K_{13} Decays and (1, 8) + (8, 1) Symmetry Breaking

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Effects of (1, 8) + (8, 1) symmetry breaking on $K_{13}$ form factors are studied in the frame of soft-pion approach and in terms of the generalized $\sigma$ models. It is shown that the soft-pion theorem for $K_{13}$ decays given by Weinstein in the presence of $(3, 3^*) + (3^*, 3)$ breaking remains unchanged when the $(1, 8) + (8, 1)$ symmetry breaking is added. It is further shown that, irrespective of whether massive gauge fields are introduced or not, the generalized $\sigma$ models give too smooth off-shell extrapolation of the amplitudes to reproduce the present experimental values of the parameters $\xi(0)$ and $\lambda_+$. As a result, it seems that either the tree approximation (the effective Lagrangian approach) or the approximate chiral $SU(2) \times SU(2)$ symmetry must be abandoned. However if a larger value for $\lambda_+$ is given by future experiments, the large negative $\xi(0)$ will naturally be understood in this scheme.

§ 1. Introduction

One of the most fruitful applications of the current algebra and PCAC hypotheses has been the derivation of low energy theorems for soft-pion processes. However, recent experiments have begun to indicate a tendency to disagree with one of the theorems, that is, with the Callan-Treiman (C-T) relation between $K_{13}$ form factors. Two important problems, though not distinguished clearly, have been investigated for $K_{13}$ decays: the breaking scheme of chiral $SU(3)$ x $SU(3)$ symmetry and off-shell behavior of various amplitudes.

In a recent paper Weinstein has presented a simple formula in $K_{13}$ decays. He has then shown that, without large $SU(3)$ violation of a certain matrix element, the experimental value $\xi \approx -1$ cannot be explained by $(3, 3^*) + (3^*, 3)$ symmetry breaking alone. He has argued also that the large $SU(3)$ violation of the matrix element would suggest large $SU(3)$ breaking of the vacuum and it is hard to understand why the large $SU(3)$-violation does not give effects in other places. Can $(1, 8) + (8, 1)$ breaking terms account for $\xi \approx -1$? Schilcher has made an affirmative conclusion, but it has been based on the unacceptable assumption that the quantity

$$\frac{1}{2} ( \sqrt{2} \epsilon_0 + \epsilon_6 ) ( \sqrt{2} \langle u_0 \rangle + \langle u_6 \rangle ),$$

where usual notations for $\epsilon_i$ and $u_i$ are adopted, is the known one and remains finite even at $\epsilon_0 = - \sqrt{2} \epsilon_6$. On the other hand Barker has derived, on the basis of a simple Lagrangian with $(1, 8) + (8, 1)$ symmetry breaking terms, the negative conclusion that one must require unphysically low masses for both the $\eta'$ and $\kappa$. 

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(scalar strange meson). However, it is not clear to what extent the results obtained are model-dependent, since the model adopted by him is too simplified.

Our aim in this paper is to study first, in § 2, model-independent effects of $(1, 8) + (8, 1)$ symmetry breaking in the soft-pion limit following Weinstein's discussion. We shall find that the conclusion obtained by Weinstein in the case of $(3, 3^*) + (3^*, 3)$ symmetry breaking alone remains unchanged. Next as the physical region of $K_{13}$ decays is far from the soft-pion point, we shall investigate the off-shell behavior of the relevant amplitude on the basis of the generalized $\sigma$ model which involves the general form of interactions between spin zero mesons. In § 3, Barker's discussion will be generalized. It will be shown that the $\kappa$ mass must be lower than the kaon mass, which is in agreement with one of his results, but nothing can be said about the $\eta'$ mass. Based on the more realistic model which includes $1^\pm$ mesons as massive gauge fields, we shall deal with $K_{13}$ decays in § 4 and prove that it is very difficult to reproduce the present experimental $\lambda_\pm$ and $\xi(0)$ values. As a result, much higher-order derivative interactions in the effective Lagrangian or other symmetry breaking than $(3, 3^*) + (3^*, 3)$ and $(1, 8) + (8, 1)$ must be required. The latter means the departure from approximate chiral $SU(2) \times SU(2)$ and the former indicates, in a sense, the incorrectness of the tree approximation and, in other words, the breakdown of the effective Lagrangian approach. Section 5 is devoted to final discussions.

