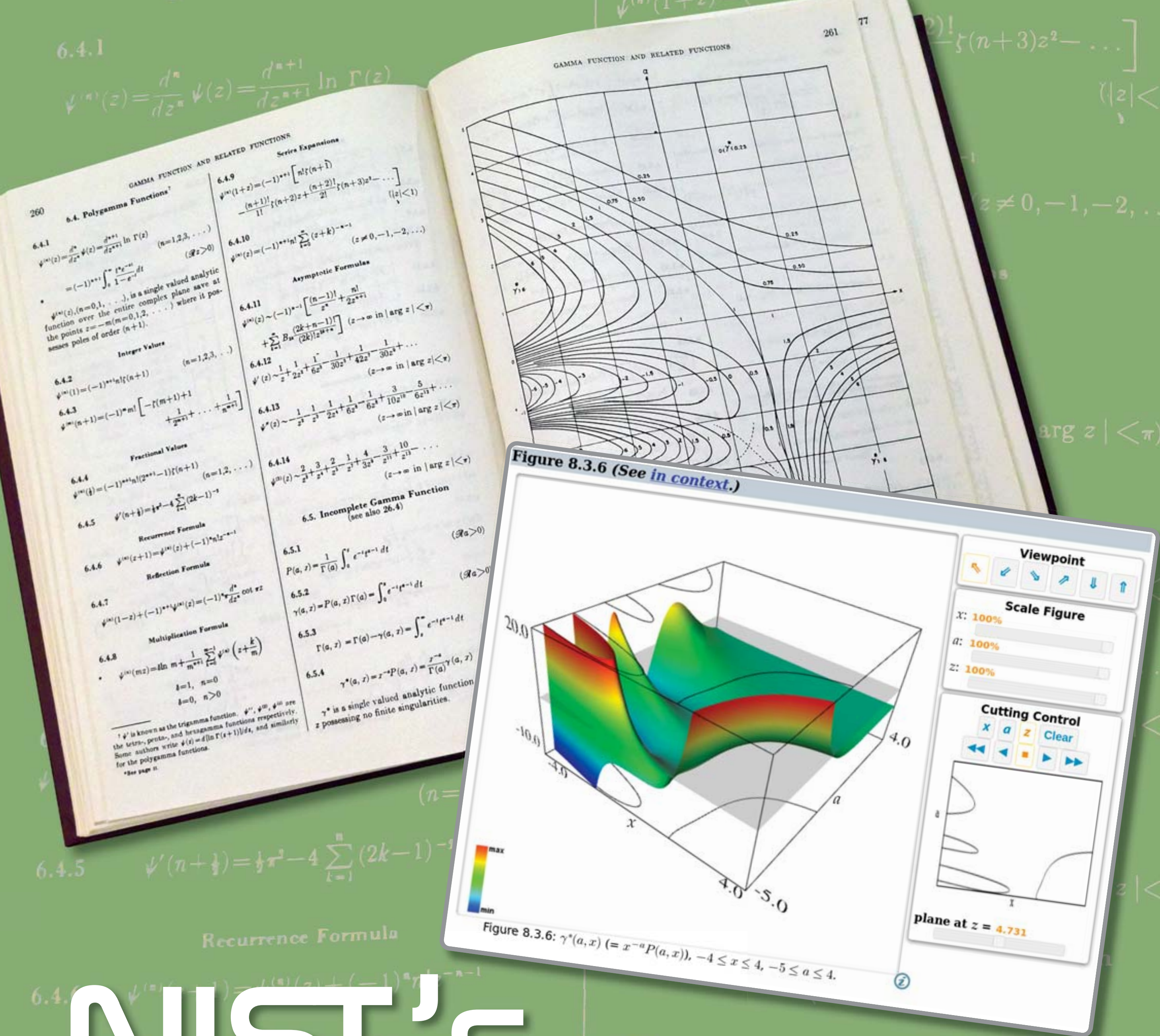
NIST's *Digital Library of Mathematical Functions* FREE

The half-century-old handbook commonly known as Abramowitz and Stegun enters the 21st century.

