HELMHOLTZ'S THEOREM WHEN THE
DOMAIN IS INFINITE AND WHEN THE
FIELD HAS SINGULAR POINTS

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SUMMARY

A direct proof, based on classical analysis, is given to extend Helmholtz's
fundamental theorem of vector analysis to cases in which the domain is infinite
and/or the field \( u(r) \) has isolated singular points. No restriction whatsoever is placed
upon the growth of \( u(r) \) as \(|r| \to \infty\), or as the singular points are approached.
Consequently all representation theorems dependent upon Helmholtz's theorem are
also freed from such restrictions.

1. Introduction

Let the vector field \( u(r) \) lie in the space \( C^k(\mathcal{D})(k \geq 0) \), where \( \mathcal{D} \) is some
domain in \( \mathbb{R}^3 \). (That is, \( u(r) \) and its partial derivatives of up to the \( k \)th order
are continuous for \( r \in \mathcal{D} \).) Then if there exists a second vector field
\( v(r) \in C^2(\mathcal{D}) \) such that

\[
\nabla^2 v = u \quad (r \in \mathcal{D}),
\]

where \( \nabla^2 \) is the Laplacian, then

\[
\begin{align*}
\nabla^2 v &= u \\
\n\nabla^2 v &= \text{grad} (\text{div} v) - \text{curl} (\text{curl} v)
\end{align*}
\]

so that \( u \) may be decomposed in the form

\[
\begin{align*}
u &= \nabla^2 v \\
\n\nabla^2 v &= \text{grad} (\text{div} v) - \text{curl} (\text{curl} v)
\end{align*}
\]

where \( \nabla \) and \( \nabla \) are given in terms of \( v \) by

\[
\begin{align*}
\chi &= \text{div} v, \\
\chi &= \text{div} v, \\
\n\chi &= \text{div} v, \\
\n\n\n\n\end{align*}
\]

A decomposition of the form (1.2) (in which \( A \) also satisfies \( \text{div} A = 0 \)) is
called a Helmholtz (or Stokes–Helmholtz) decomposition of \( u \). It is an
essential step in the proof of representation theorems for vector fields in
many branches of applied mathematics.†

A vector field \( u \) will thus have a Helmholtz decomposition if there exists

† In linear elasticity for instance the Helmholtz decomposition is required in the proof of
Lamé's decomposition of the elastodynamic field (1), and also in the proof of completeness of
the Papkovich–Neuber representation of the elastostatic field (2).

a corresponding solution \( v \) of the vector Poisson equation (1.1). For the case in which \( \mathcal{D} \) (the closure of \( \mathcal{D} \)) is a bounded regular (in the sense of (3, Chapter 4)) region of space and \( u \in C(\mathcal{D}) \cap C^1(\mathcal{D}), \) such a \( v \) is generated by the Newtonian potential formula

\[
v(r) = -\frac{1}{4\pi} \int_{\mathcal{D}} \frac{u(r') \, dv'}{|r - r'|} \tag{1.4}\]

and this in effect constitutes the conventional proof of 'Helmholtz’s theorem' (see, for example, (4)). However, the volume integral in (1.4) may fail to exist (i) if \( \mathcal{D} \) is an unbounded domain or (ii) if \( u(r) \) has a singularity at a point \( r = a \) lying in \( \mathcal{D} \). The formula (1.4) for \( v \) clearly holds good if

\[
u = O\left(\frac{1}{|r|^{2+\delta}}\right) \quad \text{as} \quad |r| \to \infty \tag{1.5}\]

in case (i), or

\[
u = O\left(\frac{1}{|r - a|^{3-\delta}}\right) \quad \text{as} \quad r \to a \tag{1.6}\]

in case (ii), for some \( \delta > 0 \). However \( u \) need not satisfy either of these conditions, in which case the formula (1.4) no longer provides \( v \) satisfying (1.1). In such a case, the argument above is not sufficient to establish Helmholtz’s theorem. Representation theorems depending upon Helmholtz’s theorem are correspondingly restricted.