§ 2. $(1, 8) + (8, 1)$ symmetry breaking in the soft-pion limit

To begin with, let us define $K_{13}$ form factors

$$\sqrt{2} \langle \pi^0(q) | V_{e,13}^\ell(0) | K^+(k) \rangle = (k + q)^s f_+(t) + (k - q)^s f_-(t)$$

and

$$i \sqrt{2} \langle \pi^0(q) | \partial_\mu V_{e,13}^\ell(0) | K^+(k) \rangle = (M^2 - \mu^2) f(t),$$

where $M(\mu)$ is the kaon (pion) mass, $t = (k - q)^2$ and $V_{e,13}^\ell$ is a strangeness-changing vector current of hadrons. We shall parametrize these form factors as follows:

$$f_\pm(t) = f_\pm(0) \left(1 + \lambda_\pm \frac{t}{\mu^2}\right)$$

and

$$f(t) = f(0) \left(1 + \lambda_\pm \frac{t}{\mu^2}\right)$$

and express, as usual,

$$\xi(t) = \frac{f_-(t)}{f_+(t)} = \frac{(M^2 - \mu^2)(\lambda_0 - \lambda_\pm)}{\mu^2 + \lambda_\pm t}.$$
in the soft-pion limit (at \( q = 0 \)), where \( A^a_s(x) \) is an axial-vector current, and has derived the following basic theorem:

\[
f_+(M^2) \left[ \left(1 - \frac{\mu^2}{M^2} \right) + \xi(M^2) \right] = \frac{2}{f^*_\pi} \langle 0 | [Q^*_a, \partial_\mu V^*_\mu(0)] | K^+ \rangle + \partial(0, M^2),
\]

where

\[
Q^*_a = \int_{x^2 = 0} d^4x A^a(0, x),
\]

and

\[
\partial(q^2, t) = \left[ \int d^4x e^{iqx} \langle 0 | T(\partial_\mu A^a_s(x) \partial_\mu V^*_\mu(0)) | K^+ \rangle - \frac{i q^2}{q^2 - \mu^2} \langle \pi^\rho | \partial_\mu V^*_\mu(0) | K^+ \rangle \right].
\]

Using the \((3, 3^*) + (3^*, 3)\) symmetry-breaking Hamiltonian

\[
\mathcal{H}_{\text{S.B.}} = \epsilon_3 u_0 + \epsilon_8 u_8,
\]

where \( u_a, v_b (a, b = 0, \cdots, 8) \) define a \((3, 3^*) + (3^*, 3)\) representation of \( SU(3) \times SU(3) \), and neglecting the \( \partial(0, M^2) \) term, he has further shown that the value \( \xi = -1 \) must require \( \langle 0 | v_{4+18} | K^+ \rangle / \langle 0 | v_8 | \pi^0 \rangle = 12 \). This implies an enormous violation of \( SU(3) \).

We shall show in the following that the same conclusion will hold even if \((1, 8) + (8, 1)\) symmetry breaking exists. The symmetry-breaking Hamiltonian is expressed in the form

\[
\mathcal{H}_{\text{S.B.}} = \epsilon_3 u_0 + \epsilon_8 u_8 + x_8 g_8,
\]

where the additional term \( g_8 \) denotes a scalar density of the \((1, 8) + (8, 1)\) representation. Current divergences can be evaluated easily from Eq. (11); then we get

\[
\partial_\mu V^*_\mu(0) = -\frac{\sqrt{3i}}{2} (\epsilon_8 u_{4+18} + x_8 g_{4+18}),
\]

\[
\partial_\mu A^*_\mu(0) = \left( \sqrt{2 \over 3} \epsilon_0 - \frac{1}{2 \sqrt{3}} \epsilon_8 \right) v_{4+18} - \frac{\sqrt{3i}}{2} x_8 h_{4+18}
\]

and

\[
\partial_\mu A^*_\mu(0) = \left( \sqrt{2 \over 3} \epsilon_0 + \frac{1}{2 \sqrt{3}} \epsilon_8 \right) v_8,
\]

where \( h_t \) represents a parity partner of \( g_t \). With these equations, we get

\[
f_+(M^2) = \langle 0 | \partial_\mu A^*_\mu(0) | K^+ \rangle = \left( \frac{\sqrt{2}}{3} \epsilon_0 - \frac{1}{2 \sqrt{3}} \epsilon_8 \right) \langle 0 | v_{4+18} | K^+ \rangle - \frac{\sqrt{3i}}{2} x_8 \langle 0 | h_{4+18} | K^+ \rangle,
\]
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\[ \frac{1}{\sqrt{2}} f_\pi \mu^2 \langle 0 | \partial_\mu A_\pi^+ | \pi^0 \rangle = \left( \sqrt{\frac{2}{3}} \epsilon_8 + \sqrt{\frac{1}{3}} \epsilon_8 \right) \langle 0 | v_4 | \pi^0 \rangle \]

and

\[ \langle 0 | [Q_8, \partial_\mu V_{\pi+0}(0)] | K^+ \rangle = -\frac{\sqrt{3}}{4} [\epsilon_8 \langle 0 | v_{\pi+0} | K^+ \rangle + i x_8 \langle 0 | h_{\pi+0} | K^+ \rangle]. \]

