Phase Transition in Bose Liquid. II

Shokichi KANNO

Department of Physics, Ibaraki University, Mito

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We apply the method proposed in a previous paper to the calculation of thermodynamical functions. This method resembles the Landau Fermi liquid theory in the sense that the energy, excitation energy, etc., are expressed by the functionals of occupation numbers of quasi-particles. It is shown that if the excitation energy of a quasi-particle $\varepsilon_p$ is proportional to $p^r$ with $2 > r > 3/2$, then the specific heat $c_p$ behaves as $\log(T_c/(T-T_c))$. We discuss also the thermal behaviour of condensation density. It is shown that the fluctuation of condensation density becomes infinite on the transition line.

§ 1. Introduction

In a previous paper referred to as I, we have shown that the interacting Bose gas can be worked out in a way similar to the Landau Fermi liquid theory by considering $\langle \{n_p\} e^{-\beta H} \{n_p\} \rangle$ which contributes most strongly to the partition function, where $\{n_p\}$ is the free state specified by the occupation numbers of particles. The relation between the excitation energy of a quasi-particle and the Landau $f$-function (the second variation of energy with respect to the occupation numbers of quasi-particles) has been obtained. In the present paper we shall obtain the thermodynamical functions by using the results of I.

We shall show that the interacting Bose gas exhibits the $\lambda$-transition when the Bose-Einstein condensation occurs. The excitation energy spectra of a quasi-particle on the transition line are not so important. The variation of energy spectra on departing from the transition line is essential to the characteristic feature of the $\lambda$-transition. This variation of energy spectra is caused by the variation of occupation number of quasi-particles and it is determined by the Landau $f$-functions. The excitation energy of a quasi-particle is closely related to the occupation of the other quasi-particles. The correlation among the quasi-particles gives rise to the $\lambda$-transition. The fraction of the number of quasi-particles contributing to the $\lambda$-transition is of order $a/\lambda_{th}$ where $a$ is the scattering length of interacting particles and $\lambda_{th}$ is the thermal wave length.

It will be shown that if the excitation energy of a quasi-particle with momentum $p$, $\varepsilon_p$, is proportional to $p^r$ with $2 > r > 3/2$, then the specific heat $c_p$ behaves as $\log(T_c/(T-T_c))$, and if $\varepsilon_p \sim p^{4/3}$, then $c_p \sim \log[\log(T_c/(T-T_c))]$. These results are different from those of Patashinskii and Pokrovskii (hereafter referred to as PP).
The condensation density will also be discussed. We shall show that the condensation density is proportional to \((T_c - T) \log(T_c/|T - T_c|)\) at constant specific volume in the vicinity of transition line. This is contrasted with the result of the ideal Bose gas where the condensation density is proportional to \((T_c - T)\). Our result is different from that of PP's theory where the condensation density is proportional to \((T_c - T) \log(T_c - T)\) at constant specific volume.

We shall calculate the fluctuation of the condensation density. This is very large and becomes infinite as \(1/(T_c - T)^{(a-r)/2)}\). This result is different from the result of PP. The fluctuation of the condensation density is very large compared with the total density fluctuation which is proportional to \((\partial \sigma/\partial \mu)_{\tau} \sim \log(T_c - T)\). This implies that the total density is regarded as constant in consideration of the fluctuation of the condensation density, and the non-condensation density also fluctuates in a way similar to the condensation density. A strong correlation exists between fluctuations of the condensation and non-condensation densities. This result is contrasted with the result of the ideal Bose gas where there is no correlation between fluctuations of the condensation and non-condensation densities.

It will be shown that the scaling property holds in our problems, i.e., the occupation number of quasi-particles, \(n_p\), is expressed in the form \(p^{-\gamma} \eta(r_c p)\) where \(r_c\) is the correlation radius. As seen in § 3, \(r_c \sim 1/|T - T_c|^{1/(a-r)}\). This result is contrasted with the scaling laws of Widom and Kadanoff, but it is consistent with the result expected from the scaling laws of the Bose liquid. If Widom and Kadanoff's scaling laws are true in the Bose liquid, then \(r_c \sim |T - T_c|^{-3/2}\) for the logarithmic singularity in the specific heat. This implies \(r = 3/2\). Widom and Kadanoff's scaling laws, however, cannot be applied directly to the Bose liquid since these laws are obtained from the Ising model in a weak external field while the Bose liquid corresponds to the spin system with easy plane in a strong field.

In this paper we assume that the singularity of thermodynamical functions at the transition point is caused by the particles with momenta smaller than a characteristic momentum \(p_0\), which is defined so that the relation \(n_p \sim p^{-\gamma}\) is violated for \(|p| \geq p_0\). The particles with larger momenta affect at most the proportionality constants of the singular terms. This assumption is reasonable since the singularity of thermodynamical functions at the transition point is related to the appearance of the long-range order correlation as seen in Ising models.