Barry I. Schneider; Bruce R. Miller; Bonita V. Saunders

Physics Today **71** (2), 48–53 (2018);
https://doi.org/10.1063/PT.3.3846



NIST's *Digital Library of Mathematical Functions*

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The half-century-old handbook commonly known as *Abramowitz and Stegun* enters the 21st century.

Barry Schneider, **Bruce Miller**, and **Bonita Saunders** are scientists in the applied and computational mathematics division at the National Institute of Standards and Technology in Gaithersburg, Maryland. They are part of a team responsible for the design, expansion, and maintenance of NIST's *Digital Library of Mathematical Functions*. They welcome any comments or corrections to the *DLMF* at dlmf-feedback@nist.gov.

Several years ago, I was invited to contemplate being marooned on the proverbial desert island. What book would I most wish to have there, in addition to the Bible and the complete works of Shakespeare? My immediate answer was: A&S [Abramowitz and Stegun].

—Michael Berry, "Why are special functions special?"
PHYSICS TODAY, April 2001, page 11

The information technology revolution is creeping into all aspects of our lives. Today more of us rely on digital information for personal and professional purposes than ever before. It is so much easier to search the Web to locate that information or to purchase the widget of one's desire than to go to a library or a retail store. And for scientists, journal articles and books are far easier to access online than anywhere else. Although there are many examples of online scientific material that go beyond their print equivalents—for instance, specialty dictionaries, encyclopedias, and millions of journal articles—the power of the Web to transcend the printed page is still in its infancy.

One classic scientific reference that the revolution has radically affected is the *Handbook of Mathematical Functions*, familiarly known as *A&S*, edited by Milton Abramowitz and Irene Stegun.[1] In this article we discuss how *A&S* was transformed into an online 21st-century resource known as the *Digital Library of Mathematical Functions*, or *DLMF*, and how that new, modern resource makes far more information available to users in ways that are quite different from the past. The *DLMF* also contains far more material—in many cases updated—than does *A&S*. (For a brief history of the evolution of *A&S* to the *DLMF*, see box 1.)

To set the stage for what follows—and to provide practical context for appreciating the utility of the *DLMF* for research and education—consider this hypothetical situation. Many years ago a researcher—let's call him Bill—was working on finding solutions to the radial Schrödinger equation for the attractive and repulsive Coulomb potential in terms of a set of square-integrable orthogonal functions. He had previously shown that it was indeed possible to expand the solutions in some complete, discrete set of functions and to relate the expansion coefficients to some set of orthogonal polynomials. For the Coulomb potential, the expansion coefficients satisfy a three-term recursion relationship described by so-called Pollaczek polynomials. When *A&S* was written, little was known about those polynomials, and Bill had to work out their properties largely on his own or with colleagues.

When Bill recently decided to generalize his older results to physical parameters not part of the original work, he found, much to his surprise and delight, that the *DLMF* has a much more extensive and up-to-date treatment of many special functions, orthogonal polynomials included, and treats the Pollaczek polynomials in some detail. Had that information been accessible when Bill first began the project, his work would have been significantly easier. Bill also found many useful references and notes in the metadata that are listed in the *DLMF*'s "info boxes." He is now actively extending the earlier research to the physically important ranges of the nuclear charge and energies for the attractive Coulomb potential.

Not your thesis adviser's handbook

Once you broaden your perspective of "a book on the Web" to include potentially much more than the adaptation of a traditionally printed book for a computer screen and embrace it as a full-fledged electronic resource on the internet, the power of hypertext and the vast computational capabilities of the Web open a rich tableau of possibilities for enhancing the book's content, utility, and ease of use. For the developer, that abundance can be frustrating, as there are so many choices and possibilities to explore; moreover, the Web's evolution takes place in fits and starts, with promising technologies sometimes stalling and at other times becoming suddenly essential.