Substituting these equations into Eq. (7), we obtain

\[ \xi(M^2) = \frac{1}{f_\pi(M^2)} \left( \frac{f_\pi}{f_K} \right) \left( 1 - \frac{f_\pi}{f_K} \frac{\mu^2}{M^2} \sqrt{2} \langle 0 | v_4 | \pi^0 \rangle \right) \left( 1 - \frac{\mu^2}{M^2} \right), \]

which is in exact accordance with the expression obtained by Weinstein. Thus it is concluded that in the soft-pion limit even the $(1, 8) + (8, 1)$ symmetry breaking may not account for the large negative value for $\xi$ unless we require again the large SU(3) violation, that is, $\langle 0 | v_{\pi+0} | K^+ \rangle / \langle 0 | v_4 | \pi^0 \rangle \approx 12$. At this stage we are left with two other possibilities. Another symmetry-breaking term may exist which transforms as, say, an $(8, 8)$ representation (see Appendix A) or the results may depend critically on the off-shell extrapolation from the soft-pion limit to the physical region. We shall deal with the latter problem in the following.

§ 3. Generalized $\sigma$ model (I)

Following our previous work, let us start with the Lagrangian

\[ \mathcal{L} = \frac{1}{4} \text{Tr} \partial_\mu B^+ \partial^\nu B^- + f(I, J, K, L), \]

where $I, J, K, L$ are independent invariants of chiral $SU(2) \times SU(2) \times U(1)$ and are composed of scalar ($S$) and pseudoscalar ($P$) meson fields which are members of a $(3, 3^*) + (3^*, 3)$ representation of $SU(3) \times SU(3)$. (For the explicit form of them, see Appendix B.) Here $B^\pm (\equiv S \pm i P)$ are $3 \times 3$ matrices. In Eq. (19) we have assumed exact $SU(2) \times SU(2)$ in order to avoid unessential effects of nonzero pion mass. It is to be noted that this Lagrangian corresponds to a generalized form of Barker’s model, since $f(I, J, K, L)$ is an arbitrary function of $I, J, K, L$. Also the Lagrangian says nothing about spin-one mesons which contribute to $f_\pi(t)$, so that it is impossible to have direct information about $\xi(t)$. Nevertheless it should be possible to say about the scalar form factor $f(t)$, which is connected with $\xi(t)$ by the relation (5), under the assumption that the symmetry-breaking parts are functions of spin zero fields only. We shall introduce the spontaneous breakdown of the symmetry in the usual manner that the diagonal elements $S_i^i$ of the scalar matrix $S$ have non-zero vacuum expectation values. The relevant weak currents are derived from the Lagrangian (19) and therefore

\[ \langle \pi^0(q) | V_{\pi+0}(0) | K^+(k) \rangle = i \beta(1 - r) \langle \pi^0(q) | \partial_\mu \pi^+ | K^+(k) \rangle 
- \frac{i}{\sqrt{2}} \langle \pi^0(q) | (\partial_\mu \pi^+) K^+ - \pi^-(\partial_\mu K^+) | K^+(k) \rangle, \]

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\[ \langle 0 | A(t) | K^+(k) \rangle = i k^2 \frac{1+r}{2} \beta \sqrt{2} \] (21)

and
\[ \langle 0 | A_s(t) | \pi^0(k) \rangle = i k^2 \frac{1}{2} f_s = i k \beta, \] (22)

where \( \beta \) denotes the vacuum expectation value of \( S_1^1, S_2^1 \) and \( \beta r \) is that of \( S_3^1 \). Deviation of \( r \) from unity indicates an \( SU(3) \) violation of the vacuum. When we assume the tree approximation, we obtain
\[ \langle \pi^0(q) | V_{t+s}(0) | K^+(k) \rangle = \frac{1}{\sqrt{2}} \left[ (k+q)^2 + \sqrt{2} \beta (1-r) \frac{g_{K^+\pi^0}(k-q)}{M^2-t} \right], \] (23)

where \( g_{K^+\pi^0} \) is the coupling constant of the decay \( K^+ \to K^0 \pi^0 \) and \( M_\pi \) is the mass of the \( \pi \) meson. Here it should be remembered that, as we have shown in Ref. 6), \( g_{K^+\pi^0} \) is not an arbitrary parameter but is related to other quantities as follows:
\[ g_{K^+\pi^0} = \frac{M^2 - M_\pi^2}{2\sqrt{2} \beta}. \] (24)

Consequently the form factors are expressed as
\[ f_+(t) = 1 \] (25)

and
\[ f_-(t) = \sqrt{2} \beta (1-r) \frac{g_{K^+\pi^0}}{M^2-t} = \frac{r-1}{2} \frac{M_\pi^2 - M^2}{M^2-t}. \] (26)

Thus the scalar form factor is
\[ f(t) = 1 + \frac{r-1}{2} \frac{M_\pi^2 - M^2}{M^2} \] (27)

and the slope \( \lambda_0 \) is given by
\[ \lambda_0 = \frac{r-1}{2} \frac{M^2}{M_\pi^2} \left( 1 - \frac{M^2}{M_\pi^2} \right). \] (28)