Gurtin (5) seems to have been the first to observe that, for the case of \( \mathcal{D} \) infinite, the restriction (1.5) is both artificial and unduly restrictive on actual applications. By employing a Green’s function first introduced by Blumenthal (6), namely

\[
\frac{1}{|r - r'|} - \frac{1}{|r'|} \tag{1.7}
\]

(instead of \( 1/|r - r'| \)), he obtained the improved result that the condition

\[
u = c + O(|r|^{-\delta}) \quad \text{as} \quad |r| \to \infty, \tag{1.8}
\]

where \( c \) is a constant vector, is sufficient to guarantee existence of the Helmholtz decomposition (1.2) when \( \mathcal{D} \) is infinite. Recently Tran-Cong (7) has further refined this approach and has shown that (1.8) may be replaced by the weaker condition

\[
u = O(|r|^v) \quad \text{as} \quad |r| \to \infty \tag{1.9}
\]

for any fixed \( v \). This work also extends Helmholtz’s theorem to a general number of spatial dimensions.

There seem to be no corresponding results for the case in which \( u(r) \) has singular points within its domain of definition; as remarked earlier, this may
also cause the conventional proof of Helmholtz's theorem to fail. However, since singular points are a common and important idealization in applied mathematics, the question as to whether Helmholtz's theorem remains true in the presence of general singularities is of interest.

Since the proof of Helmholtz's theorem reduces to the existence of a solution to a vector Poisson equation, another method of approach is to employ functional analytic techniques to establish such existence. (The method in (5, 7) uses classical analysis.) In particular, the textbook by Rektorys (8), and the four-volume work by Hörmander (9), are concerned with existence of solutions of linear partial differential equations. Typically, however, the field \( u(r) \) is required to belong to some particular linear space \( (L^p(\mathbb{D}) \text{ for example}) \) and membership of this linear space does, by definition, restrict the growth of \( u \) as \( |r| \to \infty \) or as the singular points are approached. Thus if we allow completely arbitrary growth of \( u \) in these limits, we can never have a single linear space that includes all possible behaviours. However in Hörmander's work (9, Chapter 10) an existence theorem is given which does allow such arbitrary growth; but here existence is shown in a space of distributions (rather than functions) and it is not clear whether corresponding results for the \( C^k(\mathbb{D}) \) spaces can be deduced from it. In any case, all such techniques require a large apparatus of functional analysis to be set up before any result can be established.

The purpose of the present paper is to provide a relatively brief, self-contained proof, based on classical analysis, that all restrictions on the rate of growth of \( u(r) \) (as \( |r| \to \infty \), or as singular points are approached) are unnecessary; Helmholtz's theorem remains true irrespective of such growth. This is proved by representing the required \( v \) explicitly as an infinite series. The main difficulty lies in showing that such a series exists and converges no matter what the rate of growth of \( u \) is. This is achieved in sections 2 to 5, the main theorem being Theorem 5. The proof does require, however, that the field \( u(r) \) belongs to \( C^1(\mathbb{D}) \) rather than \( C^0 \).

It follows that all representation theorems dependent upon Helmholtz's theorem are thereby freed from such restrictions.

2. Poisson's equation in the exterior region \( r \geq a \)

Consider the particular case in which \( u(r) \in C^k(\mathbb{R})(k \geq 1) \), where \( \mathbb{R} \) is the region \( r = |r| \geq a(>0) \) of \( \mathbb{R}^2 \). (We give the proof here in two dimensions as it more clearly illustrates the principles involved. The necessary modifications in the three-dimensional case are given later.) Then, in order to prove existence of a solution of (1.1), it suffices to consider the scalar equation

\[
\nabla^2 \phi = \rho, \tag{2.1}
\]

where \( \phi, \rho \) are corresponding Cartesian components of \( v, u \) and \( \rho \in C^k(\mathbb{R}) \). Take plane polar coordinates \( r, \theta \) and suppose that \( \rho(r, \theta) \) is an even
function of $\theta$ (the treatment if it is odd is similar) for each $r \geq a$. Then the assumed $C^1$-differentiability of $\rho(r, \theta)$ is sufficient for the existence of the Fourier expansion

$$\rho(r, \theta) = \frac{1}{2} \rho_0(r) + \sum_{n=1}^{\infty} \rho_n(r) \cos n\theta, \quad (2.2)$$

where

$$\rho_n(r) = \frac{1}{\pi} \int_0^{2\pi} \rho(r, \theta) \cos n\theta \, d\theta. \quad (2.3)$$