Yet the traditional handbook has virtues, such as conciseness, stability, and permanence, that are worth preserving in a Web-based resource. By adopting an enriched authoring markup language for the source documents and by leveraging hypertext links, we and other *DLMF* developers have been able

BOX 1. ABRAMOWITZ AND STEGUN AND THE *DLMF* PROJECT

In 1964 the National Bureau of Standards (NBS) published the *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*,[1] edited by Milton Abramowitz and Irene Stegun, as volume 55 of its *Applied Mathematics Series*. Originally intended as a compendium of numerical tables for evaluating the special functions—including gamma, Legendre, Jacobian, Bessel, and more than a hundred other functions commonly used in applied mathematics—it was augmented with selected properties and identities to make the tables easier to use. Ironically, that additional material was what led to the handbook's long-term popularity. Fondly referred to as *Abramowitz and Stegun*, and even simply *A&S*, it became the most widely distributed and cited NBS publication in the first 100 years of the institution's existence.

At a hefty 1046 pages, the handbook did more than concisely provide critically useful mathematical data. It served to standardize definitions and notations for special functions. Both services were at the heart of NBS's mission. And yet by the late 1990s, *A&S* was showing its age and was due for more than a simple update with errata. Important advances in the special functions had taken place: More properties of many functions had been discovered, and advances in applied mathematics had considerably enlarged the field to include many new and useful functions.

The advent of the Web had also opened unique opportunities for publishing. In 1998 NIST (the name NBS had adopted a decade earlier) initiated an effort to assemble the available information on the current state of the special functions into a freely accessible online resource that would take advantage of the then relatively new potential of the Web. Over the ensuing decade, with the generous sponsorship of NSF's Knowledge and Distributed Intelligence Initiative and with continuing internal support from NIST, 29 experts in the various special functions were contracted to write material for the new handbook.

Whereas roughly half of *A&S* had consisted of numerical tables, the revision would have none. Instead, the amount of mathematical content doubled. New chapters on Lamé, Heun, and Painlevé functions were included; so were integrals with coalescing saddles. The result was 36 chapters long (compared with 29 in *A&S*) and boasted a more extensive bibliography. Since it was a handbook, mathematical proofs were not included, but great care was taken to clarify any constraints on the validity of all formulas and to reference their original sources. Significantly more graphs and visualizations were also included. In 2010, after 11 years of dedicated effort, the enhanced online resource was released as the *Digital Library of Mathematical Functions* at https://dlmf.nist.gov. The printed companion handbook was published as the *NIST Handbook of Mathematical Functions*.[7]

to layer the presentation of information. For users interested in browsing, we strove to preserve the book's concise, telegraphic style, which emphasizes the essential, selected properties of the functions. Most users will find the information they need simply by scanning the relevant sections. When that is insufficient, they can turn to our search engine.

Once our friend Bill realized that Pollaczek polynomials were relevant to his problem, for instance, he searched for "pollaczek recurrence" to locate the specific relation he needed. That's fine as far as it goes, but in a context so full of mathematical formulas and bereft of text, the usual text search is insufficient; the search cries out for a math-aware search engine. Our search tool[2] recognizes not only mathematical notations but also the semantics of the symbols as well. Thus, as exemplified in box 2, the proper and common names of functions or their classifications can be used as search terms.

For researchers interested in digging a bit deeper, an additional layer of information reveals the symbols used, the sources of the material, cross-references, and so forth. The mathematical symbols themselves are linked to their definitions to reduce ambiguities of notation. The selective nature of a handbook virtually guarantees that not every useful relation will be presented. But by providing the specific source of each equation, the methods by which it can be derived, and other relevant expository annotations, the *DLMF* offers a researcher alternative sources of information or guidance on how to derive what she needs for her application.

