658

Progress of Theoretical Physics, Vol. 49, No. 2, February 1973

On the Application of Parafermion Algebras of Order 2

A. L. CAREY

Department of Mathematical Physics
University of Adelaide, South Australia

(Received September 11, 1972)

The representation theory of $SO(2N+1)$ is used to analyse the structure of general representations of parafermion algebras of order 2. The results are applied to overcome a difficulty in Green's application of parastatistics to the leptons.

§ 1. Introduction

The representation theory of parafermion algebras has received increasing attention over the last few years. This theory has however been restricted almost exclusively to representations characterised by the existence of a unique vacuum state vector $|\rangle$ satisfying

$$a_r |\rangle = 0, \quad r = 1, \ldots, N,$$

the so-called Fock space representations. Relation (1) has as a direct consequence:

$$\{a_r, a^\dagger_r \} |\rangle = \delta_i^j \rho |\rangle,$$

where $\rho$ is a positive integer, the order of the parastatistics. This relation greatly restricts the representations of the parafermion algebra which may be considered. However, recent work in the representation theory of the $n$-dimensional rotation group has produced some results which may be used to give a physical interpretation to a more general class of representations.

In this paper we consider parafermi statistics of order 2 utilising the fact that certain combinations of the parafermion annihilation and creation operators satisfy the commutation relations of the Lie algebra of $SO(n)$. We obtain in this way a theory of the leptons which has already been utilised in one form by Green to give a new variant of the neutrino theory of light. The formalism of Green contains a difficulty in that, the 'Fock-space' representations only allow states in which the $\beta$-lepton and $\mu$-lepton numbers differ by no more than one. To overcome this he introduces pairs of creation and annihilation operators of neutrinos of zero energy as 'dummy operators'. This expedient can however be circumvented by considering the more general representations.

We also relate our results to those obtained by Govorkov on 'non-Fock'
§ 2. Parafermion representations of \( SO(2N+1) \)

The annihilation operators \( a_r \) and their hermitian conjugates, the creation operators \( a^r \), for parafermions of arbitrary order satisfy

\[
\begin{align*}
[a_r, [a^s, a_t]] &= 2\delta^s_r a_t, \\
[a_r, [a^s, a^t]] &= 2(\delta^s_r a^t - \delta^t_r a^s),
\end{align*}
\]

as well as relations conjugate to these.

The operators \( a_r, a^r; r, s=1, \ldots, N \) satisfying (3) are said to constitute a parafermion algebra. It is known that \( a_r \), as a result of (3), the set of operators

\[
[\frac{1}{2}[a_r, a_s], \frac{1}{2}[a^r, a^s], \frac{1}{2}[a^r, a_s]]
\]

for \( r, s=1, \ldots, N \)

span a Lie algebra isomorphic to that of \( SO(2N+1) \). Finite dimensional irreducible representations of the parafermion algebra in which the \( a_r \) are conjugates of the \( a^r \) correspond to unitary irreducible representations of \( SO(2N+1) \).

We define \( a_p, p=1, 2, \ldots, 2N \) by setting \( a_{r+N} = a^r \) and the operators \( A_{RS} \) span a Lie algebra isomorphic to that of \( SO(2N+1) \). Finite dimensional irreducible representations of the parafermion algebra in which the \( a_r \) are conjugates of the \( a^r \) correspond to unitary irreducible representations of \( SO(2N+1) \).

We define \( a_p, p=1, 2, \ldots, 2N \) by setting \( a_{r+N} = a^r \) and the operators \( A_{RS} \) span a Lie algebra isomorphic to that of \( SO(2N+1) \). Finite dimensional irreducible representations of the parafermion algebra in which the \( a_r \) are conjugates of the \( a^r \) correspond to unitary irreducible representations of \( SO(2N+1) \).