From these equations, we get
\[ \xi(0) = \frac{r-1}{2} \left( 1 - \frac{M^2}{M_\pi^2} \right) + \frac{\lambda_0 M_\pi^2}{\mu^2}, \] (29)

where the second term has been added, though, exactly speaking, \( \lambda_0 \) = 0 within this model. It is not surprising that \( f(t) \) in Eq. (27) satisfies the Callan-Treiman relation
\[ f(M^2) = \frac{1+r}{2} = \frac{f_K}{f_s}. \] (30)
The experimental value\(^\gamma\)

\[
\frac{f_K}{f_+ f_+(0)} = 1.28
\]  

(31)

Fig. 1. \(f(t)\) of Eq. (27) are shown for the two cases \(M_\pi > M\) and \(M_\pi < M\), where \(r\) is fixed at 1.56.

and Eq. (25) leads \(r\) fixed at 1.56. In Fig. 1, \(f(t)\) is plotted for various values of \(M_\pi\) and \(\xi(0)\) is shown as a function of \(\lambda_+\) in Fig. 2. It is clear from these figures and Eq. (28) that \(M_\pi\) must be smaller than the kaon mass \(M\) in order to reproduce the negative value for \(\lambda_0\). For example, the recent experimental data \(\xi(0) \sim -0.62\) and \(\lambda_+ \sim 0.024\)\(^\text{1)}\) require \(\lambda_0 \sim -0.026\) which leads us to \(M_\pi = 0.68M\). It is worth noting that this conclusion does not depend on the detailed structure of the interaction term \(f(I, J, K, L)\) and hence, in spite of the more model-dependent approach, it has been obtained also by Barker. But the other result of him that the \(\eta'\) mass also cannot be reproduced is not obtained, since it depends critically on the structure of \(f(I, J, K, L)\).

\[\text{§ 4. Generalized }\sigma\text{ model (II) with massive gauge fields}\]

In this section we shall deal with a more realistic model in which derivative interactions and spin-one mesons are considered, since in fact \(f_+(t)\) has \(t\)-dependence. The most promising way may be to assume spin-one mesons as massive gauge fields. In reality, such a view has been generally successful in low energy physics,\(^\text{9)}\) so that it would be natural to expect that the characteristic features of \(K_{13}\) decays can also be explained. It will, however, be shown here that, contrary to the expectation, this scheme hardly describes the experimental data on \(\xi(0)\) and \(\lambda_+\).

Let us begin with the following Lagrangian in which vector \((v_\mu)\) and axial-vector \((a_\mu)\) mesons couple minimally:
\[ \mathcal{L} = -\frac{1}{4} \text{Tr}(F_{\mu\nu}A_\mu F^{\mu\nu} + G_{\mu\nu}A_\mu G^{\mu\nu}) + \frac{1}{2} m_0^2 \text{Tr}(v_\mu A^\mu v^\mu + \bar{d}_\mu A_\mu d^\mu)
+ \frac{1}{4} \text{Tr}(\Delta^+ B^+ A^0 B^- + \Delta^+ B^- A^0 B^+ ) + f(I, J, K, L) + \delta \sigma, \quad (32) \]

where the same notations as in Ref. 6) have been used and the last term gives non-zero pion mass. It should be emphasized that much higher order derivatives should not be considered from the viewpoint of the effective Lagrangian, since need of such derivatives means, in a sense, the breakdown of the tree approximation. The matrices \( A_i \) defined by

\[ A_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a_i \end{pmatrix}, \quad i = 1, 2, 3 \quad (33) \]

introduce possible \((1, 8) + (8, 1)\) breaking in the first three terms in Eq. (32) which gives the momentum dependence of \( f(t) \). It is assumed that the real world is described by the effective Lagrangian derived from the original (32) by the following procedure: (1) introduction of the spontaneous breakdown, (2) diagonalization of spin-one and spin-zero fields and (3) renormalization of fields. After easy calculations, one gets the relevant weak currents

\[ V_{\mu+\alpha} = \frac{(1 + a_3) \sqrt{2} m_0}{2 \tau_0} K_{\mu \rho} + \frac{1}{2} \sqrt{2} m_0 \sqrt{z_\tau} \hat{K}_{\mu \rho} + i \beta (1 - r) \partial_\mu \hat{K}^+, \quad (34) \]

\[ A_{\mu+\alpha} = \frac{(1 + a_3) \sqrt{2} m_0}{2 \tau_0} A_{\mu \rho} + \frac{1}{2} \sqrt{2} m_0 \sqrt{z_\tau} \hat{A}_{\mu \rho} + \frac{1}{2} \beta (1 - r) \partial_\mu \hat{A}^+, \quad (35) \]