What kind of enriched markup can enable those features? Most mathematicians prefer to write using LaTeX, a powerful, expressive markup system that provides high-quality typesetting, particularly for mathematics. That the system is extensible—armed with tools developers can use to define new descriptive terms and shorthands as needed—makes it notoriously difficult to convert to HTML for the Web. And yet that is exactly its strength for the *DLMF*. A system of semantic macros for many mathematical functions and concepts ensures that the functions and concepts retain their underlying meaning.

Converting that enriched LaTeX markup into an intermediate XML representation allows for the layering of information to capture a surface presentation of the functions for users and yet still provide them with access to a database that can be mined for additional details about the functions and their connections to other, related mathematics. Those extra details, or metadata, are what we use to link symbols to their definitions and to populate indices, notation lists, and info boxes. Because the tools needed for that job are not available off the shelf, we created a system called LaTeXML (see http://dlmf.nist.gov/LaTeXML), which converts the author's LaTeX into XML and thence into HTML and the mathematical markup language MathML. Box 2 illustrates the benefits of that approach.

The stability of a traditional handbook is maintained by securing, across updates and even new editions of the *DLMF*, the connection between equations, their reference numbers (such as equation 14.3.6 in box 2), and permanent URLs (so-called permalinks, such as http://dlmf.nist.gov/14.3.E6 in box 2), so that the *DLMF* can be used as a resource for decades to come. The permalinks, moreover, provide a convenient way for researchers to exchange information from the *DLMF*.

Behind the plots

The graphs and visualizations in the *DLMF* are hidden gems, often overlooked as researchers peruse the website. A re-

BOX 2. A MATH-AWARE SEARCH ENGINE AND LAYERED INFORMATION

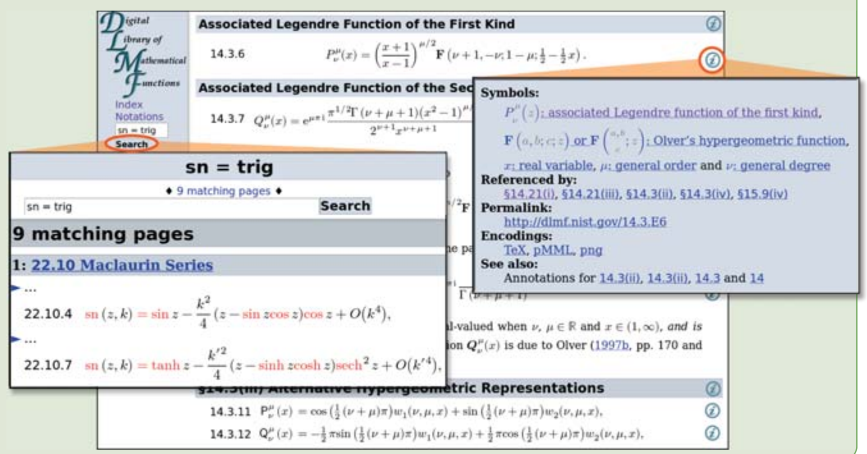
NIST's *Digital Library of Mathematical Functions* (*DLMF*) has a sophisticated search engine. Like other search engines, it can retrieve information using textual terms, but it also recognizes queries phrased as math operators, symbols, or conventional LaTeX commands. The latter are then matched against mathematical expressions in the *DLMF*.

Moreover, the search engine recognizes the meaning of the symbols. For example, both "*J*" and "Bessel" match the Bessel function J_ν. The figure's left inset shows the result of searching for the Jacobian elliptic function sn(z, k) in terms of trigonometric functions. While the "sn" matches the appropriate elliptic function, the term "trig" or "trigonometric" matches any trigonometric function, such as sine, cosine, or hyperbolic tangent. One benefit of such technologies as the mathematical markup language MathML is that the search engine can highlight the portions of formulas that match the query terms and provide some reassuring feedback to users.