In § 2 we shall give the summary of I. In § 3 using the results obtained in I, we obtain \(\alpha(p)\) and \(\gamma(p)\) which are the rate of the variation of the excitation energy caused by the change of chemical potential and that of condensation density. In § 4 the specific heat is calculated and it is shown that there appears a logarithmic singularity in \(c_p\). In § 5 the condensation density and its fluctuation are considered. In § 6 remaining problems are discussed and it is shown that the scaling property holds in our problems.
§ 2. Preliminary remark (Summary of I)

In the previous paper I we have shown that the free energy is expressed by a functional of \( \{ n_p \} \):

\[
F\{ n_p \} = E\{ n_p \} - TS\{ n_p \},
\]

where \( n_p \) is the number of quasi-particles with momentum \( p \) and satisfies

\[
N = \sum_p n_p \quad \text{or} \quad \frac{1}{V} = \int \frac{d^3 p}{(2\pi)^3} n_p + N_0,
\]

where \( V = V/N \) and \( N_0 = n_0/V \) is the condensation density. \( E\{ n_p \} \) is the energy and \( S\{ n_p \} \) is the entropy given by

\[
k \sum_p \{ n_p \log(1 + n_p^{-1}) + \log(1 + n_p) \}.
\]

\( n_p \) is obtained from the minimum of \( F\{ n_p \} \),

\[
n_p = \left[ \exp\{\beta E/\partial n_p - \mu \} - 1 \right]^{-1},
\]

where \( \mu \) is the chemical potential. If

\[
\mu = \partial E/\partial n_0,
\]

then the Bose-Einstein condensation occurs and \( N_0 \) is of order \( 1/V \).

Let \( \alpha(p) \) and \( \gamma(p) \) be defined by

\[
d(\delta E/\delta n_p - \mu) = -\alpha(p) \partial \mu + \gamma(p) \partial N_0
\]

under the fixed temperature. Bearing in mind that \( \delta E/\delta n_p \) is the functional of \( n_p \) and \( N_0 \) and does not depend on \( \mu \) or \( v \) explicitly, we have

\[
\alpha(p) + \int \frac{d^3 k}{(2\pi)^3} f(p, k) \chi_k \alpha(k) = 1,
\]

\[
\gamma(p) + \int \frac{d^3 k}{(2\pi)^3} f(p, k) \chi_k \gamma(k) = \gamma(p, 0),
\]

where \( f(p, k) = V \delta^3 E/\partial n_p \partial n_k \) and

\[
\chi_p = \beta \exp\{\beta (\delta E/\delta n_p - \mu)\} \left[ \exp\{\beta (\delta E/\delta n_p - \mu)\} - 1 \right].
\]

From (2.3) and (2.4) with \( p = 0 \), we have

\[
\partial N_0 / \partial \mu = \alpha(0)/\gamma(0).
\]

Taking the derivative of both sides of (2.1) with respect to \( v \), we obtain

\[
-\frac{1}{v^2} = \frac{1}{v^2} \int \frac{d^3 p}{(2\pi)^3} \chi_p \alpha(p) \left( \frac{\partial \mu}{\partial v} \right) \gamma(p),
\]

\[
\text{(gaseous phase)}
\]

\[
-\frac{1}{v^2} = \left[ \int \frac{d^3 p}{(2\pi)^3} \chi_p \alpha(p) + \frac{\alpha(0)}{\gamma(0)} \left\{ 1 - \int \frac{d^3 p}{(2\pi)^3} \chi_p \gamma(p) \right\} \right] \left( \frac{\partial \mu}{\partial v} \right) \gamma(p),
\]

\[
\text{(condensed phase)}
\]
where we have used (2.4) and (2.8).

\( (\partial^2 F/\partial n^2)_N \) is related to the fluctuation of condensation density, and is expressed in terms of \( \alpha(p) \) and \( \gamma(p) \) as (see (2.22), (4.7) and (4.8) of I)

\[
\frac{\partial^2 F}{\partial n^2} = \frac{\gamma(0)}{V} \left( \frac{1}{V^2} \right) \frac{\partial \mu}{\partial \nu} \int d^4p \chi(p) \alpha(p).
\] (2.11)

In this whole paper we assume that on the transition line the excitation energy \( \varepsilon_p^{(\infty)} = (\delta E/\delta n_p - \delta E/\delta n_0)_e \) is proportional to \( p^r \) with \( 2 > r \geq 3/2 \) for \( p \leq p_0 \) (a characteristic momentum) and write

\[
\varepsilon_p^{(\infty)} = A p^r.
\] (2.12)

From the graphical calculations it can be shown that

\[
f(p, k) = \begin{cases} 2l\beta A^4 |p + k|^{r - 4}, & 2 > r > 3/2, \\ 4\pi^2 \beta A^4 / \log (p_0 / |p + k|), & r = 3/2 \end{cases}
\] (2.13)

for small momenta (see § 7 of I), and

\[
p_0 \sim m^2 v_0 / \beta \sim \alpha / \lambda^2,
A \sim (m^2 v_0 / \beta)^{3-r} / m \sim (a / \lambda^2)^{3-r} / m,
\] (2.14)

where \( v_0 \) is the zero wave number Fourier transform of the potential and \( a \) is the scattering length of interacting bosons (see § 6 of I). \( \lambda \) is the thermal wave length \( (2\pi/mkT)^{1/2} \).

§ 3. Self-energy parameters; \( \alpha(p) \) and \( \gamma(p) \)

It is very difficult to solve the integral equation (2.5) and (2.6) for \( \alpha(p) \) and \( \gamma(p) \). However, we need not to know exact \( \alpha(p) \) and \( \gamma(p) \) for all momenta in order to find thermodynamical functions. If we know \( \alpha(0), \gamma(0), \int d^3p \alpha(p) \chi_p \) and \( \int d^3p \gamma(p) \chi_p \) with \( \chi_p \) defined by (2.7), then the chemical potential is given by (2.9) or (2.10).