Now there certainly exists a $\phi_0(r)$ such that

$$\nabla^2 \phi_0 = \frac{1}{2} \rho_0 \quad (2.4)$$

since this merely requires that

$$\phi_0'' + \frac{1}{r} \phi_0' = \frac{1}{2} \rho_0 \quad (2.5)$$

and this ordinary differential equation with $\rho_0 \in C^k[a, \infty)$ has a solution $\phi_0 \in C^{k+2}[a, \infty)$.

After subtracting away this solution we may suppose, from now on, that

$$\rho(r, \theta) = \sum_{n=1}^{\infty} \rho_n(r) \cos n\theta. \quad (2.6)$$

Also the formula (2.3) for $\rho_n(r)$ may be integrated by parts $k$ times to give

$$|\rho_n(r)| \leq \frac{1}{\pi} \left( \frac{1}{n^k} \right) \int_0^{2\pi} \left| \left( \frac{\partial}{\partial \theta} \right)^k \rho(r, \theta) \right| \, d\theta$$

so that

$$|\rho_n(r)| \leq m(r)/n^k, \quad (2.7)$$

where $m(r) \in C[a, \infty)$.

Now we seek a solution of Poisson's equation (2.1) in the form

$$\phi(r, \theta) = \sum_{n=1}^{\infty} \phi_n(r) \cos n\theta, \quad (2.8)$$

where $\phi_n(r)$ satisfies

$$\phi_n'' + \frac{1}{r} \phi_n' - \frac{n^2}{r^2} \phi_n = \rho_n \quad (2.9)$$

($n \geq 1$), and make the change of independent variable

$$r = ae^u \quad (0 \leq u < \infty). \quad (2.10)$$

Then (2.9) becomes

$$\Phi_n'(u) - n^2 \Phi_n(u) = R_n(u) \quad (2.11)$$
(n ≥ 1), where

\[ \Phi_n(u) = \phi_n(\alpha e^u), \]

\[ R_n(u) = a^2 e^{2u} \rho_n(\alpha e^u). \]

Note that \( R_n(u) \in C^k([0, \infty) \) and, from (2.7),

\[ |R_n(u)| \leq M(u)/n^k, \]

where \( M(u) \in C[0, \infty) \). The general solution of (2.11) can be written as

\[ \Phi_n(u) = A_n e^{nu} + B_n e^{-nu} + \frac{e^{nu}}{2n} \int_0^u e^{-nu'} R_n(u') \, du'. \]

\( (n \geq 1, \ u \geq 0) \). For the special case in which the Fourier expansion (2.2) is a finite series, the corresponding finite series (2.8) for \( \phi \) (with \( \phi_n \) given by (2.15), (2.12) and with any choices of \( A_n, B_n \)) satisfies Poisson's equation (2.1) and the problem is immediately solved. However, in the general case in which the series (2.2) for \( \rho(r, \theta) \) is infinite, the corresponding infinite series for \( \phi(r, \theta) \) does not converge in \( r \geq a \) for most choices of the constants \( A_n, B_n \). The rest of this section is devoted to showing that, whatever the behaviour of \( \rho(r, \theta) \) as \( r \to \infty \), choices of \( A_n, B_n (n \geq 1) \) do exist which allow convergence of the series (2.8) defining \( \phi(r, \theta) \); the fact that such constants do exist is far from obvious in the general case.

