The Temperature Dependence of Phonon Velocity and Roton Minimum in Liquid He II

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The temperature dependence of phonon velocity and roton minimum is investigated from the microscopic point of view, by taking into account the contributions from the interactions between quasi-particles given in previous papers. In consequence, we find that the phonon-roton interactions and the roton-roton interactions play very important roles in the temperature dependence of phonon velocity and roton minimum, respectively. The results are compared with experiments to yield fairly good agreement.

§ 1. Introduction

Recently Dietrich et al. have made measurements of the temperature dependence of roton minimum. On the other hand the temperature dependence of phonon velocity was studied experimentally by Whitney and Chase in 1962 and they found the existence of a maximum point in the curve of phonon velocity at 0.65°K. This behavior of the temperature dependence of phonon velocity has been studied extensively by Khalatnikov et al. on the basis of the phenomenological kinetic equations for the distribution functions of phonons and rotons which are regarded as classical gases, respectively. From the microscopic point of view, ter Haar et al. have developed a theory for the temperature dependence of phonon velocity on the basis of the perturbation theory at finite temperatures, by assuming Landau's phenomenological phonon-phonon interactions. They, however, considered only a part of all diagrams which contribute to the self-energy, and the neglected diagrams give divergent results.

The purpose of this paper is to calculate the temperature dependence of phonon velocity and of roton minimum by taking into account the effect of all diagrams which contribute to the second-order self-energy. The difficulty of the divergence of self-energy which arises from the diagrams ignored by ter Haar and others will be removed with the aid of the method proposed in previous papers (referred to as I and II), and later by Nishiyama.

Section 2 is devoted to the derivation of a divergence free formula for the excitation spectrum at finite temperatures. In § 3, the temperature dependence of phonon velocity is calculated quantitatively on the basis of the formula derived in § 2, and is compared with experimental results to have a good agreement. In § 4, the temperature dependence of roton minimum is calculated to yield also a fairly good agreement with experiments.
§ 2. The excitation energy at finite temperature

Consider a system of \( N \) interacting helium atoms of mass \( m \) enclosed in a cubic box of volume \( V \). The Hamiltonian of this system is described in terms of density fluctuation operator \( \rho_\kappa \) and velocity operator \( v_\kappa \) in the following form:

\[
H = E_0^B + \sum_\kappa E_\kappa B_\kappa^* B_\kappa + \sum_{p, q \neq 0} \frac{1}{3} \Gamma_\kappa (p, q) (B_p^* B_q^* B_{-p-q} + B_{-p-q} B_q B_p) \\
+ \sum_{p, q \neq 0} \Gamma_\kappa (p, q) (B_p^* B_p B_{p+q} + B_p B_{p+q} B_{-p})
\]

(2·1)

up to the order of \( N^{-1/2} \), where

\[
E_0^B = \frac{N(N-1)}{2V} \nu(0) + \frac{1}{2} \sum_{k \neq 0} \left( E_k^B - \frac{k^2}{2m} - \frac{N}{V} \nu(k) \right)
\]

gives the ground-state energy in the lowest approximation. The Bogoliubov excitation energy \( E_k^B \) is expressed by

\[
E_k^B = \frac{k^2}{2m \lambda_k} = \frac{k}{2m} \sqrt{k^2 + c^2(k)}
\]

(2·2)

and the structure factor \( \lambda_k \) in the lowest approximation is

\[
\lambda_k = k/\sqrt{k^2 + c^2(k)},
\]