Consider the integral of the form

\[ \int d^3p \chi_p \gamma(p). \]

\( \chi_p \) is large for small momenta but is negligibly small for large momenta. Therefore, the integral may be replaced by \( \int_{|p| < p_0} d^3p \chi_p \gamma(p) \) where \( p_0 \) is a characteristic momentum introduced in § 6 of I. The contributions from larger momenta do not affect the critical behaviour of thermodynamical functions in the vicinity of transition line.

The occupation number of quasi-particles are given by (2.2). In the vicinity of the transition line it follows from (2.3), (2.4) and (2.12) that

\[
\delta E/\delta n_p - \mu = (\delta E/\delta n_p - \mu)_e - \alpha(p) (d\mu/dv)(v - v_e) + O((v - v_e)^4)
= Ap^r - \alpha(p) (d\mu/dv)(v - v_e), \quad \text{(gaseous phase)}
\] (3.1)
where the subscript \( c \) implies the value on transition line. For the condensed phase the term \( \gamma(p)N_0 \) is added to the right hand side of (3·1). Let us define \( r_c \) by

\[
Ar_c^{-r} = |\alpha(r_c^{-1}) (d\mu/dv)(v-v_c)|, \quad \text{(gaseous phase)}
\]

(3·2)

\( r_c \) becomes infinite on the transition line (since \( \alpha(0) \sim 1/r_c^{n-r} \) as we shall see later). From (3·1) it follows that

\[
\delta E/\delta n_p - \mu = Ar_c^{-r}
\]

for \( p \gg 1/r_c \), and

\[
\delta E/\delta n_p - \mu = -\alpha(0) (d\mu/dv)(v-v_c) \equiv Ar_c^{-r}
\]

for \( p \ll 1/r_c \). If we note these, we find

\[
X = \int \frac{d^3p}{(2\pi)^3} \chi_p \approx \left\{ \begin{array}{ll}
  cr_c^{n-r} & \text{for } r > 3/2, \\
  c \log(r_c p_0) & \text{for } r = 3/2
\end{array} \right.
\]

(3·3)

for large \( r_c \), where \( c \) is a constant independent of \( r_c \) but depends on temperature. Hence effectively

\[
\chi_p = \frac{1}{\beta A^3 (p^r + r_c^{-r})^3}
\]

(3·4)

for small momenta.

We first consider the case \( 2 > r > 3/2 \). By the direct integration of \( f(p, k) \) given by (2·13) it follows that

\[
\int_{|k| \leq p_0} \frac{d^3k}{(2\pi)^3} \chi_k \{f(p, k) - f(0, p)\} = I_1 + Y \frac{\log(p_0/p)}{\log(r_c p_0)} + Cf(0, p)
\]

(3·5)

and

\[
\int_{|k| \leq p_0} \frac{d^3k}{(2\pi)^3} \chi_k \{f(p, k) - f(0, p)\} f(0, k) = \left\{ I_2 - Y \frac{\log(p_0/p)}{\log(r_c p_0)} \right\} f(0, p) + Z
\]

(3·6)

where we have put \( \chi_k = 1/\beta(Ak^r)^3 \) and

\[
Y = (I/\pi^r) \log(r_c p_0),
\]

(3·7)

\[
Z = 2I_2^2 A^3 p_0^{n-r-1} / (2r-3) \pi^3 \sim v_0.
\]

(3·8)

\( I_1, I_2 \) and \( C \) are certain constants (\( I_1 \) and \( I_2 \) are finite and \( C \to 0 \) in the limit \( p_0 \to \infty \)).

We note that \( Y \) and \( Z \) can be written in the following forms:

\[
Y = \int \frac{d^3k}{(2\pi)^3} \chi_k f(0, k)
\]

(3·9)

and

\[
Z = \int \frac{d^3k}{(2\pi)^3} \chi_k f(0, k)^2.
\]

(3·10)
Equation (3.9) follows from the observation that $\chi_k$ is effectively equal to $1/\beta A^3(k^r+r_c^{-s})^c$.

From (2.5) we have

$$\alpha(0) + \int \frac{d^3p}{(2\pi)^3} f(0, p) \alpha(p) \chi_p = 1$$

and

$$\alpha(0) = \alpha(p) + \int \frac{d^3k}{(2\pi)^3} \left[ f(p, k) - f(0, k) \right] \chi_k \chi(k).$$

Multiplying both sides of (3.12) by $\chi_p$ and integrating with respect to $p$, we have, using (3.5),

$$X\alpha(0) = \int \frac{d^3p}{(2\pi)^3} \chi_p \alpha(p) + \int \frac{d^3k}{(2\pi)^3} \chi_k \alpha(k) \left[ I_1 + Y \frac{\log(p_0/k)}{\log(r_c p_0)} + C f(0, k) \right].$$