We recall the following properties of \( SO(2N+1) \). In any given irreducible representation one can define weight vectors \( (\eta_1, \ldots, \eta_N) \) as ordered sets of simultaneous eigenvalues of the \( N \) commuting hermitian operators

\[
a^r\eta = \frac{1}{2}[a^r, a_r].
\]

An irreducible representation is uniquely specified by its weight vector of highest weight \( (L_1, \ldots, L_N) \) for which \( L_1 \geq L_2 \geq \cdots \geq L_N \geq 0 \). A representation with this highest weight will be denoted by \( 2N+1D_{(L_1, \ldots, L_N)} \). In this representation the weight vector of lowest weight is \( (-L_1, \ldots, -L_N) \).

Now, (1) and (2) imply that it corresponds to a weight vector \( (-p/2, \ldots, -p/2) \) which is of lowest weight in that representation. For parafermions of order 2 this weight vector is \( (-1, \ldots, -1) \) which suggests that the other representations to consider would be those labelled by \( (1, 0^{s-r}) r=0, \ldots, N \). These representations do in fact arise naturally in the theory of parafermions of order 2.
We introduce an hermitian operator $\pi$ satisfying
\begin{align*}
\pi^2 &= 1 , \\
\pi a_r a_r + a_r \pi a_r &= 0 , \\
a_r a_r' \pi + \pi a_r a_r' &= 2 \delta^r_r \pi , \\
a_r a_r + a_r a_r &= 0 , \\
a_r' a_r + a_r a_r' &= 0 ,
\end{align*}
(8)
plus relations conjugate to these. Denote by $A$ the algebra generated by the $a_r$, $a_r'$ and $\pi$. A realisation of $A$ may be obtained as follows.

Let $a_r = \pi a_r \pi$ for $r = 1, \ldots, N$, and define
\begin{align}
\alpha_r = \frac{1}{2} (a_r + a_r') , \\
\beta_r = \frac{1}{2} (a_r - a_r').
\end{align}
(9)
Relations (8) imply that the operators $\alpha_r$, $\alpha_r'$, $\frac{1}{2} [\alpha_r, \alpha_r']$, $\frac{1}{2} [\alpha_r', \alpha_r']$, $\frac{1}{2} [\alpha_r, \alpha_r']$, satisfy the commutation relations of $SO(2N+1)$ and commute with the $\beta_r$. They also imply that the operators
\begin{align}
\pi, \beta_r, \beta_r', \frac{1}{2} [\beta_r, \beta_r'], \frac{1}{2} [\beta_r', \beta_r'], \frac{1}{2} [\beta_r, \beta_r], \frac{1}{2} [\beta_r', \beta_r], \frac{1}{2} [\pi, \beta_r], \frac{1}{2} [\pi, \beta_r']
\end{align}
(10)
satisfy the commutation relations of $SO(2N+1,1)$. Thus a realisation of $A$ is obtained by taking a representation of $SO(2N+1) \times SO(2N+1,1)$ in which the operator $\pi$ is hermitian and satisfies $\pi^2 = 1$. Such a representation can be obtained by observing that complex linear combinations of the operators (10) satisfy the commutation relations of $SO(2N+2)$. Defining the weights of the representation in terms of the $N+1$ commuting hermitian operators $\pi/2$, $\frac{1}{2} [\beta_r, \beta_r']$, $r = 1, \ldots, N$, consider for example the irreducible representation of $SO(2N+2)$ of highest weight $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. Taking the tensor product of this with a representation of $SO(2N+1)$ gives the required realisation of $A$.

The relations (8) and (9) have the additional consequence that the $\alpha_r$'s and $\beta_r$'s form two sets of fermi oscillator operators and that
\begin{align}
a_r = \alpha_r + \beta_r , \\
r = 1, \ldots, N ,
\end{align}
(11)
which is simply the well-known Green Ansatz.