(2·3)

where

\[
c(k) = \sqrt{\frac{4N m \nu(k)}{V}}
\]

and \( \nu(k) \) indicates the Fourier transform of interaction potential between two helium atoms. The functions \( \Gamma_\kappa (p, q) \) and \( \Gamma_\kappa (p, q) \) in (2·1) are given by

\[
\Gamma_\kappa (p, q) = \frac{1}{8m \sqrt{N}} \nu^{1/2}(p \lambda q) \left\{ (p \cdot q) \left( 1 + \frac{1}{\lambda_p \lambda_q} \right) - (p \cdot (p + q)) \left( 1 + \frac{1}{\lambda_p \lambda_{p+q}} \right) \right. \\
- \left( q \cdot (p + q) \right) \left( 1 + \frac{1}{\lambda_q \lambda_{p+q}} \right) \}
\]

and

\[
\Gamma_\kappa (p, q) = \frac{1}{8m \sqrt{N}} \nu^{1/2}(p \lambda q) \left\{ (p \cdot q) \left( 1 - \frac{1}{\lambda_p \lambda_q} \right) - (p \cdot (p + q)) \left( 1 - \frac{1}{\lambda_p \lambda_{p+q}} \right) \right. \\
- \left( q \cdot (p + q) \right) \left( 1 + \frac{1}{\lambda_q \lambda_{p+q}} \right) \}
\]

(2·4)

The excitation energy in liquid helium II at finite temperature is calculated by making use of the method of a temperature Green function\(^{10}\) defined by
The symbol $T_r$ indicates the chronological operator concerning temperature $r$. The second-order self-energy is given by the following three diagrams shown in Fig. 1. From the three diagrams (a), (b), (c) in Fig. 1, we can readily write down the expressions

$$\Sigma^{(a)}(k, \omega_n) = 18 \times \frac{-1}{\beta} \sum_{\nu'} \sum_{p'} \left( \frac{\Gamma_\nu(k, p')}{3} \right)^2 \tilde{G}_0(p, \omega_{\nu'}) \tilde{G}_0(-p - k, -\omega_{\nu'} - \omega_n),$$

(2.6)

$$\Sigma^{(b)}(k, \omega_n) = 2 \times \frac{-1}{\beta} \sum_{\nu'} \sum_{p'} \Gamma_\nu(k, p') \tilde{G}_0(p, \omega_{\nu'}) \tilde{G}_0(p + k, \omega_n - \omega_{\nu'})$$

(2.7)

and

$$\Sigma^{(c)}(k, \omega_n) = 4 \times \frac{-1}{\beta} \sum_{\nu'} \sum_{p'} \Gamma_\nu(p, k') \tilde{G}_0(p, \omega_{\nu'}) \tilde{G}_0(p - k, \omega_{\nu'} - \omega_n)$$

(2.8)

for the proper self-energy. In the above equations, $\tilde{G}_0(p, \omega_n)$ is the Fourier transform of the free temperature Green function given by

$$\tilde{G}_0(p, \omega_n) = \frac{1}{i\omega_n - E_{p^B}},$$

(2.9)

where $\omega_n = 2n\pi/\beta$ ($n = 0, \pm 1, \cdots$). After taking the summation concerning $n$ and replacing $i\omega_n$ by $E_{k^B} + i\eta$, we get the second-order excitation energy at temperature $T = (k_B\beta)^{-1}$ in the following form:

$$\Sigma^{(a)}(k, E_{k^B}) = -2 \sum_{p} \frac{\Gamma_\nu(k, p')^3}{E_{p^B} + E_{p^B} + E_{p+k}^B - n(E_{p^B})},$$

(2.10)

$$\Sigma^{(b)}(k, E_{k^B}) = 2 \sum_{p} \frac{\Gamma_\nu(k, p')^3}{E_{p^B} - E_{p} + i\eta + n(E_{p^B})} + 4 \sum_{p} \frac{\Gamma_\nu(k, p')^3}{E_{p^B} - E_{p} - E_{p+k}^B + i\eta + n(E_{p^B})}$$

(2.11)

and
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\[ \Sigma^c(k, E_k) = -4 \sum_p \frac{\Gamma_s(p, k)^2}{E_k^B + E_p^B + E_{p+k}^B - E_p^B + \eta} (n(E_p^B) - n(E_{p+k}^B)), \tag{2·12} \]

where

\[ n(E_p^B) = \frac{1}{e^{\beta E_p^B} - 1}. \tag{2·13} \]