\[ A_{\mu+\alpha} = \sqrt{2} m_0 \sqrt{z_\tau} \hat{A}_{\mu \rho}^+ = \frac{2 \sqrt{2} m_0 \sqrt{z_\tau}}{2 \tau_0} \hat{A}_{\mu \rho}^+ + \frac{2 \beta}{\sqrt{z_\tau}} \partial_\mu \hat{A}^+, \quad (36) \]

where \( \hat{K}_{\mu \rho}^+ \), etc., denote diagonalized and renormalized fields and \( f = \sqrt{2} \beta \tau_0 / m_0 \). The renormalization constants \( z_i \) are defined by, for example,

\[ \pi^+ = \sqrt{z_\pi \hat{\pi}^+} \quad (37) \]

and are explicitly written as

\[ z_\pi = 1 + f^2, \]

\[ z_\tau = \frac{2}{1 + a_3} + \frac{2}{1 + a_3} \frac{(1 + r)}{2} f^2, \]

\[ z_\rho = \frac{2}{1 + a_3} + \frac{2}{1 + a_3} \frac{(1 - r)}{2} f^2, \quad (38) \]

and

\[ z_{K^+} = z_{K^-} = \frac{2}{1 + a_1}. \]
To evaluate the matrix element of the current (34) between the kaon and the pion, we shall make use of the equation of motion for the $K^*$:

$$\partial_{\nu}(\partial^\nu K^* - \partial^\nu \bar{K}^* - m_{K^*} K^* + j_{\nu}) = \frac{i}{2} \gamma_{\nu} K^* -$$

(39)

where $j_{\nu}$ is a source current and $m_{K^*}$ is the $K^*$ mass. The relevant interaction terms are, for the $K^*-K^+\pi^0$ coupling,

$$\mathcal{L}_{K^*-K^+\pi^0} = \frac{\partial_{\nu} \partial^\nu K^*}{2} - \frac{1}{z_{K^*}} (\partial^\nu K^*)^2 + \frac{(1 + a_s) (1 - r) m_0}{4} f \left[ (\alpha' \partial^\nu K^*)^2 + \alpha (\partial^\nu \pi^0 K^*)^2 \right]$$

(40)

and, for the $\kappa^-K^+\pi^0$ coupling,

$$\mathcal{L}_{\kappa^-K^+\pi^0} = \frac{\partial_{\nu} \partial^\nu K^*}{2} - \frac{1}{z_{K^*}} (\partial^\nu K^*)^2 + \frac{(1 + a_s) (1 - r) m_0}{4} f \left[ (\alpha' \partial^\nu K^*)^2 + \alpha (\partial^\nu \pi^0 K^*)^2 \right] + \frac{1}{2} \frac{M^2}{z_{K^*} - z_{\pi}} \kappa^-K^+\pi^0,$$

(41)

where $\alpha$, $\alpha'$, $\epsilon'$ are defined by

$$\alpha = \frac{f}{m_0 (1 + f^2)}$$

$$\alpha' = \frac{(1 + r) f}{2 m_0 [1 + ((1 + r)^2 / 4)]}$$

(42)

and

$$\epsilon' = \frac{(1 - r) f}{2 m_0 [1 + ((1 - r)^2 / 4)]}.$$

After lengthy calculations, we get

$$f_+ (t) = \frac{(1 + a_s) m_0^2 z_{K^*}}{4 \sqrt{z_{K^*} z_{K}}} \frac{(1 + a_s) (1 + r) f^2}{2 (1 + a_s) m_0^2} t - (z_K + z_{K^*})$$

$$- \frac{1}{4} (1 + a_s) (1 + r) f^2 \left( \frac{1 + r}{1 + a_s} \right) - \frac{z_K + z_{K^*}}{t - m_{K^*}^2}$$

(43)

and

$$f_- (t) = - \frac{(1 + a_s) m_0^2 z_{K^*}}{4 \sqrt{z_{K^*} z_{K}}} \frac{(1 + a_s) (1 + r) f^2}{2 (1 + a_s) m_0^2} (\mu^2 - M^2) - (z_K - z_{K^*})$$

$$+ \frac{1}{4} (1 + a_s) (1 - r) f^2 \left( \frac{1 + r}{1 + a_s} \right) - \frac{z_K - z_{K^*}}{m_{K^*}^2}$$

$$+ \frac{f^2 (1 - r)}{z_{K^*} z_K z_{K^*} (1 + a_s)} \left[ \frac{\mu^2 r}{2} \left[ -1 - \frac{2}{1 + a_s} + \frac{(1 - r) f^2}{2 (1 + a_s)} \right] \right]$$
It is easy to show that these form factors, if the pion mass is neglected, satisfy the Callan-Treiman relation

$$f_+(M^2) + f_-(M^2) = \frac{1}{J_z} \frac{1 + r}{2} = \frac{F_K}{f_*}.$$  \hspace{1cm} (45)