On any given webpage of the *DLMF*, there's more than meets the eye. Hovering over or clicking on the information icon ⓘ next to a formula reveals a variety of metadata (right inset)—data about the data—associated with the formula. In this case, the metadata include listings of defined symbols, links to original sources, cross-references, and alternative formats, such as TeX code, MathML, and images.

The permalink gives the permanent URL for the formula; here it indicates that 14.3.E6 will always refer to this formula, independent of errata, updates, or future editions of the *DLMF*. Other metadata include any relevant keywords, links to *A&S*, notes on the formula's derivation, and any other annotations, classifications, or errata.

In addition to the list of symbols shown in the info boxes, individual symbols in the formula hyperlink to their definitions in the *DLMF*—another benefit of MathML. Clicking on a symbol will clarify and disambiguate it from similar-appearing notations. One must, as the saying goes, know one's *P*s and *Q*s, as there are several distinct functions denoted by each.



searcher can use many strategies to gain insight into a function that describes physical phenomena. She might study its differential equation, compute its zeros using Newton-type methods or asymptotic expansions, or use some other numerical and analytical technique to examine the function. However, an accurate graphical representation can provide that aha moment or quickly send her back for more analysis or computation when the plot differs from what is expected. Indeed, our own plots of the complex zeros of Bessel functions of positive integer order alerted us to inaccuracies in diagrams published in Frank Olver's 1954 asymptotic expansion paper[3] and repeated in *A&S*. The new plots showed that the arrangement of zeros needed to be moved to the opposite side of their asymptotes (see http://dlmf.nist.gov/10.21.ix).

As technology has evolved for viewing graphics on the Web, so has our development of graphics for the *DLMF*. That evolution made it possible to replace the static graphs in the original handbook with close to 600 2D and 3D figures, including 200 that can be interactively rendered by users as 3D visualizations. The clarity and responsiveness of the visualizations are a product of so-called WebGL-based technology[4] and our own careful attention to plot and data accuracy. The JavaScript-enabled WebGL application programming interface (API) provides portability and allows users to view the visualizations in major Web browsers without downloading a 3D graphics plug-in.

We achieved plot accuracy—that is, accurate graphical representation of function features—by designing computational grids, or meshes, that effectively capture key function attributes, such as zeros, poles, and branch cuts.[5] To tackle the issue of data accuracy, we computed function values using at least two different methods, with codes from reliable repositories, from standard computer algebra packages, or even from *DLMF* chapter authors. We resolved any discrepancies through discussions with the chapter author and a close examination of function definitions and properties.

Although illustrative diagrams and graphs appear throughout the *DLMF*, most figures that represent function curves or surfaces can be specifically found in the graphics sections of each chapter. Clicking on a static 3D image opens a page with an array of interactive options that allow users to rotate, scale, zoom, or otherwise manipulate the graphs. Figures 1, 2, and 3 exemplify some of the capabilities.

The future of the *DLMF*

NIST is committed to the *DLMF*'s long-term maintenance and further development, both in the evolution of its mathematical content and in its presentation on the Web. We issue quarterly updates, including errata, corrections, and clarifications. As we track the evolving Web technologies—for example, HTML5, MathML, and WebGL—and different usage patterns, we add new features and capabilities. Over time, we plan to make the site increasingly accessible and to add support for mobile devices.

Occasionally, new material becomes relevant to applications and is added to the website. A revision of the chapters on orthogonal polynomials and Painlevé transcendents is under way, and a completely new chapter on orthogonal polynomials of several variables is in progress. We also continue to improve the metadata associated with functions. The metadata are now presented in human-readable form, but we are exploring ontologies to recast the data in machine-readable forms, an advance that will strengthen and extend our search engine's