From (2·10), (2·11) and (2·12), the excitation energy \( E_k(T) \) at \( T \) is written by

\[ E_k(T) = E_k^B + \Sigma^a(k, E_k^B) + \Sigma^b(k, E_k^B) + \Sigma^c(k, E_k^B). \tag{2·14} \]

D. ter Haar and others derived a similar result to calculate the absorption coefficient of the first sound in liquid helium II on the basis of the Landau phenomenological Hamiltonian. They consider only the effect of \( \Sigma^c(k, E_k^B) \) in (2·14) and do not take the contributions from \( \Sigma^a(k, E_k^B) \) and \( \Sigma^b(k, E_k^B) \) into consideration. Since \( \Sigma^a(k, E_k^B) \) and \( \Sigma^b(k, E_k^B) \) play a very important role in the temperature dependence of phonon velocity and roton minimum, as will be seen later, we must consider the effect of new terms \( \Sigma^a(k, E_k^B) \) and \( \Sigma^b(k, E_k^B) \) in (2·14). However the first terms on the right-hand side of (2·10) and (2·11) are strongly divergent. In order to avoid this serious difficulty, we make use of the method proposed in the previous paper. In this paper, the excitation spectrum at zero temperature has been calculated to yield a convergent result which is in good agreement with the experiment. Nishiyama derived later essentially the same result by making use of the ordinary perturbation formalism. Since the method of Nishiyama is more convenient for the present approach than the original method in I, we use his method to overcome the difficulty.

Let us now introduce the Feynman energy \( E_k^{F12} \) defined by

\[ E_k^{F12} = \frac{k^2}{2mS(k)}. \tag{2·15} \]

\( S(k) \) is the structure factor of liquid helium II at \( 0^\circ C \):

\[ S(k) = \langle G | \rho_{k \rho_{k^*}} | G \rangle / \langle G | G \rangle = \lambda_k \langle G | (B_{-k} + B_{k^*}) (B_k + B_{-k^*}) | G \rangle / \langle G | G \rangle, \tag{2·16} \]

where \( | G \rangle \) denotes the ground state of the total Hamiltonian (2·1). After the straightforward calculation, we have

\[ S(k) = \lambda_k + 4 \lambda_k \sum_p \left\{ \frac{\Gamma_s(k, p)^2}{(E_k^B + E_p^B + E_{p+k}^B)^3} + \frac{\Gamma_s(k, p) \Gamma_s(k, p)}{E_k^B (E_k^B + E_p^B + E_{p+k}^B)} \right\}. \tag{2·17} \]

Substitution of (2·17) into (2·15) gives

\[ E_k^{F12} = E_k^B - 4E_k^B \sum_p \left\{ \frac{\Gamma_s(k, p)^2}{(E_k^B + E_p^B + E_{p+k}^B)^3} + \frac{\Gamma_s(k, p) \Gamma_s(k, p)}{E_k^B (E_k^B + E_p^B + E_{p+k}^B)} \right\}. \tag{2·18} \]
in second-order approximation. From (2.14) and (2.18) we get
\[
E_k(T) = E_k^F + 2 \sum_p \left\{ \frac{\Gamma_a(k, p) (E_k^B - E_p^B - E_{p+k}^B) + \Gamma_s(k, p) (E_k^B + E_p^B + E_{p+k}^B)^2}{(E_k^B + E_p^B + E_{p+k}^B)^2 (E_k^B - E_p^B - E_{p+k}^B + i\eta)} \right\} \\
- 4 \sum_p \frac{\Gamma_a(k, p)^2}{E_k^B + E_p^B + E_{p+k}^B} n(E_p^B) \\
+ 4 \sum_p \frac{\Gamma_s(k, p)^2}{E_k^B - E_p^B - E_{p+k}^B + i\eta} n(E_p^B) \\
- 4 \sum_p \frac{\Gamma_s(p, k)^2}{E_k^B + E_{p+k}^B - E_p^B + i\eta} (n(E_p^B) - n(E_{p+k}^B)).
\] (2.19)

The second term on the right-hand side of (2.19) is independent of temperature \( T \) and gives a correction term for the Feynman energy \( E_k^F \) at zero temperature. This correction term is strongly convergent in contrast to the first terms in \( \Sigma^{(a)}(k, E_k^B) \) and \( \Sigma^{(b)}(k, E_k^B) \). On the other hand, the third and fourth terms on the right-hand side of (2.19) give the temperature dependence of the excitation spectrum.