Now consider the integral

$$\frac{1}{\log(r_c p_0)} \int \frac{d^3k}{(2\pi)^3} \log\left(\frac{p_0}{k}\right) \chi_k \alpha(k)$$

appearing in (3.13). In the vicinity of transition line ($r_c$ is very large in this region) $\chi_k$ is proportional to $r_c^{-s}$ and $\alpha(k)$ is proportional to $1/r_c^{sr-a}$ in the limit $k \to 0$, as we shall see later. $\chi_k$ decreases rapidly with increasing $|k|$ and we anticipate that $\alpha(k)$ does not increase so rapidly ($\{d^3k \chi_k \alpha(k)\}$ is very small compared with $\alpha(0) \chi_{k=0} p_0^a$ as we shall see). $\chi_k \alpha(k)$ is maximum at $k=0$ with the maximum value proportional to $r_c^{sr} r_c^{-sr-a} = r_c^s$ which is very large. In the integral (3.14) the factor $\log(p_0/k)$ comes originally from the right-hand side of (3.5) calculated on the transition line where $r_c \to \infty$. $\log(p_0/k)$ is, therefore, replaced by $\log(r_c p_0)$ in the limit of $k \to 0$. $\log(p_0/k)$ is a slowly varying function, and $\chi_k \alpha(k)$ has a very large maximum value at $k=0$ and rapidly decreases as $k$ increases. Thus we have approximately

$$\frac{1}{\log(r_c p_0)} \int \frac{d^3k}{(2\pi)^3} \log\left(\frac{p_0}{k}\right) \chi_k \alpha(k) \approx \int \frac{d^3k}{(2\pi)^3} \chi_k \alpha(k).$$

(3.13) now reduces to

$$X\alpha(0) - Y \int \frac{d^3k}{(2\pi)^3} \chi_k \alpha(k) - C \int \frac{d^3k}{(2\pi)^3} \chi_k f(0, k) \alpha(k) = 0$$

for large $r_c$. Here, $X$ and $Y$ are proportional to $r_c^{sr-a}$ and $\log(r_c p_0)$, respectively, so that the other constant terms independent of $r_c$ are neglected.

In a similar way, by multiplying both sides of (3.12) by $\chi_p f(0, p)$ and integrating with respect to $p$, we have
\[ Y\alpha(0) - Z \int \frac{d^3k}{(2\pi)^3} \chi_k \alpha(k) + Y \int \frac{d^3k}{(2\pi)^3} \chi_k f(0, k) \alpha(k) = 0, \quad (3.16) \]

where we have used (3.6).

From (3.11), (3.15) and (3.16) we get

\[ \int \frac{d^3p}{(2\pi)^3} \chi_p \alpha(p) \equiv \frac{Y}{Z} \quad (3.17) \]

and

\[ \alpha(0) \equiv \frac{Y^2}{ZX}. \quad (3.18) \]

For large \( r_c \), \( Z \) is constant, and \( X \) and \( Y \) are proportional to \( r_c^{3r-3} \) and \( \log(r_c p_0) \), respectively.

Let

\[ \gamma(p) = 1/X + \tau_1(p). \]

Substituting this into (2.6) and making use of (3.5), we have

\[ \tau_1(p) + \int \frac{d^3k}{(2\pi)^3} f(p, k) \tau_1(k) = -\frac{Y}{X} \log(p_0/p) + O\left(\frac{1}{X} f(p, 0)\right). \]

\( \log(p_0/p) \) is a slowly varying function so that we may replace the right-hand side by some average constant \( \bar{Y}/X \). Then we can easily see that \( \gamma(p) \equiv -(\bar{Y}/X) \times \alpha(p) \), i.e.,

\[ \gamma(p) \equiv (1/X) - (\bar{Y}/X) \alpha(p). \quad (3.19) \]

From this it follows that

\[ \gamma(0) = 1/X + O(1/X^2) \quad (3.20) \]

and

\[ 1 - \int \frac{d^3p}{(2\pi)^3} \chi_p \tau(p) = O(1/X). \quad (3.21) \]

We can see that expression (3.19) is consistent with (2.6) for \( p = 0 \) as follows. From (2.6) with \( p = 0 \) we have

\[ f(0, 0) = \gamma(0) + \int \frac{d^3k}{(2\pi)^3} f(0, k) \chi_k \tau(k). \]

Substituting (3.19) it follows that

\[ f(0, 0) = \left\{ \frac{1}{X} - \frac{\bar{Y}}{X} \frac{Y^3}{Z} \right\} + \left\{ \frac{Y}{X} - \frac{\bar{Y}}{X} \left(1 - \frac{Y^3}{Z} \right) \right\}, \quad (3.22) \]

where we have used (3.18) and the relation

\[ \int \frac{d^3k}{(2\pi)^3} f(0, k) \chi_k \alpha(k) = 1 - \frac{Y^3}{ZX} \]
which is obtained from (3·11) and (3·18). From (2·13), \( f(p, 0) \propto p^{r-3} \): This relation has been obtained originally on the transition line where \( r_e \to \infty \). In the vicinity of transition line it can be seen that \( f(0, 0) \) is proportional to \( 1/r_e^{r-3} \) and from (3·3) it follows that

\[
f(0, 0) = c/X, \tag{3·23}
\]

where \( c \) is some constant. Comparing (3·22) and (3·23) we can see that (3·22) is satisfied if \( \bar{Y} = Y + 1 - c \).