It is now convenient to rewrite Eqs. (43) and (44) by the known quantities

$$f_K = F, \quad m_*^2 = m_0^2, \quad m_{K*}^2 = z_* m_0^2,$$

$$m_{K*}^2 = z_K z_* \frac{(1 + a_2)(1 + a_3)}{4} m_0^2$$  \hspace{1cm} (46)

and

$$m_{K*}^2 = z_K z_* \frac{(1 + a_2)(1 + a_3)}{4} m_0^2.$$  

With these quantities and two observed relations

$$2m_*^2 \approx m_{K*}^2 \quad \text{and} \quad 2m_*^2 \approx m_{K*}^2,$$

the form factors are simply represented in terms of the parameters $F, f_+(0), r$ and meson masses:

$$f_+ (t) = f_+(0) \frac{m_{K*}^2 - (F/4)t}{m_{K*}^2 - t}$$  \hspace{1cm} (48)

and

$$f_- (t) = \frac{1}{4} F f_+(0) \left[ (M^2 - \mu^2) + 4m_{K*}^2 \left( \frac{3 - r}{1 + r} - \frac{1}{F} \right) \right] \frac{1}{m_{K*}^2}$$

$$+ \frac{1}{2} \frac{r - 1}{r + 1} F f_+(0) \left[ \frac{\mu^2}{2} \left( 1 + \frac{2(3r - 1)}{r^2 - 6r + 1} \left( 1 - \frac{1 + r}{F} \right) \right) \right]$$

$$+ \frac{M^2}{2} \left[ 1 + \frac{2(3-r)}{r^2 - 6r + 1} \left( \frac{1 + r}{F} \right) \right]$$

$$- \frac{t}{2} \left[ 5 + \frac{2(3-r)}{r^2 - 6r + 1} \left( 1 - \frac{1 + r}{F} \right) \right] + 2(2M_*^2 - M^2) \frac{1}{M_*^2 - t} + \frac{\mu^2 - M^2}{m_{K*}^2} f_+(t),$$  \hspace{1cm} (49)

where $f$ is fixed at unity (KSRF relation) and
As a result, the ratio of the form factors in problem becomes

\[
\frac{\xi(0)}{f_{+}(0)} = \left(\frac{1}{4} F - 1\right) \frac{M^2 - \mu^2}{m_K^2} + F - 1
+ \frac{1}{4} \frac{r-1}{r+1} F \left[ \mu^2 - 2M^2 + \frac{3r-1}{2} \mu^2 + \frac{3-r}{2} M^2 \right] \left( 1 - \frac{1+r}{F} \right) \left( \frac{4}{r^2 - 6r + 1} \right) \frac{1}{M^2} .
\]

The parameter \( r \) should be in the region

\[
3 - 2\sqrt{2} < r < 3 + 2\sqrt{2}.
\]

This condition has been deduced from the positivity condition of Eqs. (38) and (50). In Fig. 3, \( \xi(0) \) versus \( r \) is plotted for four typical values of \( M_0 \), where \( F \) is fixed at the experimental data 1.28. In this case \( \lambda_+ \) takes a rather small value 0.017. It is evident from Fig. 3 that the large negative value for \( \xi(0) \) suggests either the very small \( \kappa \) mass or the large \( SU(3) \) violation of the vacuum. As already stated, neither possibilities can be accepted. The situation would be slightly improved if we take into account a nonminimal interaction such as

\[
\text{Tr} \left[ (F_{\rho\sigma} + G_{\rho\sigma}) A^\dagger B^+ A^\dagger B^+ + (F_{\rho\sigma} - G_{\rho\sigma}) A^\dagger B^- A^\dagger B^+ \right].
\]

This term contributes to \( f_{+}(t) \) through the \( K^* K \pi \) coupling but not to \( f(t) \), so that it has effect on \( \lambda_+ \) and \( \xi(0) \), that is,

\[
\lambda_+ = (\lambda_+)_{\text{minimal}} + 0.0125 A
\]

and

\[
\xi(0) = (\xi(0))_{\text{minimal}} - 0.179 A ,
\]

where \( A \) is a parameter in an arbitrary unit. The modified \( \xi(0) \) curves are shown also in Fig. 3 for \( A=1 \) and \( A=3 \). From these curves, it is concluded that, unless we admit a large positive \( \lambda_+ \) value, say, \( \lambda_+ \approx 0.545 \) for \( A=3 \), \( \xi(0) \) never becomes less than \(-0.5\) for reasonable \( r(1 \leq r \leq 2) \). To be more clearly,
the additional term does not contribute to the combination $\sim 14\lambda_+ + \xi(0)$ which is about $-0.2$ from the recent experiment.\cite{15} In this scheme, for $M_\pi \simeq 0.89$ GeV,\cite{16} $r$ must be larger than five as shown in Fig. 3. These results are essentially due to the fact that the off-shell extrapolation derived from this model is still too smooth to explain the large deviation of the experimental value from that in the soft-pion limit. It is interesting, however, to note that if a larger value for $\lambda_+ (\lambda_+ \gtrsim 0.05)$ is given by future experiments, the large negative value for $\xi(0)$ ($\xi(0) \lesssim -0.5$) will naturally be understood in the framework of this model, since large $A$ in Eq. (54) gives large positive $\lambda_+$ and large negative $\xi(0)$.