§ 3. The temperature dependence of phonon velocity

In this section we shall investigate the temperature dependence of phonon velocity on the basis of formula (2.19) derived in the preceding section. The temperature dependent part \( \delta E_k(T) \) in (2.19) is given by
\[
\delta E_k(T) = \delta E_k^{(a)}(T) + \delta E_k^{(b)}(T) + \delta E_k^{(c)}(T),
\] (3.1)

where
\[
\delta E_k^{(a)}(T) = -4 \sum_p \frac{\Gamma_a(k, p)^2}{E_k^B + E_p^B + E_{p+k}^B} n(E_p^B),
\] (3.2)
\[
\delta E_k^{(b)}(T) = 4 \sum_p \frac{\Gamma_s(k, p)^2}{E_k^B - E_p^B - E_{p+k}^B + i\eta} n(E_p^B)
\] (3.3)

and
\[
\delta E_k^{(c)}(T) = -4 \sum_p \frac{\Gamma_s(p, k)^2}{E_k^B + E_{p+k}^B - E_p^B} (n(E_p^B) - n(E_{p+k}^B))
\] (3.4)
on taking the real part of (2.19). The symbol \( \mathcal{P} \) indicates that we are to take
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The principal values in the integrals. In order to proceed further, we replace $E_k$ involved in the right-hand side of (3·2), (3·3) and (3·4) by the experimental excitation spectrum $E_k$ at zero temperature. This replacement may be plausible if we take the higher-order corrections into consideration.

The deviation of phonon velocity from that of zero temperature is given by

$$\delta c(T) = \delta c^{(a)}(T) + \delta c^{(b)}(T) + \delta c^{(c)}(T)$$

$$= \lim_{k \to \infty} \frac{1}{k} [\delta E_k^{(a)}(T) + \delta E_k^{(b)}(T) + \delta E_k^{(c)}(T)]. \quad (3·5)$$

We now calculate first the contribution from $\delta c^{(a)}(T)$ and $\delta c^{(b)}(T)$. Since we can put

$$E_k \to \frac{ck}{2m} \quad (3·6)$$

and

$$\lambda_k \to \frac{k}{c}$$

in the limit of $k \to 0$, we have

$$\delta c^{(a+b)}(T) = \delta c^{(a)}(T) + \delta c^{(b)}(T)$$

$$= -2 \left[ \frac{1}{2} \frac{4}{(8\pi)^2} \frac{1}{N} \frac{1}{c} \sum_p f(p) \right], \quad (3·7)$$

$$f(p) = \frac{\lambda_p^4(1 + 1/\lambda_p)^3 \rho^4}{E_p(e^{\beta p} - 1)}, \quad (3·8)$$

where $c = 3\AA^{-1}$ is the phonon velocity at zero temperature. In order to carry out the integration in (3·7), we separate the region of the momentum integration by an appropriate momentum $P_e$ into two parts, that is, the phonon region and the roton region. Then we have

$$\delta c^{(a+b)}(T) = - \frac{4}{(8\pi)^2} \frac{1}{N} \frac{2}{(2\pi)^3} \left\{ \int_{\rho} \int_{P_e} f(p) \rho^4 d\rho + \int_{P_e} \int_{\rho} f(p) \rho^4 d\rho \right\}. \quad (3·9)$$

In the phonon region, we can replace $E_p$ and $\lambda_p$ by $cp/2m$ and $p/c$, respectively, and we can extend the finite range of integration to infinity without introducing any appreciable error by virtue of the existence of the statistical factor $n(E_p)$, and we have the contribution $\delta c^{(p+b)}$ from the phonon region as

$$\delta c^{(p+b)}(T) = - \frac{\pi^2}{120} \frac{V}{N} \frac{1}{c^3} \left( \frac{k_B T}{c^2/2m} \right)^4, \quad (3·10)$$

where we have used the fact that $1 \gg \lambda_p^4$. In the roton region, we can put

$$\lambda_p = 1$$

and
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\[ E_p = \Delta + \frac{1}{2\mu} (p - k_0)^2, \]  