Let us next turn to the case \( r = 3/2 \). We have from (2·13)

\[
F(p, k) = \frac{1}{4\pi} \int d\Omega, f(p, k) = 4\pi^3 \beta A^3 / \log(1/p) \quad \text{for} \mid p \mid \gg \mid k \mid.
\]

Since the logarithmic function is slowly varying, we can approximate as

\[
F(p, k) = \{4\pi^3 \beta A^3 / \log(1/p) \} \theta (p - k) + \{4\pi^3 \beta A^3 / \log(1/k) \} \theta (k - p).
\]

From this point the argument can be developed in much the same way as for the case \( 2 > r > 3/2 \). The results are (3·17), (3·18) and (3·19) where now

\[
\begin{aligned}
X &= \int \frac{d^3k}{(2\pi)^3} \chi_k \simeq \frac{1}{2\pi^2 \beta A^3} \log(r_e p_e), \\
Y &= \int \frac{d^3k}{(2\pi)^3} \chi_k f(0, k) \simeq 2 \log \{ \log(r_e p_e) \}, \\
Z &= \int \frac{d^3k}{(2\pi)^3} \chi_k f(0, k) \simeq 8\pi^3 \beta A^3 \sim v_0.
\end{aligned} \tag{3·24}
\]

From (3·17) \( \sim \) (3·21) we can see that the conditions of the second-order transition given in § 4 of I are satisfied.

We shall give the relation between \( r_e \) and \( v - v_e \) where \( v_e \) is the specific volume on transition line. We have (3·2) for the gaseous phase and

\[
Ar_e^{-r} = \alpha(r_e^{-1})(d\mu/dv) - \gamma(r_e^{-1})(dN_0/dv) \mid (v_e - v) \tag{3·25}
\]

for the condensed phase. If we substitute (2·8) and (3·19) into this, we get

\[
Ar_e^{-r} = \alpha(r_e^{-1}) - \frac{Y^2}{XZ} \frac{d\mu}{dv} \mid (v_e - v), \quad \text{(condensed phase)} \tag{3·26}
\]

where we have neglected \( \alpha(r_e^{-1})/X \) which is very small compared with \( \alpha(r_e^{-1}) \) for large \( r_e \). From (2·9) and (3·17) \( d\mu/dv \sim 1/\log(r_e) \). From (3·18) we have

\[
\alpha(p) = Y^2/ZX
\]

for \( \mid p \mid \ll r_e^{-1} \). For \( \mid p \mid \sim r_e^{-1} \) it is expected that \( \alpha(r_e^{-1}) \) is also of order \( \alpha(0) = Y^2/ZX \sim (\log r_e)^3/r_e^{3r-3} \). It follows, therefore, that

\[
r_e \sim B/\sqrt{v - v_e} \mid (v - v_e)^{3/2}
\]

which becomes infinite as \( v \rightarrow v_e \).
The proportionality constant $B$ of (3·27) in the condensed phase is larger than that in the gaseous phase as may be seen by comparing (3·2) and (3·26). As we shall see in the next section this fact leads to the result that the specific heat at constant pressure in condensed phase is larger than that in gaseous phase.

§ 4. Thermodynamical functions

From (2·9), (2·10), (3·17) and (3·21) it follows that

$$ (\partial \mu / \partial v)_T = -Z/\nu Y, \quad (4·1) $$

where the terms proportional to $1/X$ are neglected. Equation (4·1) holds in both the gaseous and condensed phases. If we note $\left(\partial \mu / \partial P\right)_T = (\partial \mu / \partial P)_T (\partial P / \partial v)_T = \nu (\partial P / \partial v)_T$, then we have

$$ P = P_e + \int_{v_e}^{v} \frac{1}{\nu} \left(\frac{\partial \mu}{\partial v} \right)_T dv, \quad (4·2) $$

where $P_e$ is the pressure on transition line. Substituting (4·1) into (4·2) we get, in the vicinity of transition line,

$$ P = P_e + (v - v_e)Z/v_e Y, \quad (4·3) $$

since $Y$ is a slowly varying function of $v$.

We first consider the case $2 > r > 3/2$. From (3·7) and (3·27) we get

$$ P = P_e + \frac{v - v_e}{v_e} Z/I \log \left( \frac{B}{|v - v_e|^{1/(3 - r)}} \right) \quad (4·4) $$

which gives the equation of state. We note that constant $B$ within the logarithm in (4·4) in condensed phase is larger than that in gaseous phase as shown in § 3.

The chemical potential as a function of pressure is obtained from (4·2):

$$ P - P_e = \int_{v_e}^{v} \frac{1}{\nu} \left(\frac{\partial \mu}{\partial v} \right)_T dv $$

$$ = \frac{1}{v_e} \int_{v_e}^{v} \left(\frac{\partial \mu}{\partial v} \right)_T dv + \int_{v_e}^{v} \left(\frac{1}{\nu} - \frac{1}{v_e} \right) \left(\frac{\partial \mu}{\partial v} \right)_T dv. $$