§ 5. Summary

In this paper we have studied effects of $(1, 8) + (8, 1)$ symmetry breaking on $K_{13}$ decays and the behavior of the amplitude extrapolated from the soft-pion limit. The results obtained are summarized as follows:

1) Since Weinstein's work\cite{31} about the effect of $(3, 3^*) + (3^*, 3)$ symmetry breaking on $\xi(M^2)$ in the soft-pion limit, additional $(1, 8) + (8, 1)$ symmetry breaking has been studied by many authors.\cite{40, 41} We have found in §1 that, even in the presence of $(1, 8) + (8, 1)$ symmetry breaking, Weinstein's result remains essentially unchanged and the value $\xi \simeq -0.1$ requires large $SU(3)$ violation of the renormalization constant and the vacuum.

2) As the physical region is far from the soft-pion point, the behavior of the off-mass shell amplitude is essential to $K_{13}$ decays. Firstly we have examined the scalar form factor based on the generalized σ model which has the general form of interaction terms without derivatives. Then we have found that under the condition $\xi \leq -0.5$, the $\kappa$ mass must be lower than the kaon mass. Although it seems similar to the conclusion obtained by Berman and Roy,\cite{10} our approach is essentially different from theirs. They have proposed that the C-T relation must be modified owing to the existence of the $\kappa$ meson, but the scalar form factor (27) in §3 satisfies the C-T relation. Secondly we have further discussed on the basis of the generalized σ model which includes spin-one mesons as massive gauge fields. However, this model has still too smooth off-shell behavior to explain both the present $\lambda_+$ and $\xi(0)$ values. In essence, one always gets the C-T relation in the world where chiral $SU(2) \times SU(2)$ is exact, and simple traditional models which have been powerful in low energy hadron physics cannot give sufficient variations of $K_{13}$ form factors.

In conclusion it seems to us that if the present experimental data remain correct, only the four possibilities will be left:

1) need of other symmetry breaking besides $(3, 3^*) + (3^*, 3)$ and $(1, 8) + (8, 1)$ ones,

2) much higher order derivatives or violation of the tree approximation,

3) existence of the $\kappa$ pole whose mass is lower than the kaon and
4) large $SU(3)$ violation of the vacuum.
The possibility 1) in which the C-T relation is modified may not only lead us to a very complicated case with many parameters but also demand change of the usual point of view. As a model of this case, we examine effects of (8, 8) symmetry breaking in Appendix A. The possibility 2) implies the failure of the simple and successful view that, at low energies, the single meson exchange mechanism plays a dominant role. The last two cases could not be accepted from the experimental and the phenomenological analyses. Anyway further investigations and experiments will be needed in order that we may understand $K_{13}$ decays naturally.

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Appendix A
We shall deal with the effect of (8, 8) symmetry breaking on the parameter $\xi$. A symmetry-breaking Hamiltonian is assumed to be

$$\mathcal{H}_{S,B} = S_0 + CS_\xi,$$  \hspace{1cm} (A1)

where $S_0$ ($S_\xi$) denotes a scalar $SU(3)$ octet (singlet) contained in the (8, 8) representation, which satisfies the following commutation relations:

$$[Q_{a}^8, S_\xi] = i \sqrt{\frac{3}{8}} P_\alpha, \quad (\alpha = 1, \cdots, 8)$$ \hspace{1cm} (A2)

$$[Q_{a}^8, S_0] = i \frac{3}{\sqrt{5}} d_{a\bar{b}r} P_r + \text{(operators belonging to 10 and 1\bar{1}0)},$$

where $P_\alpha$ is a parity partner of $S_\alpha$ ($\alpha = 1, \cdots, 8$). Similarly we get easily current divergences

$$\partial_\mu A_{\mu} = - \left( \sqrt{\frac{3}{8}} + \frac{3}{\sqrt{5}} C d_{a\bar{b}a} \right) P_\alpha + \text{(operators belonging to 10 and 1\bar{1}0)},$$ \hspace{1cm} (A3)

$$-i[Q_{a}^8, \partial_\mu V_{\mu+1356}] = i C \frac{3 \sqrt{3}}{4 \sqrt{5}} P_{+1356} + \text{(decuplet part)}. \hspace{1cm} (A4)$$