Returning to the algebra $A$, consider the representation obtained from the $^{2N+1}D_{0,2^N} \otimes ^{2N+2}D_{0,2^{N+1}}$ representation of $SO(2N+1) \times SO(2N+2)$. Denote by $P$ the parafermion algebra of order 2 generated by the $a_r$ and $a_r'$. A (reducible) representation of $P$ is defined by the above representation of $A$ and by considering the reduction into irreducible representations of $SO(2N+1)$ one obtains irreducible representations of $P$. The reduction may be effected by using the branching rule for $SO(2N+2)$ and the character formula for $SO(2N+1)^b$ and gives
\begin{align}
^{2N+1}D_{0,2^N} \otimes ^{2N+2}D_{0,2^{N+1}} = ^{2N+1}D_{0,2^N} \otimes ^{2N+1}D_{0,2^{N+1}}.
\end{align}
On the Application of Parafermion Algebras of Order 2

\[ = ^{2N+1}D_\alpha^{\gamma_j} \bigoplus ^{N+1}D_\alpha^{\gamma_{-1,0}} \bigoplus \cdots \bigoplus ^{2N+1}D_\alpha^{\gamma_j}. \]  
(12)

Denote the corresponding decomposition into irreducible subspaces by

\[ E^N \otimes E^N = V^N \oplus V^{N-1} \oplus \cdots \oplus V^r \oplus \cdots \oplus V^0. \]  
(13)

The first representation: \( ^{2N+1}D_\alpha^{\gamma_j} \) in this decomposition is the usual Fock space representation of \( P \). We now investigate the structure of the other representations and their interpretation in terms of the leptons.

§ 3. Physical interpretation of the representations of \( P \)

First we recall that the creation and annihilation operators of \( P \) satisfy, as well as (1), the relations

\[
\begin{align*}
    a_i a_j a_k + a_k a_j a_i &= 0, \\
    a_i a_j a_k + a_k a_j a_i &= 2b^j a_k, \\
    a_i a_j a_k + a_k a_i &= 2b^j a_k + 2b^i a_i. 
\end{align*}
\]  
(14)

In the application of parastatistics of order 2 to the leptons\(^1\) one has the following physical interpretation. In a state vector like \( a_i a_j a_k a_l \), the operators such as \( a_i a_j \) which are an odd number of places removed from the vacuum state vector are understood to create \( \beta \)-particles or annihilate \( \mu \)-particles, while those such as \( a_i a_j \) which are an even number of places removed from the vacuum state vector are understood to create \( \mu \)-particles or annihilate \( \beta \)-particles. This convention is suggested by the fact that

\[ a_i a_j a_k a_l = -a_i a_j a_k a_l. \]

(15)

Thus the \( \beta \)- and \( \mu \)-particles satisfy Fermi statistics among themselves. This interpretation has however, the disadvantage that the \( \beta \)-lepton and \( \mu \)-lepton numbers cannot differ by more than one. We now show that by considering the whole space \( E^N \otimes E^N \) a formalism can be developed which allows state vectors with an arbitrary number of \( \beta \)-particles or \( \mu \)-particles. Moreover, this formalism is consistent with the above place interpretation and may be generalised to the infinite dimensional case (i.e., \( N \to \infty \)).

To see this we need the following facts\(^1\) about the representations \( ^{2N+1}D_\alpha^{\gamma_j} \). Each representation \( ^{2N+1}D_\alpha^{\gamma_j} \) is uniquely specified by its lowest weight \((-1, 0^{N-r})\) and this weight is simple (i.e., the vector corresponding to it is unique up to a scalar multiple). We also know from (6) the various root vectors of \( SO(2N+1) \) (these are the weights of the fundamental representation).

In particular, the annihilation operators \( a_i \) have roots \( e_1(0, \cdots, 0, -1, 0, \cdots, 0) \). By application of elements of the Weyl group of \( SO(2N+1) \) to the lowest weight all of the \( NC_r \) weights containing \( r(-1) \)'s and \( N-r \) \( 0 \)'s may be generated. Since these weights are simple,\(^1\) we have within the space \( V^r \) one vector \( \phi_\beta \), unique
up to a scalar multiple, corresponding to each of the above weights for \( j = 1, 2, \ldots, \kappa C_r \).

These \( \phi_j \) satisfy

\[
a_s \phi_j = 0; \quad s = 1, 2, \ldots, N; \quad j = 1, 2, \ldots, \left( \frac{N}{r} \right).
\]

(16)

For, if \( a_s \phi_j \neq 0 \) it would have weight \( m + e_s \), where \( m \) is the weight of \( \phi_j \). But if \( m + e_s \) were a weight then \( V' \) would contain a vector of weight lower than \((-1^r, 0^{N-r})\) and as this is the lowest weight we conclude that (16) holds.