The first integral equals $\mu - \mu_e$ where $\mu_e$ is the chemical potential on the transition line and depends on only temperature. Using (4·1) we obtain

$$ P - P_e = \frac{1}{v_e} (\mu - \mu_e) + \frac{(v - v_e)^2 Z}{2v_e^2 Y}. $$

If we express $v - v_e$ in terms of $P - P_e$ using (4·4), we have

$$ \mu - \mu_e = v_e (P - P_e) - \frac{I}{2v_e \pi^2 Z} (P - P_e)^3 \log \left( \frac{B}{|P - P_e|^{1/(3 - r)}} \right) + O((P - P_e)^4). \quad (4·5) $$
The transition temperature $T_e$ is obtained from

$$P = P_e, \quad v = v_e \quad \text{or} \quad \mu = \mu_e,$$

where $P_e$, $v_e$ and $\mu_e$ are the functions only of temperature since we have so far been considering a fixed temperature. The relation $P = P_e$, for example, gives the temperature on transition line as a function of pressure. The pressure difference $P - P_e$ is proportional to $T - T_e(P)$ where $T_e(P)$ is the solution of $P = P_e(T)$. The volume difference $v - v_e$ is also proportional to $T - T_e(v)$, but note that $T - T_e(P) \neq T - T_e(v)$. $P_e$, $v_e$ and $\mu_e$ are obtained from

$$P_e = -\int_{v_e}^{v} \left( \frac{\partial \mu}{\partial v} \right) dv,$$

$$\frac{1}{v_e} = \int_{0}^{v_e} \frac{d^3p}{(2\pi)^3} n_p^{(e)},$$

$$\mu_e = (\delta E/\delta n_0)_{(n_p^{(e)})}^{(n_v^{(e)})},$$

where $n_p^{(e)}$ is the average occupation number of quasi-particles with momentum $p$ on the transition line. (4·7) follows from (4·2) since $P \to 0$ as $v \to \infty$, and (4·8) and (4·9) follow from (2·1) and (2·3), respectively. In the calculation of $P_e$, $v_e$ and $\mu_e$, the contributions from the quasi-particles with momenta larger than $p_0$ are important.

The specific heat at constant pressure per one particle is obtained from

$$c_p = -(\partial^2 \mu / \partial T^2)_p.$$ From (4·5) we have

$$c_p \sim \log \{B/|P - P_e|^{1/(3-r)}\} \sim \log \{B/|T - T_e|^{1/(3-r)}\}$$

which becomes logarithmically infinite on the transition line. As remarked in § 3, $c_p$ in condensed phase is larger than that in gaseous phase. This result is proved experimentally.

In the case $r = 3/2$ the thermodynamical functions are obtained in much the same way. The equation of state is

$$P - P_e \sim -(v - v_e)/\log \left( \frac{|v - v_e|}{v_e} \right)$$

and the specific heat is

$$c_p \sim \log \left( \frac{T_e}{|T - T_e|} \right).$$

The strength of the singularity in thermal quantities on the transition line is much weaker than those in the case $2 > r > 3/2$.

The results (4·4), (4·5) and (4·10) show that the transition is a $\lambda$-transition. The correlation among the quasi-particles with momenta smaller than the characteristic momentum $p_0$ gives rise to the $\lambda$-transition. The number of such quasi-particles is given by
where we have used (2.14). The Bose-Einstein condensation occurs at \( \lambda \sim d \). Therefore, the fraction of quasi-particles contributing to the \( \lambda \)-transition, \( N_\lambda/N \), is
\[
N_\lambda/N \sim a/\lambda \sim a/d .
\]
This fraction is small for a dilute gas. Hence it is legitimate to replace \( P_\epsilon \) and \( v_\epsilon \) in (4·4), (4·5) and (4·10) by those of the ideal Bose gas. For the dense gas \( (a \sim d) \), \( N_\lambda/N \) may be comparable with unity. In this case also our result of the logarithmic singularity in specific heat is true since the singular part of specific heat is independent of the spectral form \( \varepsilon_\rho/\rho^r \) except for \( r = 3/2 \) and only the values of \( T_\epsilon, P_\epsilon, \mu_\epsilon, dP_\epsilon/dT \), etc., change with density. These quantities do not affect the singularities of thermal functions and affect at most the proportionality constants of singular terms.

§ 5. Condensation density

The condensation density is obtained from (2·8), (3·18) and (3·20):
\[
\left( \frac{\partial N_\epsilon}{\partial v} \right)_T = \left( \frac{Y^1}{Z} \right) \left( \frac{\partial \mu}{\partial v} \right)_T ,
\]
(5·1)
or integrating \( 2 > r > 3/2 \)
\[
N_\epsilon \sim (\mu - \mu_\epsilon) \frac{Y^1}{Z} \approx (\mu - \mu_\epsilon) \left[ \log (r_\epsilon p_\epsilon) \right]^3 .
\]
(5·2)
From this we find
\[
N_\epsilon \sim (P_\epsilon - P) \left( \log \frac{P_\epsilon}{P - P_\epsilon} \right)^3 \sim (v_\epsilon - v) \log \frac{v_\epsilon}{v_\epsilon - v}
\]
at constant temperature. It follows that
\[
N_\epsilon \sim (T_\epsilon - T) \log \left( \frac{T_\epsilon}{T_\epsilon - T} \right)
\]
at constant specific volume, and
\[
N_\epsilon \sim (T_\epsilon - T) \left( \log \frac{T_\epsilon}{T_\epsilon - T} \right)^3
\]
at constant pressure for \( 2 > r > 3/2 \). For \( r = 3/2 \), \( N_\epsilon \) is obtained by replacing the factor \( \log (T_\epsilon/T_\epsilon - T) \) by the factor \( \log[\log (T_\epsilon/T_\epsilon - T)] \) in the above expressions. The thermal behaviour of \( N_\epsilon \) is singular on the transition line in contrast with the case of the ideal Bose gas.