If we sandwich both sides of Eqs. (A3) and (A4) between pseudoscalar meson states and the vacuum, the 10 and 1\bar{1}0 terms can be dropped and then

$$\frac{\mu^2}{\sqrt{2}} f_\pi = - \left( \sqrt{\frac{3}{8}} + \sqrt{\frac{3}{5}} C \right) \langle 0 | P_8 | \pi^0 \rangle,$$ \hspace{1cm} (A5)

$$M^f K = - \left( \sqrt{\frac{3}{8}} - \frac{\sqrt{3}}{2 \sqrt{5}} C \right) \langle 0 | P_{+1356} | K^+ \rangle. \hspace{1cm} (A6)$$
If the total SU(3) × SU(3) breaking Hamiltonian is composed of operators belonging to the (3, 3*) + (3*, 3), (1, 8) + (8, 1) and (8, 8) representations, the final result becomes

\[
\frac{\mu}{\sqrt{2}} f_\pi = \left( \sqrt{\frac{2}{3}} \epsilon_0 + \sqrt{\frac{1}{3}} \epsilon_3 \right) \langle 0 | v_{3s} | \pi^0 \rangle - \left( \sqrt{\frac{3}{8}} + \frac{3}{5} C \right) \langle 0 | P_3 | \pi^0 \rangle, \quad (A7)
\]

\[
M f_\pi = \left( \sqrt{\frac{2}{3}} \epsilon_0 - \frac{1}{2 \sqrt{3}} \epsilon_3 \right) \langle 0 | v_{4b+4s} | K^+ \rangle - \frac{\sqrt{3} i}{2} x_1 \langle 0 | h_{4b+4s} | K^+ \rangle
- \left( \sqrt{\frac{3}{8}} - \frac{3}{2 \sqrt{5}} C \right) \langle 0 | P_{4b+4s} | K^+ \rangle \quad (A9)
\]

and

\[
-i \langle 0 | [Q_3, \partial_\mu V_{4b+4s}] | K^+ \rangle = -i \frac{\sqrt{3} i}{4} \left[ \frac{\epsilon_0}{\sqrt{3}} \langle 0 | v_{4b+4s} | K^+ \rangle + i x_1 \langle 0 | h_{4b+4s} | K^+ \rangle \right]
- \frac{3 C}{\sqrt{5}} \langle 0 | P_{4b+4s} | K^+ \rangle. \quad (A10)
\]

With these equations, one gets

\[
\xi(M^2) = - \left( 1 - \frac{\mu^2}{M^2} \right) + \frac{1}{f_\pi(M^2)} \left( \frac{f_K}{f_\pi} \right) \left( 1 - \frac{f_\pi}{f_K} \frac{\mu^2}{M^2} \right) \frac{\langle 0 | v_{4b+4s} | K^+ \rangle}{\sqrt{2} \langle 0 | v_{3s} | \pi^0 \rangle} - R, \quad (A11)
\]

\[
R = \left( \sqrt{\frac{3}{8}} + \frac{3}{5} C \right) \frac{\langle 0 | P_3 | \pi^0 \rangle}{f_K M^2} \left[ \frac{\langle 0 | v_{4b+4s} | K^+ \rangle}{\langle 0 | v_{3s} | \pi^0 \rangle} - \frac{\langle 0 | P_{4b+4s} | K^+ \rangle}{\langle 0 | P_3 | \pi^0 \rangle} \right].
\]

Consequently unless \( R = 0 \), we have different results. It is of interest to see that also in this case (1, 8) + (8, 1) symmetry breaking has no effects on \( \xi(M^2) \) explicitly.

**Appendix B**

Explicit forms of independent invariants are as follows:

\[
I_1^+ = \frac{1}{2} (B^+_+ B^-_+), \quad I_1^- = \frac{1}{2i} (B^+_+ B^-_+),
\]

\[
I_3 = B^+ - B^-,
\]

\[
I_3^+ = \frac{1}{2} \epsilon_{\alpha\beta\gamma} \epsilon^{\alpha\beta} (B^+_+ B^+_\beta + B^-_\alpha B^-_\beta),
\]

\[
J^x = \frac{1}{2} \langle B^+_+ B^+_+ B^-_+ B^-_+ \rangle,
\]

and

\[
J^z = \frac{1}{2} \langle B^+_+ B^-_+ B^+_+ B^-_+ \rangle.
\]
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\[ K^+ = \frac{1}{2} (B^+ a^a B^- \bar{a} B^+ \bar{a} + B^- a B^+ \bar{a} B^- \bar{a}) , \]
\[ K^- = \frac{1}{2i} (B^+ a^a B^- \bar{a} B^+ \bar{a} - B^- a B^+ \bar{a} B^- \bar{a}) , \]
\[ L^+ = \frac{1}{2} \epsilon_{\alpha' \beta' \alpha \beta} (B^+ a^a B^- \bar{a} B^+ \bar{a} + B^- a B^+ \bar{a} B^- \bar{a}) , \]
\[ L^- = \frac{1}{2i} \epsilon_{\alpha' \beta' \alpha \beta} (B^+ a^a B^- \bar{a} B^+ \bar{a} - B^- a B^+ \bar{a} B^- \bar{a}) , \]

where indices $\alpha$, $\beta$, $\alpha'$ and $\beta'$ run through 1 and 2.

References

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