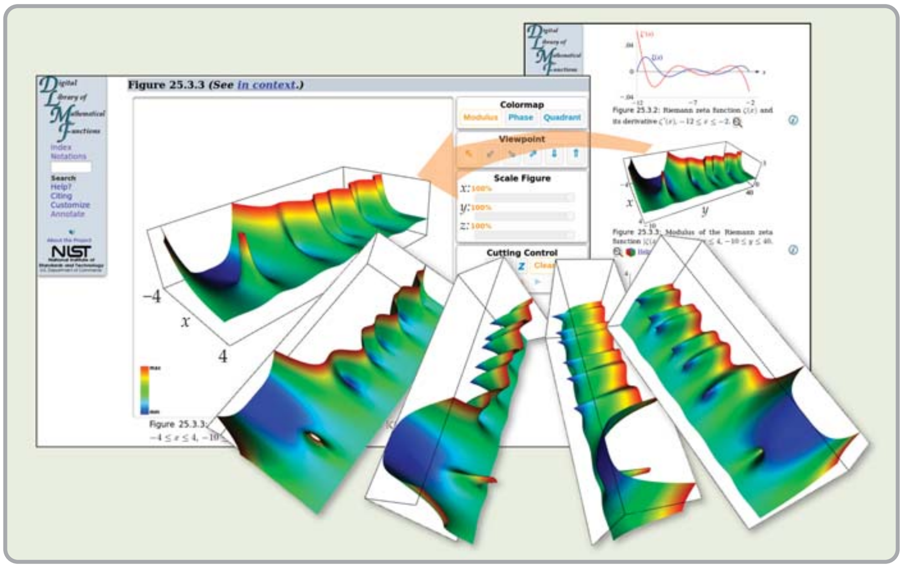


FIGURE 1. INTERACTIVE VISUALIZATIONS in the *Digital Library of Mathematical Functions* (*DLMF*). The complex shape of the modulus of the Riemann zeta function $|\zeta(x + iy)|$, an important function in number theory, quantum field theory, and other areas of mathematics and physics, illustrates the difficulty in displaying key features of a function surface with a single static image. The *DLMF* allows users to rotate a function surface in any direction and manipulate it with other tools to explore it. Clicking on the static image of the Riemann zeta function in one webpage (top right) opens a new page (top left) containing an embedded figure with control side panels. The original static image shows the simple pole at $x = 1$, $y = 0$, but obscures the nontrivial zeros on the critical line at $x = 0.5$, where the Riemann hypothesis asserts they lie. Such visualizations were designed using WebGL-based technology, which lets one view the graphics on different platforms without downloading a special browser plug-in.[4] (Adapted from *DLMF* screenshots. See chapter 25, "Zeta and Related Functions," for more information.)

capabilities and improve the ability of other, external search engines to mine features and data in the *DLMF*. In our first efforts we used MathML technology (more specifically, Presentation MathML) to represent the mathematical formulas for display in browsers. But our longer-term goal is to use Content MathML to encode representations that could be exported to applications for direct use in computations and that could further improve accessibility.

Several spin-off projects have also begun. The motivation for one—the *DLMF* Standard Reference Tables on Demand Project—is described succinctly by the feedback we received from a user in September 2016:

> It is no longer useful to include large tables of function values for the purpose of interpolation. But a set of test values for each function with various decimal precisions would be useful for developers as they try to test implementations of special functions. This could be the beginning of a benchmark for numeric software. —E. Smith-Rowland