(3.11)

where\(^1,^{18}\)

\[ \Delta = 12 \times 10^{-18} \text{ erg}, \]

\[ \mu = 10^{-34} \text{ g} \]

and

\[ k_0 = 1.9 \text{ Å}^{-1}. \]

By making use of (3.11) and by retaining the dominant terms, we have

\[ \delta c^{(a+b)}(T) \simeq 2 \left[ -\frac{1}{2} \frac{4}{c} \frac{1}{N} \frac{4k_0^4}{\Delta} \frac{2}{(2\pi)^2} \int_{P_0}^\infty dpe^{-(p-k_0)^2/\Delta} \right] \]

\[ \simeq -\frac{1}{2\pi^{3/2}} \frac{V}{N} \frac{k_0^4 (c/2m)^4}{\Delta} \sqrt{\frac{k_B T}{k_0^2/2\mu}} e^{-\frac{1}{2}k_0^2}, \]

(3.12)

where we have used the fact that

\[ \int_{p_0}^\infty dpe^{-(p-k_0)^2/\Delta} \simeq \int_{-\infty}^\infty dpe^{-(p-k_0)^2/\Delta} = \sqrt{2\pi \mu k_B T}. \]

(3.13)

Collecting (3.10) and (3.12), we can write

\[ \delta c^{(a+b)}(T) = -\frac{\pi^2}{120} \frac{8m}{N} \frac{c}{(c/2m)^8} \frac{k_B T}{k_0^2} \int_{p_0}^\infty dpe^{-(p-k_0)^2/\Delta} \]

\[ \times \frac{1}{2\pi^2} \frac{(c/2m)^4}{N} \sqrt{\frac{k_B T}{k_0^2/2\mu}} e^{-\frac{1}{2}k_0^2}. \]

(3.14)

We next calculate \( \delta c^{(c)}(T) \) in (3.5). In the limit \( |k| \rightarrow 0 \), the energy denominator and the statistical factor in (3.4) vanish simultaneously. Therefore we must expand them around \( |k| = 0 \), and retain only nonvanishing lowest terms. In the phonon region \( |p| \sim 0 \), we can write

\[ \delta c^{(c)}(T) = -\frac{1}{(2\pi)^3} \frac{8m}{N} \frac{c}{(c/2m)^8} \int dp \int_{-1}^{1} du \frac{p^4 (1-2\mu)^2}{(c/2m)} |P_p E_p| \mu \cdot \beta \cdot n(E_p) \]

\[ \times (n(E_p) + 1), \]

(3.15)

where use is made of (3.6) in the integrand and the vertex function \( \Gamma_s(p, k) \) is approximated only by the term which has the lowest power in \( p \). The angular integration gives

\[ \delta c^{(c)}(T) = -\frac{1}{(2\pi)^3} \frac{8m}{N} \frac{c}{(c/2m)^8} \int dp p \left[ 20.6 - 9 \ln \left( \frac{2}{3\pi} \frac{1}{p^3} \right) \right] n(E_p) \]

\[ \times (n(E_p) + 1), \]

(3.16)

where we have used the relation

\[ |P_p E_p| = \frac{c}{2m} (1 - 3\gamma p^3) \]
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and the expansion coefficient $\gamma = 2.2 \times 10^9 \text{g}^{-1} \text{cm}^{-2} \text{sec}^2$ is obtained with the aid of energy spectrum calculated in the previous papers I and II.\(^6\)\(^7\) Carrying out the integrations in (3·16), we find

$$\delta c^{(c)}(T) = -\frac{\pi^4}{60} \frac{V}{N} \frac{\left(k_BT\right)^4}{\left(c^2/2m\right)^2} \left(20.6 - 18 \ln \left(\frac{2}{277} \frac{1}{c} \frac{\left(c^2/2m\right)}{k_BT}\right)\right), \quad (3·17)$$

where we put $p = 3k_BT/c$ in the logarithmic term of the integrand of (3·16), and the remaining part in the integrand of (3·16) has a maximum value for this value of $p$. On the other hand, the contribution of the integral from the roton region is negligibly small. This comes from the fact that the leading term of the integrand in (3·4) is proportional to $(p-k_0)$, since the order of magnitude of the statistical factor in the roton region $p = k_0$ is given by $k_0 (p-k_0)$ and the energy denominator is $ck/2m$.