Thus the \( \phi_j \) appear to be 'vacuum states' for this representation of \( P \). However, if we write for the vector \( \phi_1 \): \(|i_r+1, \ldots, i_N\rangle\) where the labels denote the positions of the zeros in the weight vector of \( \psi_1 \), we see that an argument similar to that used to establish (16) gives \( a'|i_{r+1}, \ldots, i_N\rangle = 0 \) for \( s \varepsilon \{i_{r+1}, \ldots, i_N\} \). Thus the vector \(|i_{r+1}, \ldots, i_N\rangle\) is annihilated by certain creation operators suggesting that, if we continue to adopt the place convention described above, we must interpret \(|i_{r+1}, \ldots, i_N\rangle\) as representing a state containing \( N - r \) particles of momentum \( i_{r+1}, \ldots, i_N \) of the same type as that of the creation operator acting in the first position. This interpretation is reinforced by the following results.

Since \( a'_{i_{r}} = \frac{1}{2}[a', a_1] \) has weight \((-1^r, 0^{N-r}) + (e_{r} - e_{r})\) we see that \( a'|i_{r+1}, \ldots, i_N\rangle = 0 \) for \( s \varepsilon \{i_{r+1}, \ldots, i_N\} \). Thus the vector \(|i_{r+1}, \ldots, i_N\rangle\) is annihilated by certain creation operators suggesting that, if we continue to adopt the place convention described above, we must interpret \(|i_{r+1}, \ldots, i_N\rangle\) as representing a state containing \( N - r \) particles of momentum \( i_{r+1}, \ldots, i_N \) of the same type as that of the creation operator acting in the first position. This interpretation is reinforced by the following results.

Since \( a'_{i_{r}} = \frac{1}{2}[a', a_1] \) has weight \((-1^r, 0^{N-r}) + (e_{r} - e_{r})\) we see that \( a'|i_{r+1}, \ldots, i_N\rangle = 0 \) for \( s \varepsilon \{i_{r+1}, \ldots, i_N\} \). Thus the vector \(|i_{r+1}, \ldots, i_N\rangle\) is annihilated by certain creation operators suggesting that, if we continue to adopt the place convention described above, we must interpret \(|i_{r+1}, \ldots, i_N\rangle\) as representing a state containing \( N - r \) particles of momentum \( i_{r+1}, \ldots, i_N \) of the same type as that of the creation operator acting in the first position. This interpretation is reinforced by the following results.

Since \( a'_{i_{r}} = \frac{1}{2}[a', a_1] \) has weight \((-1^r, 0^{N-r}) + (e_{r} - e_{r})\) we see that \( a'|i_{r+1}, \ldots, i_N\rangle = 0 \) for \( s \varepsilon \{i_{r+1}, \ldots, i_N\} \). Thus the vector \(|i_{r+1}, \ldots, i_N\rangle\) is annihilated by certain creation operators suggesting that, if we continue to adopt the place convention described above, we must interpret \(|i_{r+1}, \ldots, i_N\rangle\) as representing a state containing \( N - r \) particles of momentum \( i_{r+1}, \ldots, i_N \) of the same type as that of the creation operator acting in the first position. This interpretation is reinforced by the following results.

Since \( a'_{i_{r}} = \frac{1}{2}[a', a_1] \) has weight \((-1^r, 0^{N-r}) + (e_{r} - e_{r})\) we see that \( a'|i_{r+1}, \ldots, i_N\rangle = 0 \) for \( s \varepsilon \{i_{r+1}, \ldots, i_N\} \). Thus the vector \(|i_{r+1}, \ldots, i_N\rangle\) is annihilated by certain creation operators suggesting that, if we continue to adopt the place convention described above, we must interpret \(|i_{r+1}, \ldots, i_N\rangle\) as representing a state containing \( N - r \) particles of momentum \( i_{r+1}, \ldots, i_N \) of the same type as that of the creation operator acting in the first position. This interpretation is reinforced by the following results.