From (2·11) and (4·1) we get
\[
\left( \frac{\partial^2 F}{\partial n_\epsilon^2} \right)_{N,T,P} = \gamma(0)/V .
\]
From (3·20) and (5·1) we obtain
\[ (\partial^2 F/\partial n_0^2)_{N,T,v} = 1/VX \] (5·3)
at \( n_0 = \bar{n}_0 \). If we consider the free energy as a function of \( n_0 \), then we have from (5·3)
\[ F(n_0) = F(\bar{n}_0) + \frac{1}{2} (n_0 - \bar{n}_0)^2 / VX + O((n_0 - \bar{n}_0)^3) \] (5·4)
since \( (\partial F(n_0)/\partial n_0)_T = 0 \) at \( n_0 = \bar{n}_0 \). For fixed \( V \) and \( T \), the probability having \( n_0 \) condensation particles is proportional to \( \exp[-\beta F(n_0)] \). From (5·4) the fluctuation of \( n_0 \) is easily found:
\[ \langle \Delta n_0 \rangle = \frac{(n_0 - \bar{n}_0)^2}{2(\beta \partial F / \partial n_0)} = \int d\bar{n}_0 (n_0 - \bar{n}_0)^2 e^{-\beta F(n_0)} \int d\bar{n}_0 e^{-\beta F(n_0)} = kTVX. \] (5·5)

In practice the temperature and the density also fluctuate. Thus the fluctuation of \( n_0 \) from fluctuations of \( V \) and \( T \),
\[ \langle \{ (\partial \bar{n}_0 / \partial V)_T + (\partial \bar{n}_0 / \partial T)_V \} \rangle, \] (5·6)
must be added to the right-hand side of (5·5). As is well known, \( \langle \Delta V \rangle = -kT(\partial V / \partial P)_T, \langle \Delta T \rangle = kT(C_v) \) and \( \langle \Delta V \Delta T \rangle = 0 \). From (5·2) and (4·4) we find that (5·6) is proportional to \( [\log(1/|v-c_0|)]2 > r > 3/2, \) and is negligible compared with (5·5).

From (3·3) and (3·27) it then follows that
\[ \langle \Delta n_0 \rangle \sim \begin{cases} 1/(v_c - v)^{(2r - 3)/2}, & 2 > r > 3/2, \\ \log(1/|v - v_c|), & r = 3/2. \end{cases} \] (5·7)
This becomes infinite as \( v \to v_c \).

The density fluctuation assumes the familiar expression in statistical mechanics:
\[ \langle \Delta n \rangle = -(kTn^2/V) (\partial V / \partial P)_T, \]
where \( n = 1/v \). From (4·4) it follows that
\[ \langle \Delta n \rangle \sim \begin{cases} \log(1/|v - v_c|), & 2 > r > 3/2, \\ \log[\log(1/|v - v_c|)], & r = 3/2. \end{cases} \]
The fluctuation of \( N_0 \) is very much larger than that of the density. This means that the large fluctuation of \( N_0 \) occurs at almost constant density. If we note that \( n = N_0 + n' \) where \( n' \) is the number density of excited quasi-particles, we see that \( n' \) also fluctuate as \( N_0 \), i.e., \( \Delta N_0 \approx -\Delta n' \). Thus we have
\[ \langle \Delta N_0 \rangle \approx \langle \Delta n' \rangle \approx -\langle \Delta N_0 \Delta n \rangle \] (5·8)
in the vicinity of transition line. There exists a strong correlation between the condensation density \( N_0 \) and the non-condensation density \( n' \). This is contrasted with the case of the ideal Bose gas where there is no correlation between the con-
densation density and the non-condensation density since the particles with different momenta are independent of each other. Our results (5·7) and (5·8) are also different from those of the theory of PP where \( \langle AN_0^2 \rangle \) is finite and \( \langle An^2 \rangle \approx \langle An^2 \rangle \).

\[ \text{§ 6. Discussion} \]

We have shown that the interacting Bose gas exhibits the \( \lambda \)-transition when the Bose-Einstein condensation occurs. The essential role in this regard is played by the correlation among quasi-particles, and the excitation energy spectrum determines the nature of singularity at the transition point. The quasi-particles of momentum smaller than \( p_0 \), a characteristic momentum defined in § 6 of I, contribute to the \( \lambda \)-transition. The exponent \( r \) of excitation energy \( \varepsilon_p^{(c)} = Ap^r \) on the transition line has not been determined theoretically. The experimentally observed singularity in the specific heat indicates \( 2 > r > 3/2 \).

Our method described in this paper corresponds to the Landau theory of Fermi liquid. The quasi-particles described by \( n_p \) correspond to the quasi-particles in Landau's theory: The quasi-particles have one-to-one correspondence with free particles and the occupation number of quasi-particles is given in the same form as that for free particles.

We note that the occupation number of quasi-particles \( n_p \) given by (2·2), is not equal to the average number of particles with momentum \( p \), i.e., \( n_p = \langle a_p^+ a_p \rangle = \text{Tr}(\rho a_p^+ a_p) / \text{Tr} \rho \), where \( \rho = e^{-\beta H} \). This follows, for example, from the fact that \( n_p(p \neq 0) = 0 \) at \( T = 0 \) but \( \langle a_p^+ a_p \rangle \neq 0 \) because of the depletion effect. \( \langle a_p^+ a_p \rangle \) is obtained from (A·5) of I by replacing \( A \) by \( a_p^+ a_p \). \( n_p \) has been mathematically introduced into the theory. The relation between our quasi-particles and the more conventional one defined through thermal Green's function is not yet known. The same is true for the excitation energy.