Collecting (3·14) and (3·17), we have the final result for the temperature dependent part of phonon velocity:

$$c(T) = \delta c^{(a+b)}(T) + \delta c^{(c)}(T)$$

$$= -\frac{\pi^4}{120} \frac{V}{N} \frac{\left(k_BT\right)^4}{\left(c^2/2m\right)^2} - \frac{1}{2\pi^3} \frac{V}{N} \frac{k_0^2}{c^3} \frac{(c^2/2m)^3}{\Delta} \sqrt{\frac{k_BT}{k_0^2/2m}} e^{-\frac{k_BT}{k_0^2}}$$

$$-\frac{\pi^4}{60} \frac{V}{N} \frac{\left(k_BT\right)^4}{\left(c^2/2m\right)^2} \left(20.6 - 18 \ln \left(\frac{2}{277} \frac{1}{c} \frac{\left(c^2/2m\right)}{k_BT}\right)\right). \quad (3·18)$$

Introducing the numerical values

$c = 3\text{Å}^{-1}$,

$V/N = 45.8\text{Å}$,

$\gamma = 2.2 \times 10^9 \text{g}^{-1} \text{cm}^{-2} \text{sec}^2$,

we have the result

$$\delta c(T) = 10.4T^4 \ln \left(\frac{3.1}{T}\right) - 5.7 \times 10^4 \sqrt{T} e^{-\frac{k_BT}{k_0^2}} \left(\frac{cm}{\text{sec}}\right). \quad (3·19)$$

In expression (3·19), the first term on the right-hand side is the contribution from the phonon region and the second term comes from the effect of the roton region. The first term gives increase of phonon velocity as temperature rises, and the second term, on the other hand, decreases phonon velocity. This behavior of the second term is indispensable for explaining the experimental results of the temperature dependence of phonon velocity which shows characteristic decrease. We should note here that this second term comes from (3·1) and (3·2) which are not considered in the theory of ter Haar et al. The numerical result of (3·19) is shown in Table I, and compared with the experimental result\(^5\) in Figs. 3 and 4. Although the position of the maximum point of the theoreti-
Table I. The calculated phonon velocity versus temperature in liquid helium II.

<table>
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<th>$T$ (°K)</th>
<th>$c(T)$ (m/sec)</th>
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</table>

cal curve is slightly larger than the experimental one in Fig. 3, the whole behavior of the theoretical curve is very similar to the measured one.

In conclusion, we can see that the phonon velocity increases due to the effect of phonon-phonon interaction up to 0.6°K, and the phonon-roton interaction becomes effective above 0.6°K in order to decrease the phonon velocity. We can further expect that the phonon-roton interaction also plays an important role in the temperature dependence of absorption coefficient of the first sound in liquid helium II, and this problem is now under consideration.

Fig. 3. The deviation of phonon velocity from that at zero temperature in liquid helium II: The continuous curve gives the theoretical values and the points are the experimental data of Whitney and Chase.²³

Fig. 4. The phonon velocity in liquid helium II: The continuous curve gives the theoretical values and the points are the experimental data of Whitney and Chase.²³
§ 4. The temperature dependence of roton minimum