**Lemma** Let

\[ K(u) = \max_{u' \in [0, u]} M(u'). \]

Then (i) for \( 0 \leq u \leq L < \infty \),

\[ \left| e^{nu} \int_u^L e^{-nu'} R_n(u') \, du' \right| \leq \frac{K(L)}{n^{k+1}}; \]

(ii) for \( 0 \leq u \),

\[ \left| e^{-nu} \int_0^u e^{nu'} R_n(u') \, du' \right| \leq \frac{K(u)}{n^{k+1}}. \]
Proof. (i)

\[
\left| e^{nu} \int_{u}^{L} e^{-nu'} R_n(u') \, du' \right| \leq e^{nu} \int_{u}^{L} e^{-nu'} \frac{M(u')}{n^k} \, du' \\
\leq \frac{e^{nu}}{n^k} \max_{u' \in [u, L]} \{M(u')\} \int_{u}^{L} e^{-nu'} \, du' \\
\leq \frac{K(L)}{n^{k+1}}.
\]

(ii)

\[
\left| e^{-nu} \int_{0}^{u} e^{nu} R_n(u') \, du' \right| \leq e^{-nu} \int_{0}^{u} e^{nu} \frac{M(u')}{n^k} \, du' \\
\leq \frac{e^{-nu}}{n^k} \max_{u' \in [0, u]} \{M(u')\} \int_{0}^{u} e^{nu} \, du' \\
\leq \frac{K(u)}{n^{k+1}}.
\]

The function \( K(u) \) that appears in the bounds (2.16), (2.17) is a positive, increasing and continuous function for \( u \in [0, \infty) \). There are two possibilities (which are related to the growth of \( \rho(r, \theta) \) as \( r \to \infty \)). As \( u \to \infty \) either \( K(u) \) is bounded above or \( K(u) \to +\infty \).

Theorem 1 Suppose that \( \rho(t) \in C^k(\mathcal{R}) \) with \( k \geq 2 \), and that \( K(u) \) is bounded above for \( u \in [0, \infty) \). Then \( A_n, B_n \) can be chosen in (2.15) so that (2.8) generates a \( C^k \)-function that satisfies Poisson's equation (2.1) in \( r > a \).

Proof. Let

\[
K(\infty) = \sup_{u \in [0, \infty)} K(u) = \sup_{u \in [0, \infty)} M(u).
\]

Then

\[
\int_{0}^{U} |e^{-nu} R_n(u)| \, du \leq \frac{K(\infty)}{n^{k+1}} (1 - e^{-nu}) < \frac{K(\infty)}{n^{k+1}}
\]

\((n \geq 1)\), so that all the infinite integrals

\[
\int_{0}^{\infty} e^{-nu} R_n(u) \, du
\]

exist for \( n \geq 1 \). In (2.15) take \( B_n = 0 \) and

\[
A_n = -\frac{1}{2n} \int_{0}^{\infty} e^{-nu} R_n(u) \, du.
\]
Then $\Phi_n(u)$ becomes

$$\Phi_n(u) = -\frac{e^{nu}}{2n} \int_u^\infty e^{-nu} R_n(u') \, du' - \frac{e^{-nu}}{2n} \int_0^u e^{nu} R_n(u') \, du'.$$ (2.21)

The integrals in (2.21) can be estimated by the preceding lemma to give

$$|\Phi_n(u)| \leq \frac{1}{2n} \left( \frac{K(\infty)}{n^{k+1}} \right) + \frac{1}{2n} \left( \frac{K(u)}{n^{k+1}} \right) \leq \frac{K(\infty)}{n^{k+2}}.$$ (2.22)

Thus from (2.12)

$$|\varphi_n(r) \cos n\theta| \leq \frac{K(\infty)}{n^{k+2}}$$ (2.23)

and so the series in (2.8) is uniformly convergent in $r \geq a$ by the $M$-test. Also the partial derivatives of $\varphi(r, \theta)$ of up to the $k$th order may be obtained by term-by-term differentiation, all of these series being uniformly convergent in $r \geq a$. Thus, if $k \geq 2$, the Laplacian can be applied term by term to give

$$\nabla^2 \varphi = \sum_{n=1}^\infty \nabla^2 [\varphi_n(r) \cos n\theta]$$

$$= \sum_{n=1}^\infty \rho_n(r) \cos n\theta \quad \text{(from (2.9))}$$

$$= \rho \quad \text{(from (2.6))},$$ (2.24)

which is the required result.