Since \( a'_{i_{r}} = \frac{1}{2}[a', a_1] \) has weight \((-1^r, 0^{N-r}) + (e_{r} - e_{r})\) we see that \( a'|i_{r+1}, \ldots, i_N\rangle = 0 \) for \( s \varepsilon \{i_{r+1}, \ldots, i_N\} \). Thus the vector \(|i_{r+1}, \ldots, i_N\rangle\) is annihilated by certain creation operators suggesting that, if we continue to adopt the place convention described above, we must interpret \(|i_{r+1}, \ldots, i_N\rangle\) as representing a state containing \( N - r \) particles of momentum \( i_{r+1}, \ldots, i_N \) of the same type as that of the creation operator acting in the first position. This interpretation is reinforced by the following results.
On the Application of Parafermion Algebras of Order 2

\[ a^{|i_{r+1}, \ldots, i_N} = -a^{|i_{r+1}, \ldots, i_{r-1}, i_r, i_{r+1}, \ldots, i_N}, \quad (20) \]

so that a creation operator acting in the first place creates a particle of the same type as those in the state vectors \(|i_{r+1}, \ldots, i_N\rangle\). We can annihilate particles in the states \(|i_{r+1}, \ldots, i_N\rangle\) in a manner consistent with the place interpretation for annihilation operators. Using (18) and a knowledge of the weights of the vectors \(|i_{r+1}, \ldots, i_N\rangle\) it may be shown that

\[ a_s a^{|i_{r+1}, \ldots, i_N} = 2\delta_s^i |i_{r+1}, \ldots, i_N\rangle - 2\delta_i^s |i_{r+1}, \ldots, i_N\rangle - 2\delta_{tr}^i |l, i_{r+1}, \ldots, i_N\rangle. \quad (21) \]

Therefore \(a_s\) annihilates particles of this type when it is an even number of places removed from the 'reservoir state' \(|i_{r+1}, \ldots, i_N\rangle\). Note that (16) implies that annihilation operators, an odd number of places removed from the 'reservoir state', annihilate particles of the other type.

§ 4. Comparison with Govorkov's procedure

A method of generalising the above to the infinite dimensional case has been obtained in part by Govorkov.\textsuperscript{4} It is immediate from (9) that

\[ \bar{a}_r = a_r - \beta_r. \quad (22) \]

We now consider an irreducible representation of the algebra \(B\) generated by the operators \(a_r, \beta_r\). (Irreducible representations of \(A\) give rise to irreducible representations of \(B\).) Suppose that we have a 'Fock-space' representation of \(B\), i.e., there is a unique 'vacuum state' \(\Phi\) satisfying \(a_r \Phi = \beta_r \Phi = 0, \ r = 1, \ldots, N\). Construct, recursively the vectors

\[ \Phi_{i_1 \ldots i_r} = a_{i_1} a_{i_2} a_{i_3} \cdots \Phi. \quad (23) \]

Using the defining relations for the operator \(\pi\) the following properties may be obtained:

\begin{align*}
\bar{a}_r a_s &= -a_r a_s, & a_r \bar{a}_s &= -a_r \bar{a}_s, \\
\bar{a}_r a^t &= -a^t a_r, & a_r a^t &= -a_r a^t, \\
a_r a_s &= -\bar{a}_r a_s, & a_r a^t &= -\bar{a}_r a^t + 2\delta_{rs}, \\
\bar{a}_r a^t &= -a_r a^t + 2\delta_{rs}. \quad (24)\end{align*}

Relations (24) imply in particular that the vectors \(\Phi_{i_1 \ldots i_r}\) satisfy (16), (18), (19), (20) and (21). Govorkov has demonstrated that irreducible representations of the parafermion algebra \(P\) are obtained by the action of all possible polynomials \(R(a')\) in the creation operators \(a'\), on the vectors \(\Phi_{i_1 \ldots i_r}\) (fixed \(f\)) and that this decomposes the 'Fock-space' representations of the algebra \(B\), into irreducible representations of \(P\). We know by the uniqueness of this decomposition in the finite dimensional case that Govorkov's procedure will yield the same decomposition as (12). By matching dimensions of the irreducible subspaces it is easy to
show that \( \Phi_{i_{1}, \ldots, i_{r}} \in V^{N-1} \). It is not clear that a vector corresponding to \( \Phi \) exists in \( E^{N} \otimes E^{N} \) or that it has any relation to the vacuum state \( |> \) which is contained in \( V^{N} \).