In this section we shall study the temperature dependence of roton minimum \( \Delta \). The deviation of roton minimum \( \delta \Delta (T) \) from that at zero temperature is described from (3·2), (3·3) and (3·4) by

\[
\delta \Delta (T) = \delta \Delta^{(a)} (T) + \delta \Delta^{(b)} (T) + \delta \Delta^{(c)} (T),
\]

\[
\delta \Delta^{(a)} (T) = -4 \sum_p \frac{\Gamma_s (k_0, p)^2}{(E_p + E_{p+k})} n(E_p),
\]

\[
\delta \Delta^{(b)} (T) = 4 \sum_p \frac{\Gamma_s (k_0, p)^2}{(E_p - E_{p+k})} n(E_p)
\]

and

\[
\delta \Delta^{(c)} (T) = -4 \sum_p \frac{\Gamma_s (p, k_0)^2}{(E_p + E_{p+k})} (n(E_p) - n(E_{p+k}))
\]

where

\[
|k_0| = k_0
\]

and

\[
E_{k_0} \simeq \Delta, \quad \lambda_{k_0} \simeq 1.
\]

As in the case of phonon velocity, we separate the region of the momentum-integrations of (4·2), (4·3) and (4·4) into two parts, that is, the phonon region and the roton region. We first consider the contributions from the phonon region \( (p \sim 0) \) by using the following approximate relations:

\[
\Gamma_s (k_0, p)^2 \simeq \frac{1}{(8m)^2} \frac{1}{\sqrt{N}} \frac{4k_0^2}{c^2} p,
\]

\[
E_{k_0} + E_p + E_{p+k} \simeq 2\Delta
\]

\[
\Gamma_s (k_0, p)^2 \simeq \frac{1}{(8m)^2} \frac{1}{\sqrt{N}} 4ck_s^2 p \cdot \cos \theta,
\]

\[
E_{k_0} - E_p - E_{p+k} \simeq - \frac{cb}{2m}
\]

\[
\Gamma_s (p, k_0)^2 \simeq \frac{1}{(8m)^2} \frac{1}{\sqrt{N}} \frac{4k_0^4}{c^2} p
\]

and

\[
E_{k_0} + E_{p+k} - E_p \simeq 2\Delta.
\]

The substitution of the relations in (4·6) into (4·2), (4·3) and (4·4) and the momentum-integrations give
\[\delta A^{(a)}_{(ob)}(T) = -\frac{\pi^2}{60} \frac{V k_s^4}{N c} \frac{(k_BT)^4}{(c^2/2m)^3} \delta, \quad (4.7)\]

\[\delta A^{(b)}_{(ob)}(T) = -\frac{A}{6\pi^2} \frac{V c k_s^4}{N} \frac{(k_BT)^4}{(c^2/2m)^3} \delta, \quad (4.8)\]

and

\[\delta A^{(c)}_{(ob)}(T) = -\frac{\pi^2}{60} \frac{V k_s^4}{N c} \frac{(k_BT)^4}{(c^2/2m)^3} \delta + \frac{9}{8\pi^2} \frac{V k_s^8}{N c^3} \frac{(c^2/2m)^3}{\Delta} \frac{(k_BT)^3}{(k_s^3/2\mu)^2} e^{-\beta/k_BT} , \quad (4.9)\]

where in (4.8)

\[A = \int_0^\infty dx \frac{x^2}{e^x - 1} \approx 2.4\]

and in the second term of (4.9) we have approximated in the following way:

\[\exp\left(-\beta \frac{(p_k + 2\mu)}{2\mu}\right) \approx \exp\left(-\beta \frac{p_k^2}{6\mu}\right). \quad (4.10)\]

Collecting (4.7), (4.8) and (4.9), we have the expression for the temperature dependence of roton minimum:

\[\delta A^{(a)}_{(ob)}(T) = -\frac{2.4}{8\pi^2} \frac{V c k_s^4}{N} \frac{(k_BT)^3}{(c^2/2m)^3} \left[\frac{\pi^2}{60} \frac{V k_s^4}{N c} \frac{(k_BT)^4}{(c^2/2m)^3} \delta - \frac{9}{8\pi^2} \frac{V k_s^8}{N c^3} \frac{(c^2/2m)^3}{\Delta} \frac{(k_BT)^3}{(k_s^3/2\mu)^2} e^{-\beta/k_BT} \right] \quad (4.11)\]

from the phonon region.