Note. The condition that $K(u)$ be bounded above is equivalent to the condition that

$$\frac{\partial^k \rho}{\partial \theta^k} (r, \theta) = O\left(\frac{1}{r^2}\right)$$ (2.25)

as $r \to \infty$, uniformly in $\theta$. This is similar to the restrictive conditions (as $r \to \infty$) required by conventional proofs (for example, (4)). The more interesting and difficult case, however, occurs when $K(u) \to \infty$ as $n \to \infty$; this case allows arbitrary growth of $\rho(r, \theta)$ as $r \to \infty$. The proof given in Theorem 1 will now fail since the infinite integrals in (2.21) need not exist for any value of $n$; this would occur for instance if $\rho(r, \theta)$ were such that $M(u) = e^u$.

**Theorem 2** Suppose that $\rho(r) \in C^k(\mathbb{R})$ with $k \geq 2$, and that $K(u) \to \infty$ as $u \to \infty$. Then $A_n$, $B_n$ can be chosen in (2.15) so that (2.8) generates a $C^k$-function that satisfies Poisson's equation (2.1) in $r \geq a$.

**Proof.** Since $K(u)$ is continuous and approaches $\infty$ as $u \to \infty$ there exists a
sequence \( \{L_n\} \) (which may not be unique since \( K(u) \) need not be strictly increasing) such that

\[
K(L_n) = \ln n
\]  
(2.26)

for \( n \geq n_0 \) (Fig. 1); then \( L_n \to \infty \) as \( n \to \infty \). In (2.15) take \( B_n = 0 \) for all \( n \) and

\[
A_n = -\frac{1}{2n} \int_0^{L_n} e^{-nu} R_n(u) \, du
\]  
(2.27)

for \( n \geq n_0 \) (the values of the \( A_n \) for \( n < n_0 \) can be arbitrarily assigned). Then \( \Phi_n(u) \) becomes

\[
\Phi_n(u) = -\frac{e^{nu}}{2n} \int_u^{L_n} e^{-nu'} R_n(u') \, du' - \frac{e^{-nu}}{2n} \int_0^u e^{nu'} R_n(u') \, du'
\]  
(2.28)

for \( n \geq n_0 \). Let \( U > 0 \) be chosen arbitrarily, but then fixed, and let \( u \in [0, U] \). Since \( L_n \to \infty \) as \( n \to \infty \), it follows that \( u < L_n \) for \( u \in [0, U] \) and \( n \geq N(U) \). Then for \( n \geq N(U) \), the lemma can be applied to the integrals in (2.28) to give

\[
|\Phi_n(u)| \leq \frac{1}{2n} \left( \frac{K(L_n)}{n^{k+1}} \right) + \frac{1}{2n} \left( \frac{K(u)}{n^{k+1}} \right)
\]

\[
\leq \frac{K(L_n)}{n^{k+2}} \quad \text{(since } K(u)/n) \]

\[
= \frac{\ln n}{n^{k+2}} \quad \text{(from (2.26))}
\]

for \( u \in [0, U] \) and \( n \geq N(U) \). It follows that the series

\[
\sum_{n=1}^{\infty} \phi_n(r) \cos n\theta
\]  
(2.29)

converges uniformly in the annulus \( a \leq r \leq b \) for any fixed finite \( b (>a) \). The remainder of the proof is the same as in Theorem 1 except that now all the series converge uniformly in \( a \leq r \leq b \) instead of \( r \geq a \). Since \( b \) can be arbitrarily chosen, this completes the proof.