However, we know\(^{10} \) that there is a unique vector \( \chi \) of lowest weight in \( E^{N} \). It is clear that \( \chi \otimes \chi \) satisfies the requirements of the state \( \Phi \) and utilising the decomposition (13) we find that \( \chi \otimes \chi \in V^{N} \). Then since
\[
(\alpha_{i} + \beta_{i}) \chi \otimes \chi = 0.
\]
The uniqueness of \( |> \) in \( V^{N} \) with this property implies that \( |> \) is a scalar multiple of \( \chi \otimes \chi \). In this sense we may identify \( \Phi \) with \( |> \) and hence
\[
\alpha_{r}|> = \beta_{r}|> = 0, \quad r = 1, \ldots, N.
\]

Thus we can parallel Govorkov's procedure to construct the representation \( S^{N+1}D_{\alpha^{r}, \delta^{N-r}} \) with the \( \Phi_{i_{1}, \ldots, i_{r}} \) replacing the \( |i_{1}, \ldots, i_{r}> \) and \( |> \) replacing \( \Phi \). But, since the weight of \( |i_{1}, \ldots, i_{r}> \) is simple and \( \Phi_{i_{1}, \ldots, i_{r}} \) also has this weight we conclude that in fact the \( \Phi \)'s are just the reservoir states in another form. This observation allows us to complete our description of the leptons in this formalism.

§ 5. General representations of the parafermion algebra

We construct firstly an operator which 'counts the difference between the number of \( \mu \)-leptons and \( \beta \)-leptons'. The Casimir invariant of \( SO(2N+1) \) in a particular representation \( S^{N+1}D_{\alpha^{1}, \ldots, \alpha^{N}} \) has eigenvalues given by\(^{9} \)
\[
\sigma_{s}(2N+1) = 2 \sum_{j=0}^{N} L_{j}(L_{j} + 2N - 2j + 1).
\]
For the representation labelled by \( (1^{r}, 0^{N-r}) \) this is
\[
\sigma_{s}(2N+1) = 2r(2N+1-r).
\]
Thus, for parafermi representations of order \( 2, \sigma_{s}(2N+1) \) determines or is determined by \( r \).

Using the branching formula for \( SO(2N+1) \)\(^{10} \) one has
\[
S^{N+1}D_{\alpha^{r}, \delta^{N-r}} = \bigoplus_{\epsilon_{i} \in \mathbb{Z}} S^{N}D_{\alpha^{r-1}, \delta^{N-r-1}} \quad \text{for} \quad r < N.
\]
The Casimir invariant of \( SO(2N) \) in the representation \( S^{N}D_{\alpha^{r-1}, \delta^{N-r-1}} \) has eigenvalues given by\(^{9} \)
\[
\sigma_{s}(2N) = 2 \sum_{j=1}^{N} \lambda_{j}(\lambda_{j} + 2N - 2j)
\]
and so for representations labelled by \( (1^{r-1}, 1-q, 0^{N-r}) \):
\[
\sigma_{s}(2N) = 2r(2N-r) - 2q(2N-2r+2-q).
\]
Define an operator, \( I \), which is an \( SO(2N) \) invariant, by
On the Application of Parafermion Algebras of Order 2

\[ I = \frac{1}{3} (\sigma_1 (2N+1) - \sigma_2 (2N)) - N, \]
\[ = \frac{1}{2} \{a^*, a_s\} - N, \]
\[ = \frac{1}{2} \Sigma_r (\{a^*, a_r\} - 2). \]  
(27)

From (25) and (26) \( I \) takes the values
\[ I = (r-N) \quad \text{for } q=0 \text{ and } r<N, \]
\[ N-r+1 \quad \text{for } q=1 \text{ and } r<N, \]
\[ q'(2-q') \quad \text{for } r=N \text{ and } q'=q \quad \text{for } N \text{ even}, \]
\[ q'=2-q \quad \text{for } N \text{ odd.} \]  
(28)