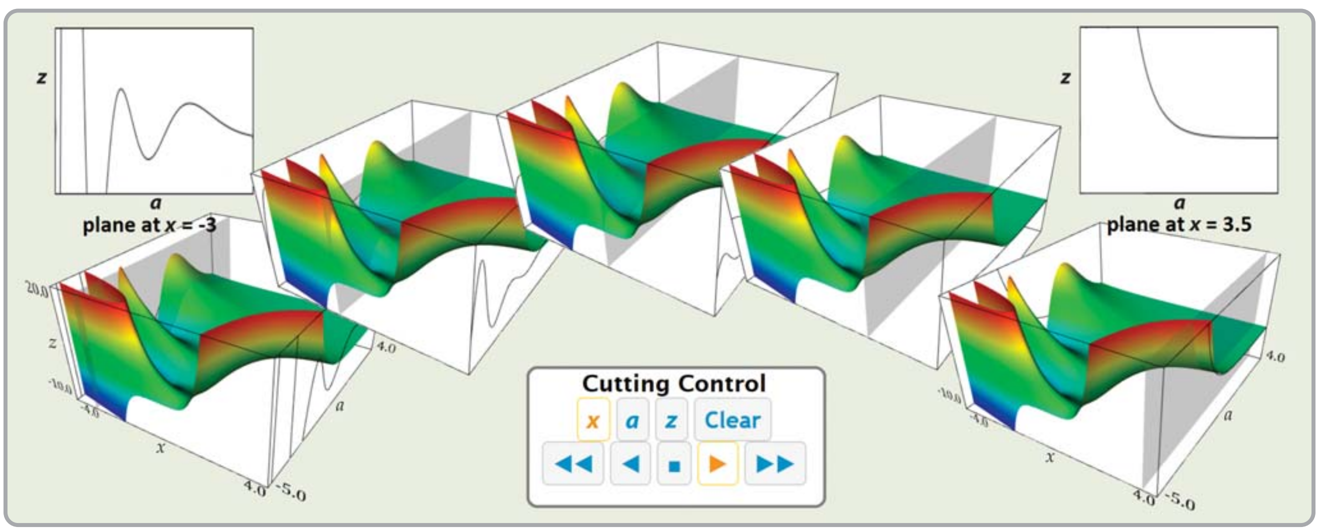


FIGURE 2. CUTTING THROUGH A SURFACE. One of the visualization options offered in the *Digital Library of Mathematical Functions* (*DLMF*) is cutting-plane control. Observing how a plane intersects a function surface can provide users with a unique perspective. These images show a cutting plane moving in the *x*-direction through a surface that represents the incomplete gamma function $\gamma^*(a, x)$. The two-dimensional curves that mark the intersection of the plane with the surface are projected onto opposite ends of each bounding box and displayed in a pop-up window. Users can manipulate standard media controls to create an animation or manually move the plane using a slider bar. In the case of figure 1's Riemann zeta function surface, a view of nontrivial zeros is revealed by cutting the surface with a plane at $x = 0.5$. (Adapted from *DLMF* screenshots. See chapter 8, "The Incomplete Gamma and Related Functions," for more information.)