The three-dimensional case

The method of proof in the three-dimensional case resembles that in two dimensions, the principal difference being that the Fourier expansion (2.3) is replaced by an expansion in surface harmonics of the form

\[
\rho(r, \theta, \psi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \rho_{nm}(r) P_n^m(\cos \theta) \cos m\psi,
\]  
(2.30)

where \( \rho(r, \theta, \psi) \) is an even function of \( \psi \), and \( r, \theta, \psi \) are spherical polar
coordinates; \( P_n^m(z) \) is an associated Legendre function (10, Chapter 5). An appropriate completeness theorem for such expansions is given by Sternberg and Smith (11 Chapter 9), and the necessary convergence results by Gregory (12). If \( \phi(r, \theta, \psi) \) is similarly expanded, then the coefficients \( \phi_{nm}(r) \) must satisfy
\[
(r^2 \phi'_{nm})' - n(n + 1) \phi_{nm} = r^2 \rho_{nm}
\]
(2.31)
for \( r \gg a \).

If we now make the substitutions
\[
r = ae^u,
\]
(2.32)
then \( \Phi_{nm}(u) \) satisfies
\[
\Phi''_m - (n + \frac{1}{2})^2 \Phi_{nm} = R_{nm},
\]
(2.34)
where
\[
R_{nm} = (ae^u)^3 \rho_{nm}(ae^u).
\]
(2.35)
Equation (2.34) is now treated in the same way as (2.11). This leads to the following theorem for the three-dimensional case.

**Theorem 3** Suppose that \( \rho(r) \in C^k(\mathcal{R}) \) with \( k \geq 2 \), where \( \mathcal{R} \) is the region \( r = |r| \geq a > 0 \) of \( \mathbb{R}^3 \). Then there exists a function \( \phi(x) \in C^k(\mathcal{R}) \) that satisfies
\[
\nabla^2 \phi = \rho \quad (r \in \mathcal{R}).
\]
(2.36)
(Such a $\phi$ exists no matter what the rate of growth of $p(r)$ as $r \to \infty$.)

**Corollary** The solution $\phi(r)$ that is established by Theorem 3 as being in $C^k(\mathcal{R})$ is also $C^{k+1}$ in the domain $r > a$.

**Proof.** If $\phi_1, \phi_2$ are any $C^2$-solutions of (2.36) in some domain $\mathcal{D}$, their difference is harmonic in $\mathcal{D}$ and hence is $C^\infty(\mathcal{D})$. Thus $\phi_1, \phi_2$ must be 'equally differentiable' in $\mathcal{D}$. Since differentiability is a local property, it suffices to compare the solution generated in Theorem 3 with local solutions generated by the Newtonian potential formula (see (5, Lemma 3.1)). This formula generates solutions that are $C^{k+1}(\mathcal{D})$ when $p \in C^k(\mathcal{D})$. Hence the solution in Theorem 3 must also be $C^{k+1}$ in $r > a$.

3. Poisson's equation in the interior region $r \leq a$ when $p(r)$ has a singularity at $r = 0$

If $u(r)$ has a (sufficiently severe) singular point, then the Newtonian potential formula (1.4) will fail just as it may fail when the domain is infinite. This case seems not to have been discussed before, but since singular points are an important idealization, the question as to whether Helmholtz's theorem remains true in general is of interest.

**Theorem 4** Suppose that $p(r) \in C^k(\mathcal{R})$ with $k \geq 2$, where $\mathcal{R}$ is the region $0 < r = |r| \leq a$ of $\mathbb{R}^3$. Then there exists a function $\phi(r) \in C^k(\mathcal{R})$ that satisfies

$$\nabla^2 \phi = p \quad (r \in \mathcal{R}). \quad (3.1)$$

Moreover $\phi \in C^{k+1}$ in $0 < r < a$.

(Such a $\phi$ exists no matter what the rate of growth of $p(r)$ as $r \to 0$.)

The proof is virtually identical to that advanced in Theorems 1, 2, 3 (and the corollary). The substitution (2.32) is merely replaced by $r = ae^{-n}$.

4. Helmholtz's theorem when the domain is infinite and when the field has singular points

The main theorem of this paper is as follows.