We next calculate the contributions from roton region (\(|p| \sim k_s\)) by using (4.5) as the approximated relations for \(E_p\) and \(\lambda_p\) in the integrands of (4.2), (4.3) and (4.4). The contributions from the roton region are given by

\[\delta A^{(a)}_{(rot)}(T) = -\frac{4}{(8m)^2} \frac{1}{N} \sum_p \frac{\lambda_{p+k_e}(2k^2 + (p + k_e)^2/\lambda_{p+k_e})}{2\Delta + E_{p+k_e}} \times n\left(\Delta + \frac{1}{2\mu} (p - k_e)^2\right), \quad (4.12)\]

\[\delta A^{(b)}_{(rot)}(T) = -\frac{4}{(8m)^2} \frac{1}{N} \sum_p \frac{\lambda_{p+k_e}(p + k_e)^4}{E_{p+k_e}} n\left(\Delta + \frac{1}{2\mu} (p - k_e)^2\right) \quad (4.13)\]

and

\[\delta A^{(c)}_{(rot)}(T) = -\frac{4}{(8m)^2} \frac{1}{N} \sum_p \frac{\lambda_{p+k_e}(p + k_e)^4}{E_{p+k_e}} n\left(\Delta + \frac{1}{2\mu} (p - k_e)^2\right) - n(E_{p+k_e}) \quad (4.14)\]
where we have used the fact that $|p| \approx k_0$. The integrations on the right-hand side of (4.12), (4.13) and (4.14) are to be carried out for two regions when $|p + k_0|$ is in the phonon region and when it is in the roton region, respectively. After the integrations for each region, we obtain

$$\delta A^{(p)}_{\text{rot}}(T) = \frac{V}{8\pi^3 N} k_0^2 \frac{(c^2/2m)^3}{\Delta} \sqrt{\frac{k_B T}{k_0^3/2\mu}} e^{-4/\kappa_B T},$$

(4.15)

$$\delta A^{(r)}_{\text{rot}}(T) = \frac{1}{8\pi^3 N} k_0^2 \frac{(c^2/2m)^3}{\Delta} \sqrt{\frac{k_B T}{k_0^3/2\mu}} e^{-4/\kappa_B T} + \text{(small terms)}$$

(4.16)

and

$$\delta A^{(0)}_{\text{rot}}(T) \approx \text{(small terms)}$$

for the contributions from the roton region. In (4.15) and (4.16), we have explicitly written down only the leading terms, and the other small terms are negligible compared with the leading terms.

Collecting (4.11), (4.15) and (4.16), we have the final result for the deviation of roton minimum from that at zero temperature:

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<th>$A(T)$ ($\times 10^{-16}$ erg)</th>
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<td>2.2</td>
<td>6.61</td>
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Table II. The calculated roton minimum versus temperature in liquid helium II.

Fig. 5. The roton minimum in liquid helium II: The continuous curve gives the theoretical values and the points are the experimental data of Dietrich et al."
By using the numerical values of the constants $\Delta$, $c$, $V/N$ and $k_0$ given in the preceding section, we obtain the result

$$\delta \mathcal{A}(T) = -9.5 \times 10^{-10} T^3 - 3.5 \times 10^{-10} T^4 - 17.5 \times 10^{-12} \sqrt{T} e^{-s.5/T} \text{ (erg)}. \quad (4.18)$$

The first and second terms in (4.18) are the contributions from the phonon region to roton minimum, and the third term in (4.18) is the correction term from the roton region. From (4.18), the numerical values for the roton minimum $\mathcal{A}(T) = \mathcal{A} + \delta \mathcal{A}(T)$ is shown in Table II and compared with experiments in Fig. 5, from which we find that the theoretical result is in fairly good agreement with the experimental one.

In conclusion we can see that the roton minimum gradually decreases due to the effect of roton-phonon interaction up to $0.7^\circ K$, and begins to decrease rapidly above $0.7^\circ K$ by the effect of roton-roton interaction.

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**References**

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