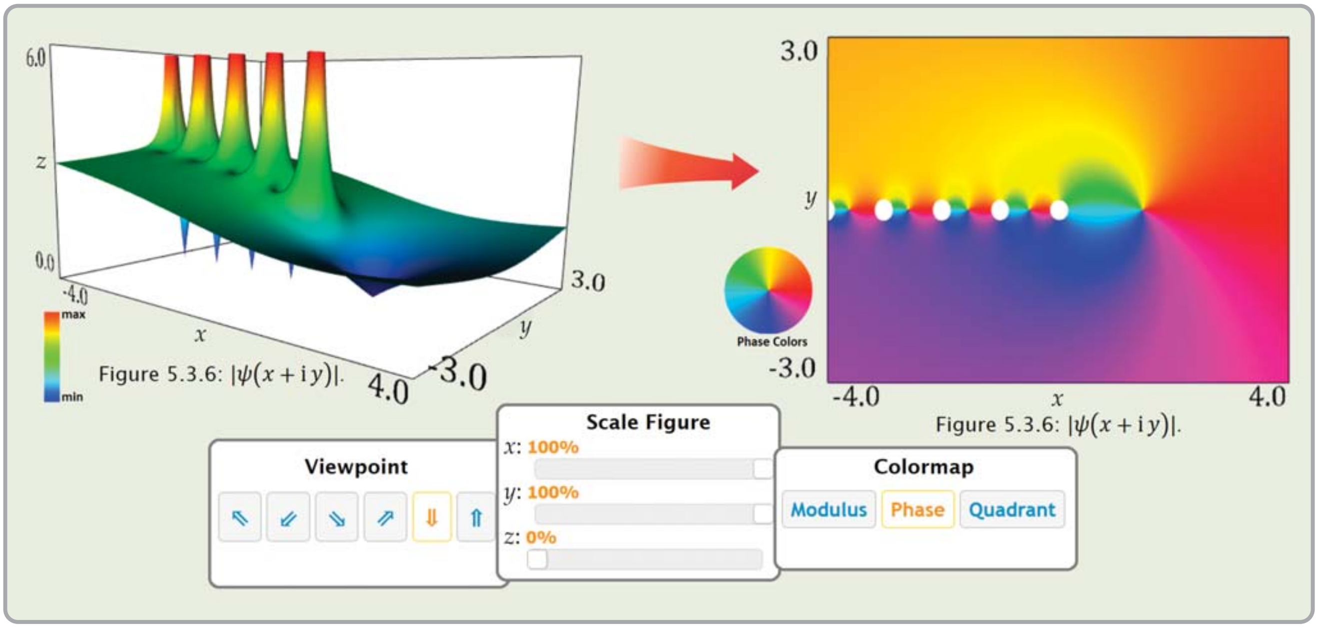


FIGURE 3. CREATING A PHASE DENSITY PLOT. One may use interactive control options to produce graphical displays that reveal significant features of a function. One display (left) represents the modulus of the complex digamma function $\psi(x + iy) = \Gamma'(x + iy)/\Gamma(x + iy)$, where Γ denotes the complex gamma function. Users can construct a phase density plot (right) of ψ by setting the controls as shown: phase colormap, top view, z scaled to zero. Like the gamma function, the ψ function has simple poles (seen as circular holes) at the nonpositive integers along $y = 0$. The phase changes by 2π radians as one completely circles any of those poles. Circling the zeros between those poles in the opposite direction yields the same 2π phase change. The phase-reversal pattern between successive poles and zeros is analogous to that of a fluid flow produced by a semi-infinite line of vortices with alternating directions of flow. (Adapted from screenshots of the *Digital Library of Mathematical Functions*.)

Our eventual goal is an online testing service, by which users can generate high-precision tables of special-function values with certified error bounds that can be compared with their own uploaded function data. A collaborative effort between the NIST applied and computational mathematics division and the University of Antwerp computational mathematics research group has already produced a fully operational beta version of the site. It's publicly available at http://dlmftables.uantwerpen.be.

Yet another project is the Digital Repository of Mathematical Formulae (DRMF), a wiki-based compendium of formulas for orthogonal polynomials and special functions to promote interaction in the OPSF community. The *DRMF* will be expandable but moderated, so that new formulas from the literature can be added to it after they have been reviewed. To create a solid foundation, the wiki is being seeded with formula data from the library and other sources, which require the extension and development of various format-conversion processors.[6]

Come explore!

With the *Digital Library of Mathematical Functions*, NIST has attempted to preserve the best attributes of *A&S*, while adapting them to the tools and technologies of the Web. It is a vibrant and evolving product that draws on new knowledge of special functions discovered since 1964 and corrects and expands the older material in the handbook. A group of associate editors who are experts in various special functions provide continuing advice on individual chapters, and a group of senior associate editors provide overall advice to the project.

We invite readers to explore the library: Hover the mouse over intriguing objects, symbols, and graphics to see behind the scenes. Open the info boxes to see what other possibly useful data may be available. We believe that the library will prove as useful to scientists and engineers of today and tomorrow as *A&S* has been since 1964.

We appreciate the helpful comments and suggestions of Eric Shirley (NIST) and William Reinhardt (University of Washington). We also acknowledge the efforts of the more than 50 authors, validators, editors, and developers of the DLMF. *A complete list of them is at http://dlmf.nist.gov/about/staff.*

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