**Theorem 5** (Helmholtz's theorem) Suppose that $u \in C^k(\mathcal{D}) \cap C(\mathcal{D})$ with $k \geq 2$, where $\mathcal{D} (\subset \mathbb{R}^3)$ is the union of a bounded domain $\mathcal{D}_1$ and an infinite exterior domain $\mathcal{D}_2$ as follows.

(i) $\mathcal{D}_1$ is any regular region of space that is interior to the sphere $|r| = b$ and contains the spherical annulus $b \geq |r| \geq a$ for some $b > a > 0$. (The conventional proof of Helmholtz's theorem holds for $u(r), r \in \mathcal{D}_1$.)

(ii) $\mathcal{D}_2$ is the exterior domain $|r| > a$.
Then there exist potentials $\chi, A \in C^k(\mathcal{D})$ such that

\begin{align}
\mathbf{u} &= \text{grad} \, \chi + \text{curl} \, A, \\
\text{div} \, A &= 0,
\end{align}

for $r \in \mathcal{D}$.

This result still holds good if $\mathbf{u}$ has a finite number of singular points lying in $\mathcal{D}_1$. (No restriction is placed on the growth of $\mathbf{u}$ at infinity or as the singular points are approached.)

**Proof.** Let $\rho(r)$ be any Cartesian component of $\mathbf{u}(r)$. Then, from (i), there exists $\phi_1(r) \in C^{k+1}(\mathcal{D}_1)$ such that

\[ \nabla^2 \phi_1 = \rho \quad (r \in \mathcal{D}_1). \tag{4.3} \]

Also, the conditions of Theorem 3 are satisfied in $\mathcal{D}_2$ so there exists $\phi_2(r) \in C^{k+1}(\mathcal{D}_2)$ such that

\[ \nabla^2 \phi_2 = \rho \quad (r \in \mathcal{D}_2). \tag{4.4} \]

In the overlap domain $a < |r| < b$ we therefore have

\[ \nabla^2 (\phi_1 - \phi_2) = 0, \tag{4.5} \]

that is, $\phi_1 - \phi_2$ is harmonic. Thus $\phi_1 - \phi_2$, like any function harmonic in the annulus $a < |r| < b$, may be expressed in the form

\[ \phi_1 - \phi_2 = H_1 + H_2, \tag{4.6} \]

where $H_1$ is harmonic in $|r| < b$ and $H_2$ is harmonic in $|r| > a$. It follows that the function $\phi(r) \in C^{k+1}(\mathcal{D})$ defined by

\[ \phi(r) = \begin{cases} 
\phi_1 - H_1 & (r \in \mathcal{D}_1), \\
\phi_2 + H_2 & (r \in \mathcal{D}_2)
\end{cases} \tag{4.7} \]

satisfies

\[ \nabla^2 \phi = \rho \quad (r \in \mathcal{D}). \tag{4.8} \]

Since $\rho$ was any Cartesian component of $\mathbf{u}$ it follows that there exists a vector field $\mathbf{v} \in C^{k+1}(\mathcal{D})$ such that

\[ \nabla^2 \mathbf{v} = \mathbf{u} \quad (r \in \mathcal{D}). \tag{4.9} \]

Helmholtz's theorem now follows with $\chi = \text{div} \, \mathbf{v}$, $A = -\text{curl} \, \mathbf{v}$.

Suppose now that $\mathbf{u}(r)$ has a singular point at $r = a \in \mathcal{D}_1$. Then the above proof holds with $\mathcal{D}_1$ modified by removal of a domain $|r-a| < \delta_1$ for any sufficiently small $\delta_1$. Also the conditions of Theorem 4 are satisfied in some region $0 < |r-a| \leq \delta_2$ with $0 < \delta_1 < \delta_2$. The existence of a solution of Poisson's equation (4.8) for $r \in \mathcal{D}, r \neq a$ then follows by the same argument as that given above. Since we have supposed only a finite number of such
singular points, each one can be treated in this manner and this completes
the proof.

REFERENCES