The number density of condensed quasi-particles, \( N_0 \), becomes zero on the transition line and becomes the total density \( N/V \) at \( T = 0 \) since \( N = n_0 + \sum_{p \neq 0} n_p \), and \( n_p \) becomes zero at \( T = 0 \). Thus, in its behaviour, \( N_0 \) resembles the superfluid density \( \rho_s \). (The average \( \langle a_0^+ a_0 \rangle \) does not become the total number of particles, \( N_s \) at \( T = 0 \) because of the depletion effect.) We cannot say, however, that \( N_0 \) is the superfluid density. The superfluid density is originally introduced in the dynamical description of the fluid. In order to clarify the relation between the superfluid density and the condensation density we should study the dynamical behaviour of the liquid in our scheme.

The scaling laws of Widom\textsuperscript{9} and Kadanoff\textsuperscript{10} are obtained for the Ising model in a weak external field. These laws cannot directly be applied to the Bose liquid since the Bose liquid corresponds to the spin system with easy plane in a strong external field.\textsuperscript{9} In our theory also the scaling property is derived for the occupation number \( n_p \), i.e., \( n_p \) is expressed in the form \( n_p \sim p^{-\gamma} g(r_p,p) \). The proof goes as follows: From (3·12) we have
\[ 1 - r_e^{3r-3}\alpha(x_p/r_e) + \int_{|x_k|<r_e p} \frac{d^2x_k}{(2\pi)^2} \left\{ f(x_p, x_k) - f(0, x_k) \right\} \frac{r_e^{3r-3}\alpha(x_k/r_e)}{\beta A^2 x_k^{3r}}, \]

where we have put \( x_p = r_e p \) and \( x_k = (\beta A^2 p^{3r})^{-1} \), and used the relations \( f(p, k) = r_e^{3r-3} f(x_p, x_k) \) and \( \alpha(0) \sim 1/r_e^{3r-3} \). The last integral is convergent as \( r_e p_0 \to \infty \) if \( \alpha(p) \) increases slowly, and not so rapidly as \( p^3 \), for large \( p \). Thus we have the solution of the form \( \alpha(x_p/r_e) = r_e^{3r-3} A(x_p) \) for large \( r_e \). From (2.2)

\[ \frac{kT}{n_p} = \frac{\beta E}{\delta n_p - \mu + A p^3 - \alpha(p) (\mu - \mu_c)} \]

(gaseous phase)

\[ \sim r_e^{3r-3} [x_p^3 - A(x_p)], \]

where we have used the relation \( (\mu - \mu_c) r_e^{3r-3} \sim 1 \) (see (3.27)). Thus we find \( n_p \sim p^{-r} g(x_p) \). A similar discussion can be made in the condensed phase.

The scaling relation \( n_p \sim p^{-r} g(r_e p) \) means that the one-particle density matrix \( C(r) = \langle \psi'(0) \psi(r) \rangle \) has the form first proposed by Fisher:{\textsuperscript{41}}

\[ C(r) = G(r/r_e) / |r|^{3r-3}, \quad |r|/p_0 \gg 1. \]  

(6·1)

The scaling relation for \( n_p \) is violated for \( |p| \geq p_0 \) and (6·1) is also violated for \( |r| \leq p_0^{-1} \). In fact, \( C(0) = \langle \psi'(0) \psi(0) \rangle = n \) (number density) is not of the form (6·1). Thus the inverse of the characteristic momentum gives the distance where the scaling relation (6·1) begins to be violated.

The relation \( r_e \sim |T - T_c|^{-1/(3-r)} \) is contrasted with the scaling laws of Widom{\textsuperscript{5}} and Kadanoff{\textsuperscript{4}}. Define the critical exponents \( \alpha \) and \( \nu \) by \( \alpha = \frac{1}{3} - \frac{r}{(3-r)} \) and \( r_e \sim |T - T_c|^{-\nu} \). Then from the assumption (6·1), it can be shown for Bose liquid that{\textsuperscript{5}}

\[ (3-r)\nu \geq 1 - \alpha. \]  

(6·2)

This inequality is due to the same fact as that in the weak-scaling theory of Stell{\textsuperscript{11}}{\textsuperscript{*}}: the breakdown of the homogeneity relation (6·1) for small \( r \) gives rise to the inequality (6·2). Our results \( \nu = 1/(3-r) \) and \( \alpha = 0 \) are consistent with (6·2).

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\footnote{The author is grateful to Professor M. Suzuki for pointing out that the inequality is related to the weak-scaling theory of Stell.}
4) L. P. Kadanoff, Physics 2 (1966), 263.
7) For the diagram technique, see, for example, A. A. Abrikosov, L. P. Gorkov and I. E. Dzyaloshinskii, Methods of Quantum Field Theory in Statistical Mechanics (Prentice-Hall, Englewood Cliffs, New Jersey, 1963), as well as Ref. 2).

Note added in proof: It has been noticed that the Appendix C of the previous paper I contains some errors. They are concerned with the figures and the interpretation of them but do not affect the final results. For example, the statement in the third line of p. 2234 “The singularity...such as (d),” is incorrect. This diagram also contributes to the singular factors. In Fig. 16 there are double counting of graphs. But $\Gamma$ can be expanded in terms of $\Gamma_L$ in a fashion similar to Fig. 16 and the final results remain unaltered.