Thus for \( r<N, I \) distinguishes between the representations of \( SO(2N) \) contained in the decomposition (10) of the representation of \( A \), taking the values \(-N, -N+1, \ldots, N\). Furthermore, using (24) we have \( I(|i_r+1 \cdots i_N\rangle) = -(N-r) \) and \( I(a^* |i_r+1 \cdots i_N\rangle) = N-r+1 \). Thus \((2q-1)I\) gives the absolute value of the difference between the \( \mu \)-lepton and \( \beta \)-lepton numbers. It remains to show that we can construct state vectors containing arbitrary numbers of \( \mu \)-leptons and \( \beta \)-leptons from the vacuum state \( |\rangle \).

To do this we determine the action of \( \pi \) on the state vectors in this representation. The requirement
\[ \alpha_r |\rangle = \beta_r |\rangle = 0, \quad r=1, \ldots, N \]
and the relation \( \pi a^r = a^r \pi \) imply the following:
\[ \pi |\rangle = \varepsilon |\rangle, \]
\[ \pi (a^{k_1} \cdots a^{k_s} |i_r+1 \cdots i_N\rangle) = \begin{cases} \varepsilon a^{k_1} \cdots a^{k_s} |i_r+1 \cdots i_N\rangle & \text{s even}, \\ \varepsilon a^{k_1} \cdots a^{k_s} a^{k_{s+1}} |k_i i_{r+1} \cdots i_N\rangle & \text{s odd} \end{cases} \]
(29)

where \( \varepsilon = \pm 1 \).

Now, (29) suggests that \( \pi \) be interpreted as an operator which interchanges the 'statistical type' of the particles, i.e., it changes \( \mu \)-leptons to \( \beta \)-leptons and vice versa. This enables us to construct all possible state vectors from the creation operators \( a^r \) and \( \pi \) by acting on the vacuum state \( |\rangle \) and so obtain a general procedure for constructing representations of \( P \). Firstly construct all the 'reservoir states' recursively by defining
\[ |i_{r+1} \cdots i_N\rangle = \pi a^r |i_r+1 \cdots i_N\rangle \]
(30)
and observing that we can choose \( \pi \) so that \( \pi |\rangle = |\rangle \). The irreducible representations of the algebra \( P \) are then obtained by acting on the reservoir states with polynomials in the creation operators. The number of particles in any state vector is given by the operator
\[ K = \Sigma (a^*_s + 1) \]
and the operator $\nu = (-1)^{\nu-r}(2q-1)I$ satisfying $\pi \nu = -\nu \pi$ gives the difference between the $\beta$-lepton and $\mu$-lepton numbers. Relation (27) defines $\nu$ for infinite dimensional representations. So, using the place interpretation with say $a'|r\rangle$ representing one $\beta$-lepton, we can determine whether $|i_{r+1}, \cdots, i_N\rangle$ represents $N-r$ $\beta$-leptons or $N-r$ $\mu$-leptons from the sign of the eigenvalue of $\nu$. This further determines whether a creation operator acting on $|i_{r+1}, \cdots, i_N\rangle$ creates a $\mu$-lepton or $\beta$-lepton.

Finally, we can define operators which satisfy the commutation relations for $SU(2)$ in terms of $\pi$ and the $a^r$s.

Let

$$J_1 = \frac{1}{2} \Sigma_r (a^r a_r + a^r a_r),$$

$$J_2 = \frac{1}{2} \Sigma_r (a^r a_r - a^r a_r)$$

and

$$J_3 = \frac{1}{2} \Sigma_r (\{a^r, a_r\} - 2). \quad (= \frac{1}{2}I)$$

The commutation relations may be verified by using the relations (24) so that $A$ contains an $SU(2)$ subalgebra.

The author is indebted to Professor H.S. Green for fruitful discussions and comments while this work was in progress and to Dr. A.J. Bracken for some helpful discussions.

References

6) H. S. Green, Phys. Rev. 90 (1953), 270.
7) H. S. Green, Prog. Theor. Phys. 47 (1972), 1400.